Math 241 - Calculus III
Spring 2012, section CL1
§ 16.2. Line integrals

In these notes, we discuss line integrals and how they allow us to find the center of mass of a piece of wire.

1 Length of a curve

Let C be a curve in space, with parametrization $\vec{r}\colon [a,b] \to \mathbb{R}^3$. The element of arc length (i.e. the length of an "infinitesimal" piece of the curve) is

$$\mathrm{ds} = |\vec{r}'(t)|\,\mathrm{dt}.$$

This represents the speed $|\vec{r}'(t)|$ of motion at time t, times the infinitesimal amount of time dt. The length of the curve C is the sum of all infinitesimal "chunks" of length along C:

$$\begin{aligned} \text{length} &= \int_C \mathrm{ds} \\ &= \int_a^b |\vec{r}'(t)|\,\mathrm{dt}. \end{aligned}$$

Notation 1.1. Let us denote by $s(t)$ the length of the curve up to time t, in other words the part of the curve where the parameter ranges over $[a,t]$. This is given by the integral

$$s(t) = \int_a^t |\vec{r}'(\tau)|\,\mathrm{d}\tau$$

and is called the **arc length**.

2 Mass of a curve

Assume the curve C represents a piece of wire with density function ρ, which depends on the position. Then an infinitesimal "chunk" of wire has mass

$$\mathrm{dM} = \rho\,\mathrm{ds}$$

which is the density (in mass/length) times the infinitesimal length ds. The mass M of the entire curve C is the sum of all infinitesimal "chunks" of mass along C:

$$\begin{aligned} M &= \int_C \mathrm{dM} \\ &= \int_C \rho\,\mathrm{ds} \\ &= \int_a^b \rho(\vec{r}(t))|\vec{r}'(t)|\,\mathrm{dt}. \end{aligned}$$

In particular, if the density ρ is a constant k, then the mass is

$$\begin{aligned} M &= \int_C \rho \, \mathrm{ds} \\ &= \int_C k \, \mathrm{ds} \\ &= k \int_C \mathrm{ds} \\ &= k \text{ length} \end{aligned}$$

as expected.

Example 2.1. Let C be the upper half of the circle of radius 1 meter (see figure 1). Assume the density function is $\rho(x, y) = y$ kg/m. Find the mass of the curve C.

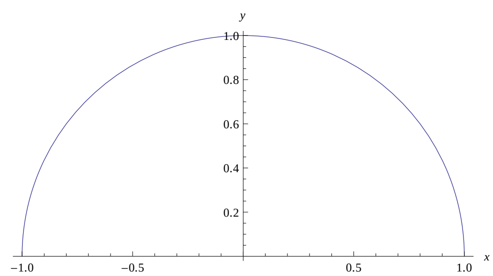

Figure 1: Curve C.

Solution. Parametrize the curve by the angle:

$$\vec{r}(t) = (\cos t, \sin t)\,,\ t \in [0, \pi].$$

Then the derivative is

$$\vec{r}'(t) = (-\sin t, \cos t)$$

and its magnitude is

$$|\vec{r}'(t)| = \sin^2 t + \cos^2 t = 1.$$

The mass M is

$$\begin{aligned} M &= \int_C \mathrm{dM} \\ &= \int_C \rho \,\mathrm{ds} \\ &= \int_0^\pi y|\vec{r}'(t)|\,\mathrm{dt} \\ &= \int_0^\pi (\sin t)(1)\,\mathrm{dt} \\ &= [-\cos t]_0^\pi \\ &= -\cos\pi + \cos 0 \\ &= -(-1) + 1 \\ &= 2 \text{ kg.} \end{aligned}$$

3 Center of mass

The center of mass of an object is the weighted average of its position. In other words, parts that weigh more count more towards the position of the center of mass.

For a discrete system consisting of particles of mass M_i at positions $\vec{r}_i$, the total mass is $M = \sum_i M_i$ and the center of mass is

$$\frac{\sum_i M_i \vec{r}_i}{\sum_i M_i} = \frac{\sum_i M_i \vec{r}_i}{M}.$$

For a continuous object like a curve C (or a surface, or a solid) the sum is replaced by an integral and the center of mass is

$$\frac{\int_C \vec{r}\,\mathrm{dM}}{\int_C \mathrm{dM}} = \frac{\int_C \vec{r}\,\mathrm{dM}}{M}.$$

Example 3.1. Let C the be upper half circle as in the previous example, with the same density $\rho(x, y) = y$. Find the center of mass of C.

Solution. We use the same parametrization (by angle). The x-component of the center of mass is 0, by symmetry. More precisely, the integral

$$\int_C x\rho\,\mathrm{ds} = \int_C xy\,\mathrm{ds}$$

is 0 because the contribution from $x \geq 0$ cancels the contribution from $x \leq 0$.

The y-component of the center of mass is the weighted average of y, that is

$$\begin{aligned}\frac{\int_C y\,\mathrm{dM}}{M} &= \frac{\int_C y\rho\,\mathrm{ds}}{M}\\ &= \frac{\int_0^\pi y(y)(1)\,\mathrm{dt}}{2}\\ &= \frac{\int_0^\pi \sin^2 t\,\mathrm{dt}}{2}\\ &= \frac{\pi/2}{2}\\ &= \frac{\pi}{4} \approx 0.785.\end{aligned}$$

Therefore the center of mass of C is at $(0, \frac{\pi}{4})$ meters.