Math 241 - Calculus III
Spring 2012, section CL1
§ 16.3. Conservative vector fields and simply connected domains

In these notes, we discuss the problem of knowing whether a vector field is conservative or not.

1 Conservative vector fields

Let us recall the basics on conservative vector fields.

Definition 1.1. Let $\vec{F}: D \to R^n$ be a vector field with domain $D \subseteq \mathbb{R}^n$. The vector field $\vec{F}$ is said to be **conservative** if it is the gradient of a function. In other words, there is a differentiable function $f: D \to \mathbb{R}$ satisfying $\vec{F} = \nabla f$. Such a function f is called a **potential function** for $\vec{F}$.

Example 1.2. $\vec{F}(x, y) = (x, y)$ is conservative, since it is $\vec{F} = \nabla f$ for the function $f(x, y) = \frac{1}{2}(x^2 + y^2)$.

Example 1.3. $\vec{F}(x, y) = (y \cos x, \sin x)$ is conservative, since it is $\vec{F} = \nabla f$ for the function $f(x, y) = y \sin x$.

The fundamental theorem of line integrals makes integrating conservative vector fields along curves very easy. The following proposition explains in more detail what is nice about conservative vector fields.

Proposition 1.4. *The following properties of a vector field $\vec{F}$ are equivalent.*

1. *$\vec{F}$ is conservative.*
2. *$\int_C \vec{F} \cdot d\vec{r}$ is path-independent, meaning that it only depends on the endpoints of the curve C.*
3. *$\oint_C \vec{F} \cdot d\vec{r} = 0$ around any closed curve C.*

Example 1.5. Find the line integral $\int_C \vec{F} \cdot d\vec{r}$ of the vector field $\vec{F}(x, y) = (y \cos x, \sin x)$ along the curve C parametrized by $\vec{r}(t) = (\frac{\pi \ln t}{\ln 16}, t^{\frac{3}{2}})$, $1 \leq t \leq 4$.

Solution. We know that $\vec{F}$ is conservative, with potential function $f(x, y) = y \sin x$. The endpoints of C are $\vec{r}(1) = (0, 1)$ and $\vec{r}(4) = (\frac{\pi \ln 4}{\ln(4^2)}, 8) = (\frac{\pi}{2}, 8)$. The fundamental theorem of calculus yields

$$\begin{aligned}
\int_C \vec{F} \cdot d\vec{r} &= \int_C \nabla f \cdot d\vec{r} \\
&= f(\vec{r}(4)) - f(\vec{r}(1)) \\
&= f(\frac{\pi}{2}, 8) - f(0, 1) \\
&= 8 \sin \frac{\pi}{2} - 1 \sin 0 \\
&= \boxed{8}.
\end{aligned}$$

2 Necessary conditions

To know if a vector field $\vec{F}$ is conservative, the first thing to check is the following criterion.

Proposition 2.1. *Let $D \subseteq \mathbb{R}^2$ be an open subset and let $\vec{F}\colon D \to R^2$ be a continuously differentiable vector field with domain D. If $\vec{F} = (P, Q)$ is conservative, then it satisfies the condition*

$$P_y = Q_x$$

everywhere on D.

Proof. Assume there is a differentiable function $f\colon D \to \mathbb{R}$ satisfying $\vec{F} = \nabla f$ on D. Because f is twice continuously differentiable (meaning it has all second partial derivatives $f_{xx}, f_{xy}, f_{yx}, f_{yy}$ and they are all continuous), Clairaut's theorem applies, meaning the mixed partial derivatives agree. Since the first partial derivatives of f are $(f_x, f_y) = (P, Q)$, we obtain

$$\begin{aligned} f_{xy} &= f_{yx} \\ (f_x)_y &= (f_y)_x \\ P_y &= Q_x. \end{aligned}$$

□

Example 2.2. The vector field $\vec{F} = (P, Q) = (ye^x, x^2y^5)$ is **not** conservative, because the partial derivatives are

$$\begin{aligned} P_y &= e^x \\ Q_x &= 2xy^5 \neq P_y. \end{aligned}$$

Remark 2.3. Most vector fields are **not** conservative. If we pick functions P and Q "at random", then in general they will not satisfy $P_y = Q_x$.

Remark 2.4. The analogue of proposition 2.1 in higher dimension $\mathbb{R}^n$ holds as well, since Clairaut's theorem holds in any dimension. However, there are more conditions to write down in higher dimension, because there are more mixed second derivatives. Namely, there are $\binom{n}{2} = \frac{n(n-1)}{2}$ conditions. In $\mathbb{R}^3$ for example, a conservative vector field $\vec{F} = (P, Q, R)$ satisfies the 3 conditions

$$\begin{aligned} P_y &= Q_x \\ P_z &= R_x \\ Q_z &= R_y. \end{aligned}$$

In §16.5, we will encode this information in something called the curl of $\vec{F}$.

Remark 2.5. Proposition 2.1 says that of $\vec{F} = (P, Q)$ is conservative, then the function $Q_x - P_y$ is identically zero. This function $Q_x - P_y$ is sometimes called the (2-dimensional) curl of $\vec{F}$, which will play an important role in §16.4. With that terminology, proposition 2.1 says "conservative implies curl is zero". Conveniently, that same statement will be the 3-dimensional analogue of proposition 2.1 in §16.5.

Question 2.6. If a vector field $\vec{F} = (P, Q)$ satisfies the condition $P_y = Q_x$, is it automatically conservative?

Answer: NO.

Example 2.7. Consider the vector field $\vec{F} = \frac{1}{x^2+y^2}(-y, x)$ on the domain

$$D = \mathbb{R}^2 \setminus \{(0,0)\} = \{(x,y) \in \mathbb{R}^2 \mid (x,y) \neq (0,0)\}.$$

Is $\vec{F}$ conservative?

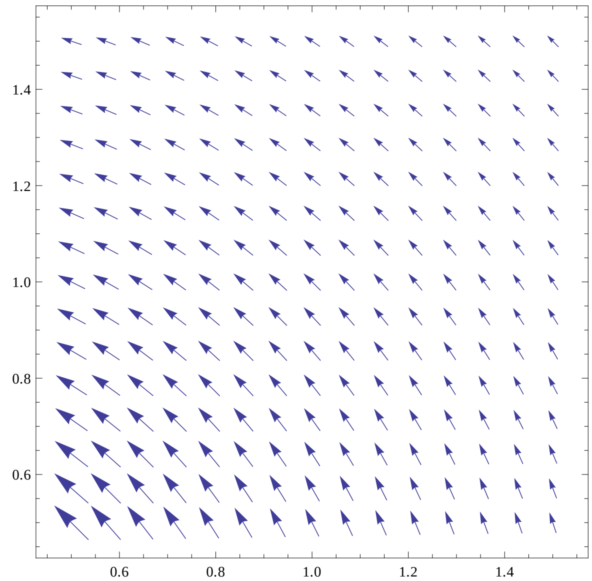

Figure 1: Plot of $\vec{F}$ in the first quadrant.

Writing $P = \frac{-y}{x^2+y^2}$ and $Q = \frac{x}{x^2+y^2}$, let us compute

$$\begin{aligned} P_y &= \frac{(-1)(x^2+y^2) - (-y)(2y)}{(x^2+y^2)^2} \\ &= \frac{-x^2 - y^2 + 2y^2}{(x^2+y^2)^2} \\ &= \frac{y^2 - x^2}{(x^2+y^2)^2} \\ Q_x &= \frac{(1)(x^2+y^2) - (x)(2x)}{(x^2+y^2)^2} \\ &= \frac{x^2 + y^2 - 2x^2}{(x^2+y^2)^2} \\ &= \frac{y^2 - x^2}{(x^2+y^2)^2} \end{aligned}$$

so that the vector field satisfies $P_y = Q_x$.

However, we will show that $\vec{F}$ is not conservative. Consider the unit circle C, positively oriented (counterclockwise), parametrized by

$$\vec{r}(t) = (\cos t, \sin t), \ 0 \leq t \leq 2\pi.$$

The velocity vector is

$$\vec{r}'(t) = (-\sin t, \cos t),\ 0 \le t \le 2\pi.$$

and the integral of $\vec{F}$ along C is

$$\begin{aligned}
\int_C \vec{F} \cdot d\vec{r} &= \int_0^{2\pi} \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t)\,\mathrm{dt} \\
&= \int_0^{2\pi} \left(\frac{-\sin t}{\cos^2 t + \sin^2 t}, \frac{\cos t}{\cos^2 t + \sin^2 t}\right) \cdot (-\sin t, \cos t)\,\mathrm{dt} \\
&= \int_0^{2\pi} (-\sin t, \cos t) \cdot (-\sin t, \cos t)\,\mathrm{dt} \\
&= \int_0^{2\pi} \sin^2 t + \cos^2 t\,\mathrm{dt} \\
&= \int_0^{2\pi} \mathrm{dt} \\
&= 2\pi \ne 0.
\end{aligned}$$

Since we have found a closed curve along which the integral of $\vec{F}$ is non-zero, $\vec{F}$ cannot be conservative.

3 Sufficient conditions

Depending on the shape of the domain D, the condition $P_y = Q_x$ is sometimes enough to guarantee that the field is conservative.

Proposition 3.1. *Let $\vec{F}\colon \mathbb{R}^2 \to \mathbb{R}^2$ is a continuously differentiable vector field defined on $\mathbb{R}^2$. If $\vec{F} = (P, Q)$ satisfies the condition $P_y = Q_x$, then $\vec{F}$ is conservative.*

Example 3.2. Consider the vector field $\vec{F} = (y \cos xy + 10x, x \cos xy + 3y^2)$ defined on $\mathbb{R}^2$. Determine whether $\vec{F}$ is conservative, and if it is, find a potential function for it.

Solution. Writing $P = y \cos xy + 10x$ and $Q = x \cos xy + 3y^2$, we compute

$$\begin{aligned}
P_y &= \cos xy + y(-x \sin xy) \\
&= \cos xy - xy \sin xy \\
Q_x &= \cos xy + x(-y \sin xy) \\
&= \cos xy - xy \sin xy
\end{aligned}$$

so that $\vec{F}$ satisfies the condition $P_y = Q_x$. Moreover, it is defined on all of $\mathbb{R}^2$, hence it is conservative. Let us find a potential function $f(x, y)$ for $\vec{F}$. We want

$$\begin{aligned}
f_x &= P = y \cos xy + 10x \\
f_y &= Q = x \cos xy + 3y^2.
\end{aligned}$$

Using the first equation, we obtain

$$
\begin{aligned}
f &= \int P \,\mathrm{dx} \\
&= \int y \cos xy + 10x \,\mathrm{dx} \\
&= y \frac{1}{y} \sin xy + 5x^2 + g(y) \\
&= \sin xy + 5x^2 + g(y)
\end{aligned}
$$

whose derivative with respect to y is

$$x \cos xy + g'(y).$$

Using the second equation, we equate this with Q:

$$
\begin{aligned}
x \cos xy + g'(y) &= x \cos xy + 3y^2 \\
\Rightarrow g'(y) &= 3y^2 \\
\Rightarrow g(y) &= y^3 + c.
\end{aligned}
$$

Choosing the constant $c = 0$, we obtain $g(y) = y^3$ and thus the potential function

$$\boxed{f(x, y) = \sin xy + 5x^2 + y^3}.$$

Asking for $\vec{F}$ to be defined (and continuously differentiable) on all of $\mathbb{R}^2$ is somewhat restrictive. In fact, that condition can be loosened, which gives the following improvement on proposition 3.1.

Theorem 3.3. *Let $D \subseteq \mathbb{R}^2$ be open and simply connected, and let $\vec{F} \colon D \to \mathbb{R}^2$ is a continuously differentiable vector field with domain D. If $\vec{F} = (P, Q)$ satisfies the condition $P_y = Q_x$, then $\vec{F}$ is conservative (on D).*

Theorem 3.3 did not apply to example 2.7, because the domain $D = \mathbb{R}^2 \setminus \{(0, 0)\}$ of $\vec{F}$ was not simply connected. Indeed, D is a punctured plane, and any curve going once around the puncture cannot be retracted to its basepoint.

When theorem 3.3 does not apply because the domain is not simply connected, then we cannot conclude from the condition $P_y = Q_x$ alone. A more subtle analysis is required.

Example 3.4. Consider the vector field $\vec{F} = \frac{1}{x^2+y^2}(x, y)$ on the domain

$$D = \mathbb{R}^2 \setminus \{(0, 0)\}.$$

Is $\vec{F}$ conservative?

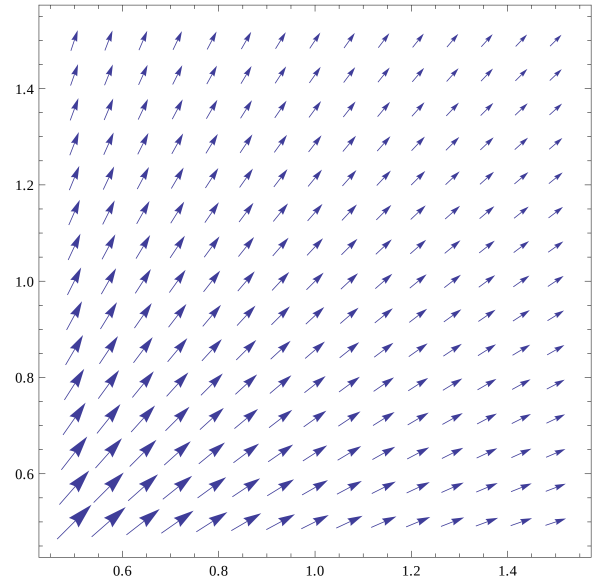

Figure 2: Plot of $\vec{F}$ in the first quadrant.

Writing $P = \frac{x}{x^2+y^2}$ and $Q = \frac{y}{x^2+y^2}$, let us compute

$$\begin{aligned} P_y &= \frac{-x(2y)}{(x^2+y^2)^2} \\ &= \frac{-2xy}{(x^2+y^2)^2} \\ Q_x &= \frac{-y(2x)}{(x^2+y^2)^2} \\ &= \frac{-2xy}{(x^2+y^2)^2} \end{aligned}$$

so that the vector field satisfies $P_y = Q_x$.

That means there is hope that $\vec{F}$ is conservative, although theorem 3.3 is not helpful here, because the domain D is not simply connected.

Turns out $\vec{F}$ **is** conservative. Let us try to find a potential function $f\colon D \to \mathbb{R}$, which would satisfy

$$\begin{aligned} f_x &= P = \frac{x}{x^2+y^2} \\ f_y &= Q = \frac{y}{x^2+y^2}. \end{aligned}$$

Using the first equation, we obtain

$$\begin{aligned}
f &= \int P \, \mathrm{dx} \\
&= \int \frac{x}{x^2+y^2} \, \mathrm{dx} \\
&= \int \frac{1}{u} \frac{\mathrm{du}}{2} \quad \text{substituting } u = x^2+y^2 \\
&= \frac{1}{2} \ln u + g(y) \quad \text{because } u > 0 \text{ always} \\
&= \frac{1}{2} \ln(x^2+y^2) + g(y) = \ln \sqrt{x^2+y^2} + g(y)
\end{aligned}$$

whose derivative with respect to y is

$$\begin{aligned}
&= \frac{1}{2} \frac{2y}{x^2+y^2} + g'(y) \\
&= \frac{y}{x^2+y^2} + g'(y).
\end{aligned}$$

Using the second equation, we equate this with Q:

$$\begin{aligned}
\frac{y}{x^2+y^2} + g'(y) &= \frac{y}{x^2+y^2} \\
\Rightarrow g'(y) &= 0 \\
\Rightarrow g(y) &= c.
\end{aligned}$$

Choosing $g(y) \equiv 0$, we obtain a function

$$f(x,y) = \frac{1}{2} \ln(x^2+y^2)$$

which is indeed a potential for $\vec{F}$, defined on D. Therefore $\vec{F}$ is conservative.

4 Restricting to a smaller domain

Does theorem 3.3 suggest that on a non simply connected domain, it is harder to find conservative vector fields? No, there are still plenty of them.

Example 4.1. Let $D = \mathbb{R}^2 \setminus \{(0,0)\}$ be the punctured plane and let $\vec{F} \colon D \to \mathbb{R}^2$ be the vector field defined by $\vec{F}(x,y) = (x,y)$. Then $\vec{F}$ is conservative, since it is $\vec{F} = \nabla f$ for the function $f(x,y) = \frac{1}{2}(x^2+y^2)$ defined on D.

Example 4.2. Again, let $D = \mathbb{R}^2 \setminus \{(0,0)\}$ and let $\vec{F} \colon D \to \mathbb{R}^2$ be the vector field defined by $\vec{F}(x,y) = (y \cos x, \sin x)$. Then $\vec{F}$ is conservative, since it is $\vec{F} = \nabla f$ for the function $f(x,y) = y \sin x$ defined on D.

More generally, pick any differentiable function $f \colon D \to \mathbb{R}$, then ∇f is a conservative vector field on D. In particular, if we start with a nice function $f \colon \mathbb{R}^2 \to \mathbb{R}$ such as $f(x,y) = y \sin x$, then ∇f is a conservative vector field on $\mathbb{R}^2$, as in examples 1.2 and 1.3. Moreover, its restriction to any open subset $D \subseteq \mathbb{R}^2$ is still conservative, with the same potential function (restricted to D), as in examples 4.1 and 4.2.

Definition 4.3. Let X and Y be sets, $f\colon X \to Y$ a function, and $S \subseteq$ a subset of X. The **restriction** of f to S is the function $f|_S\colon S \to Y$ which is the same as f, but viewed as a function defined on S only.

With this terminology, the comments above can be summarized as "the restriction of a conservative vector field is conservative".

We are not merely playing with words. Restricting to a smaller domain can change things drastically.

Example 4.4. Consider the vector field $\vec{F} = \frac{1}{x^2+y^2}(-y, x)$ on the domain which is the "right half-plane"

$$D = \{(x, y) \in \mathbb{R}^2 \mid x > 0\}.$$

Is $\vec{F}$ conservative?

In 2.7, we have checked that $\vec{F}$ satisfies $P_y = Q_x$. Moreover, the domain D is simply connected. Therefore, by theorem 3.3, $\vec{F}$ **is** conservative.

Let us find a potential function $f\colon D \to \mathbb{R}$, which must satisfy

$$f_x = P = \frac{-y}{x^2+y^2}$$
$$f_y = Q = \frac{x}{x^2+y^2}.$$

Using the second equation, we obtain

$$\begin{aligned} f &= \int Q \, \mathrm{dy} \\ &= \int \frac{x}{x^2+y^2} \, \mathrm{dy} \\ &= \int \frac{x}{x^2\left(1+(\frac{y}{x})^2\right)} \, \mathrm{dy} \quad \text{which is fine because } x \neq 0 \\ &= \int \frac{1}{1+(\frac{y}{x})^2} \frac{\mathrm{dy}}{x} \\ &= \arctan \frac{y}{x} + g(x) \end{aligned}$$

whose derivative with respect to x is

$$\begin{aligned} &= \frac{1}{1+(\frac{y}{x})^2} \frac{-y}{x^2} + g'(x) \\ &= \frac{-y}{x^2+y^2} + g'(x). \end{aligned}$$

Using the first equation, we equate this with P:

$$\begin{aligned} \frac{-y}{x^2+y^2} + g'(x) &= \frac{-y}{x^2+y^2} \\ \Rightarrow g'(x) &= 0 \\ \Rightarrow g(x) &= c. \end{aligned}$$

Choosing $g(x) \equiv 0$, we obtain the potential function

$$f(x, y) = \arctan \frac{y}{x}$$

defined on D since x is never zero on D.

Here is the geometric interpretation. The function $f(x, y) = \arctan \frac{y}{x}$ is the angle θ in polar coordinates. The angle function cannot be defined continuously on the punctured plane $\mathbb{R}^2 \setminus \{(0, 0)\}$, because the angle would have to jump suddenly by 2π after going around the origin once. However, the angle function can be defined continuously on small patches that do not go around the origin.

This interpretation explains the line integral $\int_C \vec{F} \cdot d\vec{r} = 2\pi$ along the unit circle, which we computed in example 2.7. By the fundamental theorem of line integrals, this integral is adding up the variations in angle as we move along the circle C. After one complete turn, the variations in angle add up to 2π.

Moral: The property of a vector field $\vec{F}$ being conservative or not **depends on the domain** on which we are considering $\vec{F}$, not just the formula which defines $\vec{F}$.

That distinction is illustrated by examples 2.7 and 4.4, where the vector field $\vec{F} = \frac{1}{x^2+y^2}(-y, x)$ is conservative on the right half-plane, but **not** conservative on the punctured plane.