A-INFINITY STRUCTURE ON EXT-ALGEBRAS

Abstract. We give an introduction to A-infinity algebras in these notes, which is a generalisation of differential graded algebras. We show that for a graded algebra A, the Ext-algebra $\operatorname{Ext}^*_A(k_A, k_A)$ has an A-infinity structure that contains sufficient information to recover A. On the other hand, we will present an example where the usual associative algebra structure on $\operatorname{Ext}^*_A(k_A, k_A)$ cannot recover A. We also show that the A-infinity structure is closely related to Massey products.

0.1. **Differential graded algebras**

We begin by reviewing the definition of a differential graded algebra. Throughout the notes, we use k to denote the ground field unless otherwise stated.

Definition 0.1. A **differential graded algebra** (in short DG algebra) A over a commutative ring k is a $\mathbb{Z}$-graded k-algebra

$$A = \bigoplus_{p \in \mathbb{Z}} A^p$$

together with a differential d of degree 1 such that

$$d(ab) = (da)b + (-1)^p a(db)$$

for all $a \in A^p$ and $b \in A$. In particular, A is a complex of k-modules with differentials $d^n : A^n \to A^{n+1}$, and the cohomology ring HA of a DG k-algebra A is a graded associative ring over k with

$$HA^n = \ker(d^n)/\operatorname{im}(d^{n+1}).$$

Example 0.2 (Ext-algebra as the cohomology of a DG algebra)**.** Let A be a connected graded associative algebra over k, and let k_A be the trivial A-module concentrated in degree 0.

The Ext-algebra $\operatorname{Ext}^*_A(k_A, k_A)$ is the cohomology ring of $\operatorname{End}_A(P)$, where P is a free A-resolution of k_A. $\operatorname{End}_A(P)$ is a DG algebra with

$$\operatorname{End}_A(P)_p = \prod_{n \in \mathbb{Z}} \operatorname{Hom}_A(P_n, P_{n+p})$$

and differential d given by

$$d_p(f) = f\partial + (-1)^{p+1}\partial f,$$

with $f \in \operatorname{End}_A(P)_p$ being a map of degree p.

0.2. **Recovering the associative algebra from the Ext-algebra**

For a connected graded associative algebra A over k, we have seen that the classical Ext-algebra $\operatorname{Ext}^*_A(k_A, k_A)$ is the cohomology ring of the DG algebra $\operatorname{End}_A(P)$. Our question is to recover the algebra A from $\operatorname{Ext}^*_A(k_A, k_A)$. Consider the following example:

Example 0.3. Let $A = k\langle x_1, x_2\rangle/(f)$, with $f = x_1x_2 + x_2x_1$ in degree 2. One can show that the minimal free resolution of k_A has the form

$$\cdots \to 0 \to Ar \to Ae_1 \oplus Ae_2 \to A \to k \to 0,$$

with e_i maps to x_i and r maps to the relation, and

$$\operatorname{Ext}^s_A(k_A, k_A) = \begin{cases} k & s = 0, \\ k(-1) \oplus k(-1) & s = 1, \\ k(-2) & s = 2, \\ 0 & \textit{else}. \end{cases}$$

Write $E = \mathrm{Ext}^s_A(k_A, k_A)$. In general, we know that E^1 is dual to A_1 and E^2 is dual to the relation $R = (f) = \oplus_{n\geqslant 2} R_n$ in A. Moreover, restricting the multiplication on E to $E^1 \otimes E^1$, we get a map

$$E^1 \otimes E^1 \to E^2$$

that is dual to the inclusion $R_2 \to A_1 \otimes A_1$. In this sense, we can recover A from the Ext-algebra E. See [1, Section 6] for more details of the example.

Note that if we set the relation f in degree $q > 2$, then the multiplication on E is trivial for degree reasons. Nevertheless, we have the inclusion $R_n \to (A_1)^{\otimes n}$, whose dual is the "higher multiplication"

$$(E^1)^{\otimes q} \to E^2_{-q}$$

of resolution degree $2 - q$.

We will make the definition precise in the following section.

0.3. *A*-infinity algebras

Definition 0.4. An ***A*-infinity** algebra over a base field k is a $\mathbb{Z}$-graded vector space

$$A = \bigoplus_{p\in\mathbb{Z}} A^p$$

together with a family of graded k-linear maps

$$m_n : A^{\otimes n} \to A,$$

of degree $2 - n$ for $n \geqslant 1$, satisfying the Stasheff identities SI(n):

$$\sum (-1)^{r+st} m_u(id^{\otimes} r \otimes m_s \otimes id^{\otimes t}) = 0,$$

where the sum runs over all decompositions $n = r + s + t$, and $u = r + 1 + t$.

For n small, the identities have the form:

SI(1) $m_1 m_1 = 0$;
SI(2) $m_1 m_2 = m_2(m_1 \otimes id + id \otimes m_1)$;
SI(3) $m_2(id \otimes m_2 - m_2 \otimes id) = m_1 m_3 + m_3(m_1 \otimes id \otimes id + id \otimes m_1 \otimes id + id \otimes id \otimes m_1)$.

In particular, a DG algebra is an A-infinity algebra with $m_1 = d$ and $m_2 = m$ and $m_n = 0$ for $n > 2$.

We can have a grading on the spaces A^p too, with

$$A^p = \bigoplus_{i\in G} A^p_i$$

indexed by an abelian group G. This grading i is called the Adams grading, and is denoted by a lower index. The structure maps m_n are required to respect the Adams grading.

In our examples, we always have $G = \mathbb{Z}$. In this case, we say that the A-infinity algebra A is **Adams connected** if $A_0 = k$, and $A = \oplus_{n\geqslant 0} A_n$ or $A = \oplus_{n\leqslant 0} A_n$.

A morphism of A-infinity algebras consists of a family of k-linear graded maps

$$f_n : A^{\otimes n} \to B$$

satisfying the Stasheff morphism identities. A morphism f is a quasi-isomorphism if f_1 is a quasi-isomorphism.

Theorem 0.5. *Let A be an A-infinity algebra, and let HA be the cohomology ring of A. Then there is an A-infinity structure on HA with $m_1 = 0$ and m_2 induced by the multiplication on A. And there is a quasi-isomorphism $HA \to A$ lifting the identity map of HA. This A-infinity structure on HA is unique up to quasi-isomorphism.*

By "lifting the identity map", we mean that there is also a projection map $p : A \to HA$ that induces a quasi-isomorphism. We will see in the proof that we have choose the projection p as a vector space splitting. Then the section map $HA \to A$ will respect the chosen projection p. The maps are not canonically defined.

Before we sketch a proof of the theorem, let us review the example.

Example 0.6. For $A = k\langle x_1, x_2\rangle/(f)$ with $f \in A^q$. We see that the only non-zero multiplication on E is m_q. And one can show that the restriction of m_q to $(E^1)^{\otimes q}$ is dual to the inclusion $R_n \to A_1^{\otimes n}$. The result is made more general in [1, Theorem A]

We sketch a proof of the theorem in the case when A is a DG algebra.

Sketch of proof. We start with an intuitive idea. Let $\lambda_2 : A \otimes A \to A$ be multiplication on A. We want to define maps λ_n on A inductively by the formula

$$\lambda_n = \sum_{s+t=n, s,t\geqslant 1} \lambda_2(\lambda_s \otimes \lambda_t),$$

assuming that λ_s and λ_t are defined for smaller s and t. One immediately sees that there are two problems here: the cohomology plays no role in this formula, and the degrees do not match. The right formula that will fix the problems is as follows:

$$\lambda_n = \sum_{s+t=n, s,t\geqslant 1} \lambda_2(G\lambda_s \otimes G\lambda_t),$$

where G is a homotopy on A from the identity map id_A to the projection p onto HA. Here we identify HA with $\oplus H^n$ as a subspace of A, and choose a splitting $A^n = B^n \oplus H^n \oplus L^n$, and p is the projection onto the summand $H \subseteq A$. Since $L^{n-1} \cong B^n$, we can choose G to respect the splitting: $G|_{B^n} \cong L^{n-1} \subseteq A^{n-1}$ and $G|_{H^n \oplus L^n} = 0$. For $n = 1$, we set $G\lambda_1$ formally to be the identity map. Now one can check that the maps

$$p(\lambda_n|_{HA}) : HA \to HA$$

endows HA with an A-infinity structure. □

0.4. *A*-infinity algebras and Massey products

Let A be a DG algebra. Let α_1, α_2, and α_3 be classes in HA represented by a_{01}, a_{12}, and a_{23} in A. Assume that $\alpha_1\alpha_2 = \alpha_2\alpha_3 = 0$. Set $a_{02} = G(a_{01}a_{12})$ and $a_{13} = G(a_{12}a_{23})$. Then $\partial(a_{02}) = a_{01}a_{12}$ and $\partial(a_{13}) = a_{12}a_{23}$. Up to signs, this is what we need to define the three-fold Massey product $\langle\alpha_1, \alpha_2, \alpha_3\rangle$ in HA, so $(-1)^b m_n(\alpha_1 \otimes \alpha_2 \otimes \alpha_3) \in \langle\alpha_1, \alpha_2, \alpha_3\rangle$. In general, this fact holds for higher products too.

Theorem 0.7. *Let A be a DG algebra. Let $\alpha_1, \dots, \alpha_n$ be classes in HA such that the n-fold Massey product $\langle\alpha_1, \dots, \alpha_n\rangle$ is defined. Then*

$$(-1)^b m_n(\alpha_1 \otimes \cdots \otimes \alpha_n) \in \langle\alpha_1, \dots, \alpha_n\rangle,$$

where $b = 1 + deg(\alpha_{n-1}) + deg(\alpha_{n-3}) + deg(\alpha_{n-5}) + \cdots$.

Remark 0.8. Recall that the homotopy $G : A \to A$ depends on a splitting of A, so we can have different homotopies G that produce different classes in the Massey product, but this process does not necessarily produce all the classes in the Massey product.

Example 0.9. Let p be an odd prime, and let k be a field of characteristic p. Take $A = k[x]/(x^p)$ with x in Adams degree $2d$. Then the Ext-algebra of A is

$$\mathrm{Ext}_A^*(k_A, k_A) \cong \Lambda(y_1) \otimes k[y_2],$$

with y_1 in degree $(1, -2d)$ and y_2 in degree $(2, -2dp)$. Moreover, we have $m_p(y_1 \otimes \cdots \otimes y_1) = y_2$, and one can compute that the p-fold Massey product $\langle y_1, \dots, y_1\rangle = \{(-1)^{(p+1)/2}y_2\}$.

References

[1] D.-M. Lu, J. H. Palmieri, Q.-S. Wu, J. J. Zhang. *A*-infinity structure on Ext-algebras. arXiv:math/0606144