Math 241 - Calculus III
Spring 2012, section CL1
§ 14.7. Global minima and maxima

In these notes, we discuss the problem of finding the global (or absolute) minima and maxima of a function. First, let us fix some terminology.

Definition. Let $f\colon D \to \mathbb{R}$ be a function, with domain $D \subseteq \mathbb{R}^n$. A point $x_0 \in D$ is called:

- a **local minimum** of f if $f(x_0) \leq f(x)$ holds for all x is some neighborhood of x_0;
- a **local maximum** of f if $f(x_0) \geq f(x)$ holds for all x is some neighborhood of x_0;
- a **global minimum** (or absolute minimum) of f if $f(x_0) \leq f(x)$ holds for all $x \in D$;
- a **global maximum** (or absolute maximum) of f if $f(x_0) \geq f(x)$ holds for all $x \in D$.

1 One-dimensional case

Assume for now $n = 1$, that is f is a function of one variable, with domain $D \subseteq \mathbb{R}$. An important tool in finding global extrema of a function is the following theorem.

Theorem 1.1 (Extreme value theorem)**.** *Let $f\colon [a,b] \to \mathbb{R}$ be a continuous function on the closed interval $[a,b]$. Then f reaches a (global) minimum and a (global) maximum on $[a,b]$.*

Remark 1.2. The assumption that the interval be closed is important. For example, consider the function $f\colon (0,1] \to \mathbb{R}$ defined by $f(x) = \frac{1}{x}$. Then f does **not** have a global maximum on $(0,1]$. Here the theorem does not apply, because the interval $(0,1]$ is not closed.

Remark 1.3. The assumption that the interval be bounded, i.e. a and b are finite numbers, is also important. For example, consider the function $f\colon [0,+\infty) \to \mathbb{R}$ defined by $f(x) = x$ (or your favorite non-constant polynomial). Then f does **not** have a global maximum on $[0,+\infty)$. Here the theorem does not apply, because the interval $[0,+\infty)$ is not bounded – though it **is** closed.

Upshot: The extreme value theorem provides a method for finding the (global) extrema of a function $f\colon [a,b] \to \mathbb{R}$.

1. Look for critical points of f in the interval (a,b), i.e. the interior of the domain of f.
2. Look at the values of f at the endpoints $x = a$ and $x = b$, i.e. the boundary of the domain of f.

Example 1.4. Let $f\colon [0,3] \to \mathbb{R}$ be the function defined by

$$f(x) = x^2 - 4x + 5$$

as graphed in figure 1. Find the maximal and minimal values of f, and where they occur.

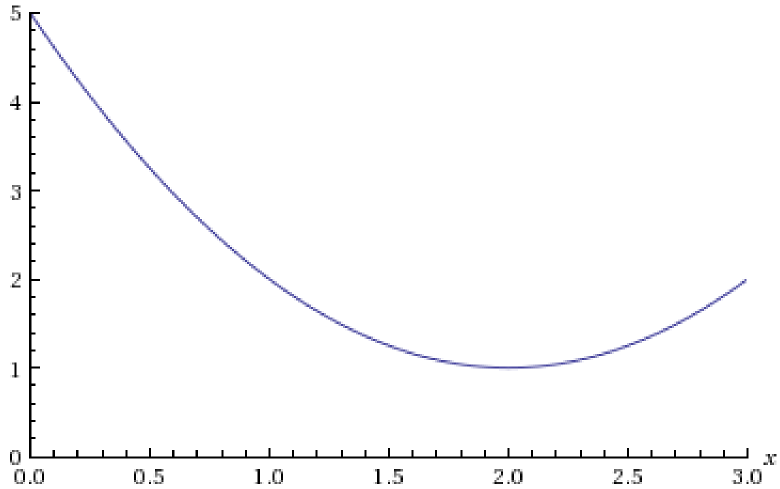

Figure 1: Graph of $f(x) = x^2 - 4x + 5$ on the interval $[0, 3]$.

Solution. Let us find the critical points of f:

$$f'(x) = 2x - 4 = 2(x - 2).$$

Setting $f'(x) = 0$, we obtain

$$2(x - 2) = 0$$

so that $x = 2$ is the only critical point of f. (Note that 2 is in $[0, 3]$, the domain of f.) There, the value of f is

$$f(2) = 4 - 8 + 5 = 1.$$

At the endpoints $x = 0$ and $x = 3$, the values of f are

$$f(0) = 5$$

$$f(3) = 9 - 12 + 5 = 2.$$

Therefore the (global) **minimum** of f is $\boxed{1}$, which occurs at $\boxed{x = 2}$.

The (global) **maximum** of f is $\boxed{5}$, which occurs at $\boxed{x = 0}$.

2 In higher dimension

The theorem above has an analogue in higher dimension.

Theorem 2.1 (Extreme value theorem). *Let $f\colon D \to \mathbb{R}$ be a continuous function with domain $D \subseteq \mathbb{R}^n$. Assume D is* ***closed*** *and* ***bounded****. Then f reaches a (global) minimum and a (global) maximum on D.*

Here "closed" means that D contains all its limit points, and "bounded" means that D is contained within some disk of finite radius.

Here are examples illustrating those two notions.

Closed and bounded.

- The disk $\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq r^2\}$ of radius r.
- The rectangle $\{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq 2, 0 \leq y \leq 1\}$.
- The circle $\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = r^2\}$ of radius r.
- The line segment $\{(x, y) \in \mathbb{R}^2 \mid x + y = 5, 0 \leq x \leq 5\}$.

Closed but NOT bounded.

- The first quadrant $\{(x, y) \in \mathbb{R}^2 \mid x \geq 0, y \geq 0\}$.
- The upper half-plane $\{(x, y) \in \mathbb{R}^2 \mid y \geq 0\}$.
- The infinite horizontal strip $\{(x, y) \in \mathbb{R}^2 \mid 0 \leq y \leq 1\}$.
- The x-axis $\{(x, y) \in \mathbb{R}^2 \mid y = 0\}$.
- More generally, any line $\{(x, y) \in \mathbb{R}^2 \mid ax + by = c\}$.
- All of $\mathbb{R}^2$.

Bounded but NOT closed.

- The "open" disk $\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < r^2\}$ of radius r.
- The punctured disk $\{(x, y) \in \mathbb{R}^2 \mid 0 < x^2 + y^2 \leq r^2\}$ of radius r, missing its center.
- The rectangle $\{(x, y) \in \mathbb{R}^2 \mid 0 < x \leq 2, 0 \leq y \leq 1\}$ missing its left edge.
- The line segment $\{(x, y) \in \mathbb{R}^2 \mid x + y = 5, 0 \leq x < 5\}$ missing its endpoint $(5, 0)$.

NEITHER closed NOR bounded.

- The "open" first quadrant $\{(x, y) \in \mathbb{R}^2 \mid x > 0, y > 0\}$.
- The first quadrant $\{(x, y) \in \mathbb{R}^2 \mid x \geq 0, y > 0\}$ missing the half-x-axis.
- The "open" upper half-plane $\{(x, y) \in \mathbb{R}^2 \mid y > 0\}$.
- The infinite strip $\{(x, y) \in \mathbb{R}^2 \mid 0 \leq y < 1\}$ missing its upper edge.
- The half-x-axis $\{(x, y) \in \mathbb{R}^2 \mid x > 0, y = 0\}$ missing its endpoint $(0, 0)$.

Upshot: The extreme value theorem provides a method for finding the (global) extrema of a function $f\colon D \to \mathbb{R}$ assuming the domain D is closed and bounded.

1. Look for critical points of f inside the domain D.
2. Look at the values of f on the boundary of the domain D.

The idea is exactly as in the one-dimensional case, although the implementation is slightly more complicated.

Example 2.2. Let $f\colon D \to \mathbb{R}$ be the function defined by

$$f(x) = x^2 - 2xy + 2y$$

having domain the rectangle $D = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq 3, 0 \leq y \leq 2\}$, as graphed in figure 2. Find the minimal and maximal values of f, and where they occur.

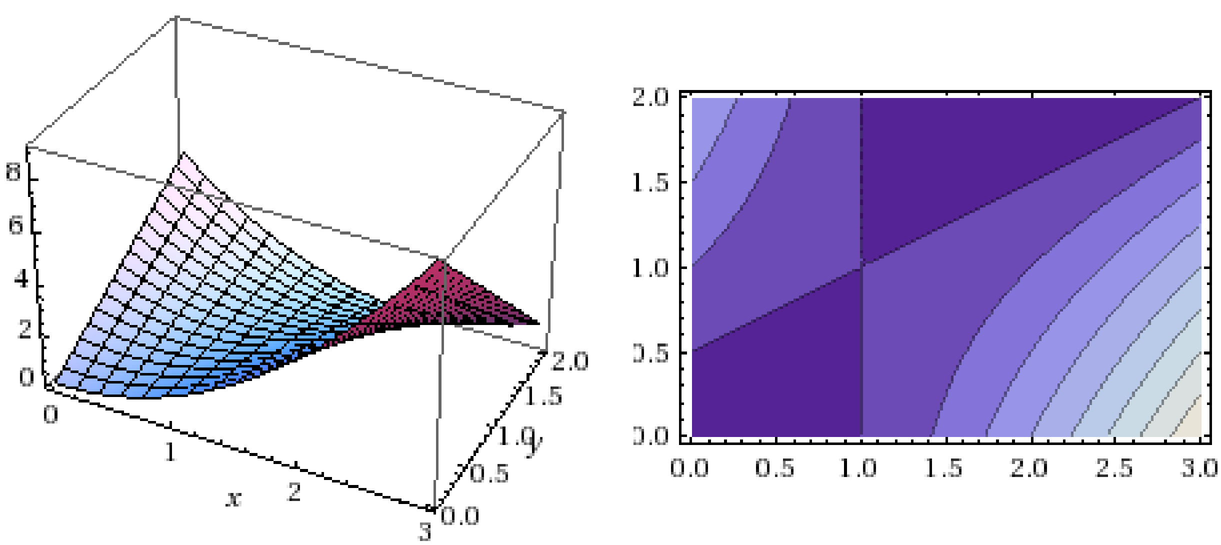

Figure 2: Graph and contour plot of $f(x, y) = x^2 - 2xy + 2y$ on the rectangular domain D.

Solution. Let us find the critical points of f inside D:

$$f_x(x, y) = 2x - 2y$$
$$f_y(x, y) = -2x + 2.$$

Setting $\nabla f(x) = 0$, we obtain the system of equations

$$2x - 2y = 0$$
$$-2x + 2 = 0$$

whose only solution is $(x, y) = (1, 1)$. Note that the point $(1, 1)$ is inside D. There, the value of f is

$$f(1, 1) = 1 - 2 + 2 = 1.$$

The boundary of D has four sides.

Left side $x = 0$. The function there is

$$f(0, y) = 2y$$

which reaches a minimum of 0 at $y = 0$ and a maximum of 4 at $y = 2$.

Right side $x = 3$. The function there is

$$f(3, y) = 9 - 6y + 2y = 9 - 4y$$

which reaches a minimum of 1 at $y = 2$ and a maximum of 9 at $y = 0$.

Bottom side $y = 0$. The function there is

$$f(x, 0) = x^2$$

which reaches a minimum of 0 at $x = 0$ and a maximum of 9 at $x = 3$.

Top side $y = 2$. The function there is

$$f(x, 2) = x^2 - 4x + 4$$

which has one critical point at $x = 2$, with value $f(2, 2) = 0$. At the endpoints, the function has values

$$f(0, 2) = 4$$
$$f(3, 2) = 1.$$

Putting all the information together, we conclude the following (see figure 3).

The (global) **minimum** of f is $\boxed{0}$, which occurs at $\boxed{(x, y) = (0, 0)}$ and $\boxed{(2, 2)}$.

The (global) **maximum** of f is $\boxed{9}$, which occurs at $\boxed{(x, y) = (3, 0)}$.

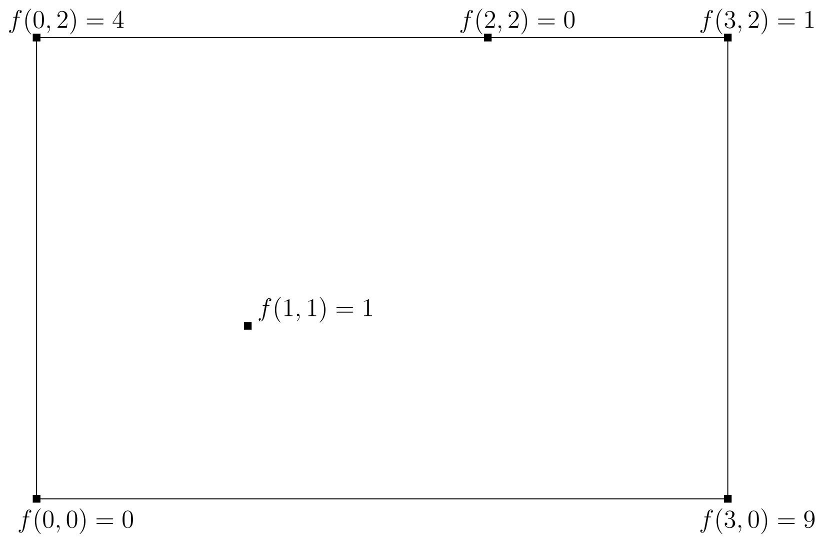

Figure 3: Certain values of f.

3 Unbounded domains

When the domain D of the function f is not bounded (or not closed), we can still use the extreme value theorem with the following strategy.

1. Restrict the function to some closed and bounded subset D'.
2. Analyze what happens in D' and outside of D'.

Example 3.1. Find the distance from the plane $2x - y + z = 3$ to the origin.

Remark 3.2. The problem is easily solved using projections, as seen in Chapter 12. However, the method described below also works for surfaces other than planes.

Solution. The plane can be expressed by $z = 3 - 2x + y$, in other words, using x and y as parameters. The distance squared from the point $(x, y, 3 - 2x + y)$ to the origin is

$$f(x, y) = x^2 + y^2 + (3 - 2x + y)^2$$

which we want to minimize. Note that the domain of f is $D = \mathbb{R}^2$. Let us find the critical points of f:

$$f_x(x, y) = 2x + 2(3 - 2x + y)(-2) = 10x - 4y - 12$$
$$f_y(x, y) = 2y + 2(3 - 2x + y)(1) = -4x + 4y + 6.$$

Setting $\nabla f(x) = 0$, we obtain the system of equations

$$10x - 4y = 12$$
$$-4x + 4y = -6$$

whose only solution is $(x, y) = (1, -\frac{1}{2})$. Geometrically, this unique critical point must a minimum, and so the minimum value of f is

$$\begin{aligned} f(1, -\frac{1}{2}) &= (1)^2 + \left(-\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 \\ &= 1 + \frac{1}{4} + \frac{1}{4} \\ &= \frac{3}{2}. \end{aligned}$$

The distance from the plane to the origin is therefore $\boxed{\sqrt{\frac{3}{2}}}$.

How to show rigorously that f reaches its minimum at the critical point $(1, -\frac{1}{2})$? Consider the disk of radius 2

$$D' = \{(x, y) \mid x^2 + y^2 \leq 4\}$$

which is closed and bounded. On D', f has one critical point $(1, -\frac{1}{2})$. On the boundary of D', which is the circle of radius 2

$$\partial D' = \{(x, y) \mid x^2 + y^2 = 4\}$$

the function f has a lower bound:

$$f(x, y) = x^2 + y^2 + (3 - 2x + y)^2 \geq x^2 + y^2 = 4.$$

Therefore, the minimum of f on D' is $f(1, -\frac{1}{2}) = \frac{3}{2}$. Outside D', that is on the region $x^2+y^2 > 4$, the same bound works:

$$f(x, y) = x^2 + y^2 + (3 - 2x + y)^2 \geq x^2 + y^2 > 4.$$

This proves that f reaches a global minimum on $\mathbb{R}^2$ at the point $(1, -\frac{1}{2})$, with value $f(1, -\frac{1}{2}) = \frac{3}{2}$. □

Remark 3.3. One can at least check that $(1, -\frac{1}{2})$ is a local minimum, using the second derivative test:

$$\begin{aligned} f_{xx}(x, y) &= 10 \\ f_{xy}(x, y) &= -4 \\ f_{yy}(x, y) &= 4 \end{aligned}$$

$$\begin{vmatrix} f_{xx}(1, -\frac{1}{2}) & f_{xy}(1, -\frac{1}{2}) \\ f_{xy}(1, -\frac{1}{2}) & f_{yy}(1, -\frac{1}{2}) \end{vmatrix} = \begin{vmatrix} 10 & -4 \\ -4 & 4 \end{vmatrix} = 24 > 0$$

$$f_{xx}(1, -\frac{1}{2}) = 10 > 0.$$

Thus $(1, -\frac{1}{2})$ is a local minimum.