Math 241 - Calculus III
Spring 2012, section CL1
§ 15.9. Change of variables in multiple integrals

1 General setup

Assume we want to compute the double integral

$$\iint_R f dA = \iint_R f(x,y) dx dy$$

on some (complicated) region R in the xy-plane. By changing to a different system of coordinates (u,v), the integral may be simplified, either because the function $f(x,y)$ or the region R becomes simpler.

Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ denote the transformation giving the (x,y)-coordinates in terms of the new (u,v)-coordinates, i.e.

$$(x,y) = T(u,v).$$

Let S be the (hopefully easier) region in the uv-plane corresponding to the region R in the xy-plane, i.e. $T(S) = R$. Then the change of variables formula is

$$\iint_R f dA = \iint_R f(x,y) dx dy = \iint_S f(T(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du dv$$

where the **Jacobian** of the transformation T is

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix}.$$

Example 1.1. When passing to polar coordinates, the transformation is

$$T(r,\theta) = (r\cos\theta, r\sin\theta)$$

and its Jacobian is

$$\frac{\partial(x,y)}{\partial(r,\theta)} = r.$$

If the region of integration is, for example, the upper half-annulus between radii 3 and 4

$$R = \{(x,y) \in \mathbb{R}^2 \mid y \geq 0, 9 \leq x^2 + y^2 \leq 16\}$$

then the corresponding region S in polar coordinates is the rectangle

$$S = \{(r,\theta) \in \mathbb{R}^2 \mid 3 \leq r \leq 4, 0 \leq \theta \leq \pi\}.$$

If the function being integrated is, for example, $f(x,y) = x + y^2$, then the integral is

$$\begin{aligned} \iint_R f(x,y) dx dy &= \iint_R (x + y^2) dx dy \\ &= \iint_S (r\cos\theta + r^2 \sin^2\theta) \left| \frac{\partial(x,y)}{\partial(r,\theta)} \right| dr d\theta \\ &= \iint_S (r\cos\theta + r^2\sin^2\theta) r dr d\theta \\ &= \int_0^\pi \int_3^4 (r\cos\theta + r^2\sin^2\theta) r dr d\theta. \end{aligned}$$

2 Worked example

Let us solve problem ♯15 from section § 15.9.

Example 2.1. Let R be the region in the first quadrant bounded by the lines $y = x$ and $y = 3x$ and the hyperbolas $xy = 1$ and $xy = 3$. Note that those four curves also bound a region in the third quadrant, which we are ignoring.

We want to compute the integral

$$\iint_R xy\, dA.$$

a. Compute the integral using the change of variables $x = \frac{u}{v}$, $y = v$.

Let us compute the Jacobian:

$$\begin{aligned}
x_u &= \frac{1}{v} \\
x_v &= -\frac{u}{v^2} \\
y_u &= 0 \\
y_v &= 1 \\
\frac{\partial(x,y)}{\partial(u,v)} &= \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} \\
&= \begin{vmatrix} \frac{1}{v} & -\frac{u}{v^2} \\ 0 & 1 \end{vmatrix} \\
&= \frac{1}{v}.
\end{aligned}$$

The region S is bounded by the curves

$$\begin{aligned}
& y = x \Leftrightarrow v = \frac{u}{v} \Leftrightarrow v^2 = u \\
& y = 3x \Leftrightarrow v = \frac{3u}{v} \Leftrightarrow v^2 = 3u \\
& xy = 1 \Leftrightarrow u = 1 \\
& xy = 3 \Leftrightarrow u = 3
\end{aligned}$$

and satisfying

$$\begin{aligned}
x > 0 &\Leftrightarrow \frac{u}{v} > 0 \\
y > 0 &\Leftrightarrow v > 0.
\end{aligned}$$

This means that the bounding curves are $v = \sqrt{u}$ and $v = \sqrt{3u}$ (and not the negative roots).

The integral is

$$\begin{aligned}
\iint_R xy\,dA &= \iint_S \frac{u}{v} v \left|\frac{1}{v}\right| dvdu \\
&= \int_1^3 \int_{\sqrt{u}}^{\sqrt{3u}} \frac{u}{v} dvdu \\
&= \int_1^3 u \left[\ln v\right]_{v=\sqrt{u}}^{v=\sqrt{3u}} du \\
&= \int_1^3 u \left(\ln\sqrt{3u} - \ln\sqrt{u}\right) du \\
&= \int_1^3 u \left(\frac{1}{2}\ln 3u - \frac{1}{2}\ln u\right) du \\
&= \frac{1}{2}\int_1^3 u\left(\ln 3 + \ln u - \ln u\right) du \\
&= \frac{1}{2}\int_1^3 u \ln 3 du \\
&= \frac{1}{2}\ln 3\left[\frac{u^2}{2}\right]_1^3 \\
&= \frac{1}{2}\ln 3\,\frac{1}{2}(9-1) \\
&= \boxed{2\ln 3}.
\end{aligned}$$

b. Compute the integral using your favorite change of variables.

Because of how the region R is defined, my favorite change of variables is $u = xy$, $v = \frac{y}{x}$. Unlike in part (a), u and v are expressed in terms of x and y, not the other way around. We will use the useful property

$$\frac{\partial(x,y)}{\partial(u,v)} = \frac{\partial(u,v)}{\partial(x,y)}^{-1}$$

to compute the Jacobian:

$$\begin{aligned}
u_x &= y \\
u_y &= x \\
v_x &= -\frac{y}{x^2} \\
v_y &= \frac{1}{x} \\
\frac{\partial(u,v)}{\partial(x,y)} &= \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} \\
&= \begin{vmatrix} y & x \\ -\frac{y}{x^2} & \frac{1}{x} \end{vmatrix} \\
&= \frac{y}{x} - \left(-\frac{y}{x}\right) \\
&= 2\frac{y}{x} \\
&= 2v \\
\frac{\partial(x,y)}{\partial(u,v)} &= \frac{\partial(u,v)}{\partial(x,y)}^{-1} \\
&= \frac{1}{2v}.
\end{aligned}$$

The region S is bounded by the curves

$$\begin{aligned}
&y = x \Leftrightarrow \frac{y}{x} = 1 \Leftrightarrow v = 1 \\
&y = 3x \Leftrightarrow \frac{y}{x} = 3 \Leftrightarrow v = 3 \\
&xy = 1 \Leftrightarrow u = 1 \\
&xy = 3 \Leftrightarrow u = 3.
\end{aligned}$$

The integral is

$$\begin{aligned}
\iint_R xy\,dA &= \iint_S u \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dvdu \\
&= \iint_S u \left|\frac{1}{2v}\right| dvdu \\
&= \frac{1}{2}\int_1^3 \int_1^3 \frac{u}{v} dvdu \\
&= \frac{1}{2}\int_1^3 u\,[\ln v]_1^3\,du \\
&= \frac{1}{2}\int_1^3 u\,(\ln 3 - \ln 1)\,du \\
&= \frac{1}{2}\int_1^3 u\ln 3\,du \\
&= \frac{1}{2}\ln 3\left[\frac{u^2}{2}\right]_1^3 \\
&= \frac{1}{2}\ln 3\,\frac{1}{2}(9-1) \\
&= \boxed{2\ln 3}.
\end{aligned}$$

Remark 2.2. Had we not used the trick "Jacobian of inverse equals inverse of Jacobian", we would have had to find the inverse transformation explicitly:

$$\begin{aligned}
&uv = y^2 \Leftrightarrow y = \sqrt{uv} \quad \text{because } y > 0 \\
&\frac{u}{v} = x^2 \Leftrightarrow x = \sqrt{\frac{u}{v}} \quad \text{because } x > 0.
\end{aligned}$$

With those formulas, we can compute the Jacobian explicitly:

$$
\begin{aligned}
x_u &= \frac{1}{2}\sqrt{\frac{1}{uv}} \\
x_v &= -\frac{1}{2}\sqrt{\frac{u}{v^3}} \\
y_u &= \frac{1}{2}\sqrt{\frac{v}{u}} \\
y_v &= \frac{1}{2}\sqrt{\frac{u}{v}} \\
\frac{\partial(x,y)}{\partial(u,v)} &= \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} \\
&= \begin{vmatrix} \frac{1}{2}\sqrt{\frac{1}{uv}} & -\frac{1}{2}\sqrt{\frac{u}{v^3}} \\ \frac{1}{2}\sqrt{\frac{v}{u}} & \frac{1}{2}\sqrt{\frac{u}{v}} \end{vmatrix} \\
&= \frac{1}{4}\begin{vmatrix} \sqrt{\frac{1}{uv}} & -\sqrt{\frac{u}{v^3}} \\ \sqrt{\frac{v}{u}} & \sqrt{\frac{u}{v}} \end{vmatrix} \\
&= \frac{1}{4}\left(\sqrt{\frac{1}{v^2}} + \sqrt{\frac{1}{v^2}}\right) \\
&= \frac{1}{4}\left(\frac{2}{v}\right) \\
&= \frac{1}{2v}.
\end{aligned}
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