Math 241 - Calculus III
Spring 2012, section CL1
§ 16.5. Conservative vector fields in $\mathbb{R}^3$

In these notes, we discuss conservative vector fields in 3 dimensions, and highlight the similarities and differences with the 2-dimensional case. Compare with the notes on § 16.3.

1 Conservative vector fields

Let us recall the basics on conservative vector fields.

Definition 1.1. Let $\vec{F}: D \to R^n$ be a vector field with domain $D \subseteq \mathbb{R}^n$. The vector field $\vec{F}$ is said to be **conservative** if it is the gradient of a function. In other words, there is a differentiable function $f: D \to \mathbb{R}$ satisfying $\vec{F} = \nabla f$. Such a function f is called a **potential function** for $\vec{F}$.

Example 1.2. $\vec{F}(x, y, z) = (y^2z^3, 2xyz^3, 3xy^2z^2)$ is conservative, since it is $\vec{F} = \nabla f$ for the function $f(x, y, z) = xy^2z^3$.

Example 1.3. $\vec{F}(x, y, z) = (3x^2z, z^2, x^3+2yz)$ is conservative, since it is $\vec{F} = \nabla f$ for the function $f(x, y, z) = x^3z + yz^2$.

The fundamental theorem of line integrals makes integrating conservative vector fields along curves very easy. The following proposition explains in more detail what is nice about conservative vector fields.

Proposition 1.4. *The following properties of a vector field $\vec{F}$ are equivalent.*

1. *$\vec{F}$ is conservative.*
2. *$\int_C \vec{F} \cdot d\vec{r}$ is path-independent, meaning that it only depends on the endpoints of the curve C.*
3. *$\oint_C \vec{F} \cdot d\vec{r} = 0$ around any closed curve C.*

Example 1.5. Find the line integral $\int_C \vec{F} \cdot d\vec{r}$ of the vector field $\vec{F}(x, y, z) = (3x^2z, z^2, x^3 + 2yz)$ along the curve C parametrized by

$$\vec{r}(t) = \left(\frac{\ln t}{\ln 2}, t^{\frac{3}{2}}, t\cos(\pi t) \right), \ 1 \leq t \leq 4.$$

Solution. We know that $\vec{F}$ is conservative, with potential function $f(x, y) = x^3z + yz^2$. The endpoints of C are

$$\vec{r}(1) = (0, 1, -1)$$
$$\vec{r}(4) = (\frac{\ln 4}{\ln 2}, 8, 0) = (2, 8, 0).$$

The fundamental theorem of line integrals yields

$$\begin{aligned}\int_C \vec{F} \cdot d\vec{r} &= \int_C \nabla f \cdot d\vec{r} \\ &= f(\vec{r}(4)) - f(\vec{r}(1)) \\ &= f(2,8,0) - f(0,1,-1) \\ &= 0 - 1 \\ &= \boxed{-1}.\end{aligned}$$

2 Necessary conditions

To know if a vector field $\vec{F}$ is conservative, the first thing to check is the following criterion.

Proposition 2.1. *Let $D \subseteq \mathbb{R}^3$ be an open subset and let $\vec{F}\colon D \to R^3$ be a continuously differentiable vector field with domain D. If $\vec{F}$ is conservative, then it satisfies $\operatorname{curl} \vec{F} = \vec{0}$.*

Explicitly, $\vec{F} = (F_1, F_2, F_3)$ satisfies the three conditions

$$\begin{aligned}\partial_1 F_2 &= \partial_2 F_1 \\ \partial_1 F_3 &= \partial_3 F_1 \\ \partial_2 F_3 &= \partial_3 F_2\end{aligned}$$

everywhere on D.

Proof. Assume there is a differentiable function $f\colon D \to \mathbb{R}$ satisfying $\vec{F} = \nabla f$ on D. Because f is twice continuously differentiable (meaning it has all second partial derivatives and they are all continuous), Clairaut's theorem applies, meaning the mixed partial derivatives agree. Since the first partial derivatives of f are $(f_x, f_y, f_z) = (\partial_1 f, \partial_2 f, \partial_3 f) = (F_1, F_2, F_3)$, we obtain

$$\begin{aligned}\partial_1\partial_2 f &= \partial_2\partial_1 f \\ \partial_1 F_2 &= \partial_2 F_1 \\ \partial_1\partial_3 f &= \partial_3\partial_1 f \\ \partial_1 F_3 &= \partial_3 F_1 \\ \partial_2\partial_3 f &= \partial_3\partial_2 f \\ \partial_2 F_3 &= \partial_3 F_2.\end{aligned}$$

These conditions are equivalent to $\operatorname{curl} \vec{F} = \vec{0}$, because of the formula:

$$\begin{aligned}\operatorname{curl} \vec{F} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_1 & \partial_2 & \partial_3 \\ F_1 & F_2 & F_3 \end{vmatrix} \\ &= \vec{i}(\partial_2 F_3 - \partial_3 F_2) - \vec{j}(\partial_1 F_3 - \partial_3 F_1) + \vec{k}(\partial_1 F_2 - \partial_2 F_1) \\ &= (\partial_2 F_3 - \partial_3 F_2, \partial_3 F_1 - \partial_1 F_3, \partial_1 F_2 - \partial_2 F_1).\end{aligned}$$

□

Example 2.2. The vector field $\vec{F} = (x, y, 5x)$ is **not** conservative, because its curl

$$\begin{aligned}
\operatorname{curl} \vec{F} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_x & \partial_y & \partial_z \\ x & y & 5x \end{vmatrix} \\
&= \vec{i}(0-0) - \vec{j}(5-0) + \vec{k}(0-0) \\
&= (0, -5, 0)
\end{aligned}$$

is not the zero vector field. In terms of partial derivatives, this is saying $\partial_z(x) \neq \partial_x(5x)$.

Remark 2.3. Most vector fields are **not** conservative. If we pick functions F_1, F_2, F_3 "at random", then in general they will not satisfy the conditions $\partial_1 F_2 = \partial_2 F_1$, $\partial_1 F_3 = \partial_3 F_1$, $\partial_2 F_3 = \partial_3 F_2$.

Definition 2.4. A vector field $\vec{F}$ is called **irrotational** if it satisfies $\operatorname{curl} \vec{F} = \vec{0}$.

The terminology comes from the physical interpretation of the curl. If $\vec{F}$ is the velocity field of a fluid, then $\operatorname{curl} \vec{F}$ measures in some sense the tendency of the fluid to rotate.

With that terminology, proposition 2.1 says that a conservative vector field is always irrotational.

Remark 2.5. That statement also holds in 2 dimensions: A conservative vector field is always irrotational. A vector field $\vec{F} = (F_1, F_2)$ is called irrotational if its "scalar curl" or "2-dimensional curl" $\partial_1 F_2 - \partial_2 F_1$ is zero.

Question 2.6. If a vector field is irrotational, is it automatically conservative?

Answer: NO.

Example 2.7. Recall the 2-dimensional vector field $\frac{1}{x^2+y^2}(-y, x)$ with domain the punctured plane

$$\mathbb{R}^2 \setminus \{(0,0)\} = \{(x,y) \in \mathbb{R}^2 \mid (x,y) \neq (0,0)\}.$$

In § 16.3, we saw that this vector field is irrotational but not conservative.

We can reuse that example in 3 dimensions by making the third component zero. Consider the vector field $\vec{F} = \frac{1}{x^2+y^2}(-y, x, 0)$ with domain

$$D = \mathbb{R}^3 \setminus \{z\text{-axis}\} = \{(x,y,z) \in \mathbb{R}^3 \mid (x,y) \neq (0,0)\}.$$

Then $\vec{F}$ is irrotational, i.e. it satisfies $\operatorname{curl} \vec{F} = \vec{0}$. However, $\vec{F}$ is **not** conservative, because the line integral of $\vec{F}$ along a loop around the z-axis is 2π and not zero.

3 Sufficient conditions

Depending on the shape of the domain D, the condition $P_y = Q_x$ is sometimes enough to guarantee that the field is conservative.

Proposition 3.1. *Let $\vec{F} \colon \mathbb{R}^3 \to \mathbb{R}^3$ is a continuously differentiable vector field (whose domain is all of $\mathbb{R}^3$). If $\vec{F}$ satisfies $\operatorname{curl} \vec{F} = \vec{0}$, then $\vec{F}$ is conservative.*

Example 3.2. Consider the vector field $\vec{F} = (3x^2y^2z + 5y^3, 2x^3yz + 15xy^2 - 7z, x^3y^2 - 7y + 4z^3)$ with domain $\mathbb{R}^3$. Determine whether $\vec{F}$ is conservative, and if it is, find a potential function for it.

Solution. First we compute

$$\begin{aligned}
\operatorname{curl} \vec{F} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_x & \partial_y & \partial_z \\ 3x^2y^2z + 5y^3 & 2x^3yz + 15xy^2 - 7z & x^3y^2 - 7y + 4z^3 \end{vmatrix} \\
&= \vec{i}\left(2x^3y - 7 - (2x^3y - 7)\right) - \vec{j}(3x^2y^2 - 3x^2y^2) + \vec{k}\left(6x^2yz + 15y^2 - (6x^2yz + 15y^2)\right) \\
&= (0, 0, 0).
\end{aligned}$$

Moreover, $\vec{F}$ is defined (and smooth) on all of $\mathbb{R}^3$, hence it is conservative. Let us find a potential function $f(x, y, z)$ for $\vec{F}$. We want

$$\begin{aligned}
f_x &= F_1 = 3x^2y^2z + 5y^3 \\
f_y &= F_2 = 2x^3yz + 15xy^2 - 7z \\
f_z &= F_3 = x^3y^2 - 7y + 4z^3.
\end{aligned}$$

Using the first equation, we obtain

$$\begin{aligned}
f &= \int F_1 \,\mathrm{dx} \\
&= \int 3x^2y^2z + 5y^3 \,\mathrm{dx} \\
&= x^3y^2z + 5xy^3 + g(y, z)
\end{aligned}$$

whose derivative with respect to y is

$$2x^3yz + 15xy^2 + g_y(y, z).$$

Using the second equation, we equate this with F_2:

$$\begin{aligned}
2x^3yz + 15xy^2 + g_y(y, z) &= 2x^3yz + 15xy^2 - 7z \\
\Rightarrow g_y(y, z) &= -7z \\
\Rightarrow g(y, z) &= \int -7z \, dy \\
&= -7yz + h(z).
\end{aligned}$$

Plugging this back into the expression for f, we obtain

$$f = x^3y^2z + 5xy^3 - 7yz + h(z)$$

whose derivative with respect to z is

$$x^3y^2 - 7y + h'(z).$$

Using the third equation, we equate this with F_3:

$$\begin{aligned}
x^3y^2 - 7y + h'(z) &= x^3y^2 - 7y + 4z^3 \\
\Rightarrow h'(z) &= 4z^3 \\
\Rightarrow h(z) &= \int 4z^3 dz \\
&= z^4 + c.
\end{aligned}$$

Choosing the constant $c = 0$, we obtain $h(z) = z^4$ and thus the potential function

$$\boxed{f(x, y, z) = x^3y^2z + 5xy^3 - 7yz + z^4}.$$

4 Simply connected domains

Asking for $\vec{F}$ to be defined (and continuously differentiable) on all of $\mathbb{R}^3$ is somewhat restrictive. That condition can be loosened.

Definition 4.1. A subset D of $\mathbb{R}^n$ is called **simply connected** if it is path-connected moreover, and every loop in D can be contracted to a point.

Example 4.2. $\mathbb{R}^3$ itself is simply connected.

Example 4.3. The first octant $\{(x, y, z) \in \mathbb{R}^3 \mid x, y, z > 0\}$ is simply connected.

Example 4.4. The open ball $\{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 < 1\}$ is simply connected.

Example 4.5. The closed ball $\{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 \leq 1\}$ is simply connected.

Example 4.6. The (surface of the) sphere $\{(x, y, z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 = 1\}$ is simply connected. See the nice picture here:

`http://en.wikipedia.org/wiki/Simply-connected#Informal_discussion`.

Example 4.7. **Interesting fact:** The punctured space

$$\mathbb{R}^3 \setminus \{(0, 0, 0)\} = \{(x, y, z) \in \mathbb{R}^3 \mid (x, y, z) \neq (0, 0, 0)\}$$

is simply connected. This might seem confusing, since the punctured plane $\mathbb{R}^2 \setminus \{(0, 0)\}$ is **not** simply connected. That is because in 3 dimensions, there is enough room to move a loop around the puncture and then contract it to a point. Therefore the informal idea that “simply connected means no holes” is not really accurate.

Example 4.8. The “thick sphere” $\{(x, y, z) \in \mathbb{R}^3 \mid 1 < x^2 + y^2 + z^2 < 4\}$ between radii 1 and 2 is simply connected.

Example 4.9. $\mathbb{R}^3$ with a line removed, for example

$$D = \mathbb{R}^3 \setminus \{z\text{-axis}\} = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 \neq 0\}$$

is **not** simply connected. Indeed, any curve in D going once around the z-axis cannot be contracted to a point.

Example 4.10. The “thick cylinder” $\{(x, y, z) \in \mathbb{R}^3 \mid 1 \leq x^2 + y^2 \leq 4\}$ between radii 1 and 2 is **not** simply connected, for the same reason.

Example 4.11. The solid torus $\{((3 + u\cos\alpha)\cos\theta, (3 + u\cos\alpha)\sin\theta, u\sin\alpha) \mid \alpha, \theta \in \mathbb{R}, 0 \leq u \leq 1\}$ is **not** simply connected. A curve going once around the “hole” in the middle (e.g. u, α constant, θ goes from 0 to 2π) cannot be contracted to a point.

Example 4.12. The (surface of the) torus $\{((3+\cos\alpha)\cos\theta, (3+\cos\alpha)\sin\theta, \sin\alpha) \mid \alpha, \theta \in \mathbb{R}\}$ is **not** simply connected. A curve going once around the “hole” in the middle (e.g. α constant, θ goes from 0 to 2π) cannot be contracted to a point. Also, a curve going once around the “tire” (e.g. θ constant, α goes from 0 to 2π) cannot be contracted to a point.

With that notion, we obtain the following improvement on proposition 3.1.

Theorem 4.13. *Let $D \subseteq \mathbb{R}^3$ be open and simply connected, and let $\vec{F}\colon D \to \mathbb{R}^3$ is a continuously differentiable vector field with domain D. If $\vec{F} = (P, Q)$ satisfies the condition* $\operatorname{curl} \vec{F} = \vec{0}$*, then $\vec{F}$ is conservative (on D).*

Theorem 4.13 did not apply to example 2.7, because the domain $D = \mathbb{R}^3 \setminus \{z\text{-axis}\}$ of $\vec{F}$ was not simply connected.

When theorem 4.13 does not apply because the domain is not simply connected, then we cannot conclude from the condition $\operatorname{curl} \vec{F} = \vec{0}$ alone. A more subtle analysis is required.

Example 4.14. Recall the 2-dimensional vector field $\frac{1}{x^2+y^2}(x, y)$ with domain the punctured plane

$$\mathbb{R}^2 \setminus \{(0,0)\} = \{(x, y) \in \mathbb{R}^2 \mid (x, y) \neq (0, 0)\}.$$

In § 16.3, we saw that this vector field conservative, with potential function $f(x, y) = \frac{1}{2}\ln(x^2 + y^2)$.

Again, we can turn this example into a 3-dimensional example by making the third component zero. Consider the vector field $\vec{F} = \frac{1}{x^2+y^2}(x, y, 0)$ with domain

$$D = \mathbb{R}^3 \setminus \{z\text{-axis}\} = \{(x, y, z) \in \mathbb{R}^3 \mid (x, y) \neq (0, 0)\}.$$

Then $\vec{F}$ is conservative, with the same potential function $f(x, y) = \frac{1}{2}\ln(x^2 + y^2)$. However, computing $\operatorname{curl} \vec{F} = \vec{0}$ was **not** enough to conclude that $\vec{F}$ is conservative, since its domain D is not simply connected.

Example 4.15. Consider the vector field $\vec{F} = \frac{1}{x^2+y^2+z^2}(x, y, z)$. Is $\vec{F}$ conservative? If it is, find a potential for $\vec{F}$.

Solution. As shorthand notation, write $\rho = \sqrt{x^2 + y^2 + z^2}$ for the distance to the origin, and note $\frac{\partial \rho}{\partial x} = \frac{x}{\rho}$. Now we compute

$$\begin{aligned}
\operatorname{curl} \vec{F} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_x & \partial_y & \partial_z \\ \frac{x}{\rho^2} & \frac{y}{\rho^2} & \frac{z}{\rho^2} \end{vmatrix} \\
&= \vec{i}\left(z(-2\rho^{-3})\frac{y}{\rho} - y(-2\rho^{-3})\frac{z}{\rho}\right) - \vec{j}\left(z(-2\rho^{-3})\frac{x}{\rho} - x(-2\rho^{-3})\frac{z}{\rho}\right) \\
&\qquad + \vec{k}\left(y(-2\rho^{-3})\frac{x}{\rho} - x(-2\rho^{-3})\frac{y}{\rho}\right) \\
&= -2\rho^{-4}(zy - yz, xz - zx, yx - xy) \\
&= (0, 0, 0).
\end{aligned}$$

Moreover, $\vec{F}$ is defined (and smooth) on the punctured space $\mathbb{R}^3 \setminus \{(0, 0, 0)\}$, which is simply connected. Therefore $\vec{F}$ is conservative.

Let us find a potential f for $\vec{F}$. We want

$$f_x = F_1 = \frac{x}{\rho^2}$$
$$f_y = F_2 = \frac{y}{\rho^2}$$
$$f_z = F_3 = \frac{z}{\rho^2}.$$

Using the first equation, we obtain

$$\begin{aligned} f &= \int F_1 \, \mathrm{dx} \\ &= \int \frac{x}{x^2+y^2+z^2} \, \mathrm{dx} \;\; \text{Take } u = x^2+y^2+z^2,\ du = 2xdx \\ &= \int \frac{1}{u}\frac{1}{2} du \\ &= \frac{1}{2}\ln u + g(y,z) \\ &= \frac{1}{2}\ln(x^2+y^2+z^2) + g(y,z) \end{aligned}$$

whose derivative with respect to y is

$$\frac{y}{x^2+y^2+z^2} + g_y(y,z).$$

Using the second equation, we equate this with F_2:

$$\begin{aligned} \frac{y}{x^2+y^2+z^2} + g_y(y,z) &= \frac{y}{x^2+y^2+z^2} \\ \Rightarrow g_y(y,z) &= 0 \\ \Rightarrow g(y,z) &= 0 + h(z). \end{aligned}$$

Plugging this back into the expression for f, we obtain

$$f = \frac{1}{2}\ln(x^2+y^2+z^2) + h(z)$$

whose derivative with respect to z is

$$\frac{z}{x^2+y^2+z^2} + h'(z).$$

Using the third equation, we equate this with F_3:

$$\begin{aligned} \frac{z}{x^2+y^2+z^2} + h'(z) &= \frac{z}{x^2+y^2+z^2} \\ \Rightarrow h'(z) &= 0 \\ \Rightarrow h(z) &= c. \end{aligned}$$

Choosing the constant $c = 0$, we obtain $h(z) = 0$ and thus the potential function

$$\boxed{f(x,y,z) = \frac{1}{2}\ln(x^2+y^2+z^2)}.$$

5 Summary

Suppose we are given a vector field $\vec{F}$ in 3 dimensions and we want to know if $\vec{F}$ is conservative.

1. Compute $\operatorname{curl} \vec{F}$. If $\operatorname{curl} \vec{F} \neq \vec{0}$, then $\vec{F}$ is certainly **not** conservative.
2. If $\operatorname{curl} \vec{F} = \vec{0}$ and the domain of $\vec{F}$ is all of $\mathbb{R}^3$ (or more generally: a simply-connected region), then $\vec{F}$ is certainly conservative.
3. If $\operatorname{curl} \vec{F} = \vec{0}$ and the domain of $\vec{F}$ is not simply connected, then one cannot conclude: $\vec{F}$ could be conservative or not. One must work harder to answer the question.