Lectures on DG-Categories

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1 Introduction

The purpose of these four lectures is to provide an introduction to the theory of dg-categories.

There are several possible points of view to present the subject, and my choice has been to emphasised its relations with the localization problem (in the sense of category theory). In the same way that the notion of complexes can be introduced for the need of derived functors, dg-categories will be introduced here for the need of a *derived version* of the localization construction. The purpose of the first lecture is precisely to recall the notion of the localization of a category and to try to explain its bad behaviour throught several examples. In the second part of the first lecture I will introduce the notion of dg-categories and quasi-equivalences, and explain how they can be used in order to state a refined version of the notion of localization. The existence and properties of this new localization will be studied in the next lectures.

The second lecture is concerned with reminders about model category theory, and its applications to the study of dg-categories. The first part is a very brief overview of the basic notions and results of the theory, and the second part presents the specific model categories appearing in the context of dg-categories.

Lecture three goes into the heart of the subject and is concerned with the study of the homotopy category of dg-categories. The key result is a description of the set of morphisms in this homotopy category as the set of isomorphism classes of certain objects in a derived category of bi-modules. This result possesses several important consequences, such as the existence of localizations and of derived internal Homs for dg-categories. The very last part of this third lecture presents the notion of triangulated dg-categories, which is a refined (and better) version of the usual notion of triangulated categories.

The last lecture contains a few applications of the general theory explaining how the problems with localization mentioned in the first lecture are solved when working with dg-categories. We start to show that triangulated dg-categories have functorial cones, unlike the case of triangulated categories. We also show that many invariants (such as K-theory, Hochschild homology, . . .) are invariant of dg-categories, though

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it is know that they are not invariant of triangulated categories. We also give a gluing statement, providing a way to glue objects in dg-categories in a situation where it is not possible to glue objects in derived categories. To finish I will present the notion of saturated dg-categories and explain how they can be used in order to define a *secondary K-theory*.

2 Lecture 1: DG-Categories and Localization

The purpose of this first lecture is to explain one motivation for working with dg-categories concerned with the localization construction in category theory (in the sense of Gabriel–Zisman, see below). I will start by presenting some very concrete problems often encountered when using the localization construction. In a second part I will introduce the homotopy category of dg-categories, and propose it as a setting in order to define a better behaved localization construction. This homotopy category of dg-categories will be further studied in the next lectures.

2.1 The Gabriel–Zisman Localization

Let C be a category and S be a subset of the set of morphisms in C[1]. A *localization of C with respect to S* is the data of a category $S^{-1}C$ and a functor

$$l : C \longrightarrow S^{-1}C$$

satisfying the following property: for any category D the functor induced by composition with l

$$l^* : \underline{Hom}(S^{-1}C, D) \longrightarrow \underline{Hom}(C, D)$$

is fully faithful and its essential image consists of all functors $f : C \longrightarrow D$ such that $f(s)$ is an isomorphism in D for any $s \in S$ (here $\underline{Hom}(A,B)$ denotes the category of functors from a category A to another category B).

Using the definition it is not difficult to show that if a localization exists then it is unique, up to an equivalence of categories, which is itself unique up to a unique isomorphism. It can also be proved that a localization always exists. One possible proof of the existence of localizations is as follows. Let I be the category with two objects 0 and 1 and a unique morphism $u : 0 \to 1$. In the same way, let $\bar{I}$ be the category with two objects 0 and 1 and with a unique isomorphism $\bar{u} : 0 \to 1$. There exists a natural functor $I \longrightarrow \bar{I}$ sending 0 to 0, 1 to 1 and u to $\bar{u}$. Let now C be a category and S be a set of morphisms in C. For any $s \in S$, with source $x \in C$ and target $y \in C$, we define a functor $i_s : I \longrightarrow C$ sending 0 to x, 1 to y and u to s. We get this way a diagram of categories and functors

[1] In these lectures I will not take into account set theory problems, and will do as if all categories were *small*. I warn the ready that, at some point, we will have to consider *non-small* categories, and thus that these set theory problems should be solved somehow. On possible solution is for instance by fixing various Grothendieck universes (see [2, Exp. 1]).

$$\begin{array}{ccc} C & & \\ \uparrow{\scriptstyle \coprod i_s} & & \\ \coprod_{s\in S} I & \longrightarrow & \coprod_s \bar{I}. \end{array}$$

We consider this as a diagram in the category of categories (objects are categories and morphisms are functors), and we form the push-out

$$\begin{array}{ccc} C & \longrightarrow & C' \\ \uparrow{\scriptstyle \coprod i_s} & & \uparrow \\ \coprod_{s\in S} I & \longrightarrow & \coprod_s \bar{I} \end{array}$$

It is not hard to show that for any category D the category of functors $\underline{Hom}(C',D)$ is *isomorphic* to the full sub-category of $\underline{Hom}(C,D)$ consisting of all functors sending elements of S to isomorphisms in D. In particular, the induced functor $C \longrightarrow C'$ is a localization in the sense we defined above.

The only non-obvious point with this argument is the fact that the category of categories possesses push-outs and even all kind of limits and colimits. One possible way to see this is by noticing that the category of small categories is monadic over the category of (oriented) graphs, and to use a general result of existence of colimits in monadic categories (see e.g. [8, II-Prop. 7.4]).

In general localizations are extremely difficult to describe in a useful manner, and the existence of localizations does not say much in practice (though it is sometimes useful to know that they exist). The push-out constructions mentioned above can be explicited to give a description of the localization C'. Explicitly, C' has the same objects as C itself. Morphisms between two objects x and y in C' are represented by strings of arrows in C

$$x \longrightarrow x_1 \longleftarrow x_2 \longrightarrow x_3 \longleftarrow \cdots \longleftarrow x_n \longrightarrow y,$$

for which all the arrows going backwards are assumed to be in S. To get the right set of morphisms in C' we need to say when two such strings define the same morphism (see [9, Sect. I.1.1] for details). This description for the localization is rather concrete, however it is most often useless in practice.

The following short list of examples show that localized categories are often encountered and provide interesting categories in general.

Examples:

(a) If all morphisms in S are isomorphisms then the identity functor $C \to C$ is a localization.

(b) If S consists of all morphisms in C, then $S^{-1}C$ is the groupoid completion of C. When C has a unique object with a monoid M of endomorphisms, then $S^{-1}C$ has unique object with the group M^+ as automorphisms (M^+ is the group completion of the monoid M).

(c) Let R be a ring and $C(R)$ be the category of (unbounded) complexes over R. Its objects are families of R-modules $\{E^n\}_{n\in\mathbb{Z}}$ together with maps $d^n : E^n \to E^{n+1}$ such that $d^{n+1}d^n = 0$. Morphisms are simply families of morphisms commuting with the d's. Finally, for $E \in C(R)$, we can define its n-th cohomology by $H^n(E) := Ker(d^n)/Im(d^{n-1})$, which is an R-module. The construction $E \mapsto H^n(E)$ provides a functor H^n from $C(R)$ to R-modules.
A morphism $f : E \longrightarrow F$ in $C(R)$ is called a *quasi-isomorphism* if for all $i \in \mathbb{Z}$ the induced map
$$H^i(f) : H^i(E) \longrightarrow H^i(F)$$
is an isomorphism. We let S be the set of quasi-isomorphisms in $C(R)$. Then $S^{-1}C(R)$ is the *derived category* of R and is denoted by $D(R)$. Understanding the hidden structures of derived categories is one of the main objectives of dg-category theory.
Any R-module M can be considered as a complex concentrated in degree 0, and thus as an object in $D(R)$. More generally, if $n \in \mathbb{Z}$, we can consider the object $M[n]$ which is the complex concentrated in degree $-n$ and with values M. It can be shown that for two R-modules M and N there exists a natural isomorphism
$$Hom_{D(R)}(M,N[n]) \simeq Ext^n(M,N).$$
(d) Let Cat be the category of categories: its objects are categories and its morphisms are functors. We let S be the set of categorical equivalences. The localization category $S^{-1}Cat$ is called the *homotopy category of categories*. It can be shown quite easily that $S^{-1}Cat$ is equivalent to the category whose objetcs are categories and whose morphismes are isomorphism classes of functors (see Exercise 2.1.2).
(e) Let Top be the category of topological spaces and continuous maps. A morphism $f : X \longrightarrow Y$ is called a *weak equivalence* if it induces isomorphisms on all homotopy groups (with respect to all base points). If S denotes the set of weak equivalences then $S^{-1}Top$ is called the *homotopy category of spaces*. It can be shown that $S^{-1}Top$ is equivalent to the category whose objects are CW-complexes and whose morphisms are homotopy classes of continuous maps.

One comment before going on. Let us denote by $Ho(Cat)$ the category $S^{-1}Cat$ considered in example (4) above. Let C be a category and S be a set of morphisms in C. We define a functor
$$F : Ho(Cat) \longrightarrow Set$$
sending a category D to the set of all isomorphism classes of functors $C \longrightarrow D$ sending S to isomorphisms. The functor F is therefore a sub-functor of the functor h^C corepresented by C. Another way to consider localization is by stating that the functor F is corepresentable by an object $S^{-1}C \in Ho(Cat)$. This last point of view is a bit less precise as the original notion of localizations, as the object $S^{-1}C$ satisfies a universal property only on the level of isomorphism classes of functors and not on the level of categories of functors themselves. However, this point of view is often useful and enough in practice.

Exercise 2.1.1 *Let C and D be two categories and S (resp. T) be a set of morphisms in C (resp. in D) containing the identities.*

(a) *Prove that the natural functor*

$$C \times D \longrightarrow (S^{-1}C) \times (T^{-1}D)$$

is a localization of $C \times D$ with respect to the set $S \times T$. In other words localizations commutes with finite products.

(b) *We assume that there exist two functors*

$$f : C \longrightarrow D \qquad C \longleftarrow D : g$$

with $f(S) \subset T$ and $g(T) \subset S$. We also assume that there exists two natural transformations $h : fg \Rightarrow id$ and $k : gf \Rightarrow id$ such that for any $x \in C$ (resp. $y \in D$) the morphism $k(y) : g(f(x)) \to x$ (resp. $h(y) : f(g(y)) \to y$) is in S (resp. in T). Prove that the induced functors

$$f : S^{-1}C \longrightarrow T^{-1}D \qquad S^{-1}C \longleftarrow T^{-1}D : g$$

are equivalences inverse to each other.

(c) *If S consists of all morphisms in C and if C has a final or initial object then $C \longrightarrow *$ is a localization of C with respect to S.*

Exercise 2.1.2 *Let Cat be the category of categories and functors, and let $[Cat]$ be the category whose objects are categories and whose morphisms are isomorphism classes of functors (i.e. $Hom_{[Cat]}(C,D)$ is the set of isomorphism classes of objects in $\underline{Hom}(C,D)$). Show that the natural projection*

$$Cat \longrightarrow [Cat]$$

is a localization of Cat along the subset of equivalences of categories (prove directly that it has the correct universal property).

2.2 Bad Behavior of the Gabriel–Zisman Localization

In these lectures we will be mainly interested in localized categories of the type $D(R)$ for some ring R (or some more general object, see lecture 2). I will therefore explain the bad behaviour of the localization using examples of derived categories. However, this bad behaviour is a general fact and also applies to other examples of localized categories.

Though the localization construction is useful to construct interesting new categories, the resulting localized categories are in general badly behaved. Often, the category to be localized has some nice properties, such as the existence of limits and colimits or being abelian, but these properties are lost after localization. Here is a sample of problems often encountered in practice.

(a) The derived category $D(R)$ lacks the standard categorical constructions of limits and colimits. There exists a non-zero morphism $e : \mathbb{Z}/2 \longrightarrow \mathbb{Z}/2[1]$ in $D(\mathbb{Z})$, corresponding to the non-zero element in $Ext^1(\mathbb{Z}/2,\mathbb{Z}/2)$ (recall that $Ext^i(M,N) \simeq [M,N[i]]$, where $N[i]$ is the complex whose only non-zero part is N in degree $-i$, and $[-,-]$ denotes the morphisms in $D(R)$). Suppose that the morphism e has a kernel, i.e. that a fiber product

$$\begin{array}{ccc} X & \longrightarrow & \mathbb{Z}/2 \\ \downarrow & & \downarrow e \\ 0 & \longrightarrow & \mathbb{Z}/2[1] \end{array}$$

exists in $D(\mathbb{Z})$. Then, for any integer i, we have a short exact sequence

$$0 \longrightarrow [\mathbb{Z},X[i]] \longrightarrow [\mathbb{Z},\mathbb{Z}/2[i]] \longrightarrow [\mathbb{Z},\mathbb{Z}/2[i+1]],$$

or in other words

$$0 \longrightarrow H^i(X) \longrightarrow H^i(\mathbb{Z}/2) \longrightarrow H^{i+1}(\mathbb{Z}/2).$$

This implies that $X \longrightarrow \mathbb{Z}/2$ is a quasi-isomorphism, and thus an isomorphism in $D(\mathbb{Z})$. In particular $e = 0$, which is a contradiction.
A consequence of this is that $D(R)$ is not an abelian category, though the category of complexes itself $C(R)$ is abelian.

(b) The fact that $D(R)$ has no limits and colimits might not be a problem by itself, as it is possible to think of interesting categories which do not have limits and colimits (e.g. any non-trivial groupoid has no final object). However, the case of $D(R)$ is very frustrating as it seems that $D(R)$ is very close to having limits and colimits. For instance it is possible to show that $D(R)$ admits *homotopy limits and homotopy colimits* in the following sense. For a category I, let $C(R)^I$ be the category of functors from I to $C(R)$. A morphism $f : F \longrightarrow G$ (i.e. a natural transformation between two functors $F,G : I \longrightarrow C(R)$) is called a *levelwise quasi-isomorphism* if for any $i \in I$ the induced morphism $f(i) : F(i) \longrightarrow G(i)$ is a quasi-isomorphism in $C(R)$. We denote by $D(R,I)$ the category $C(R)^I$ localized along levelwise quasi-isomorphisms. The constant diagram functor $C(R) \longrightarrow C(R)^I$ is compatible with localizations on both sides and provides a functor

$$c : D(R) \longrightarrow D(R,I).$$

It can then been shown that the functor c has a left and a right adjoint denoted by

$$Hocolim_I : D(R,I) \longrightarrow D(R) \qquad D(R) \longleftarrow D(R,I) : Holim_I,$$

called the *homotopy colimit* and the *homotopy limit* functor. Homotopy limits and colimits are very good replacement of the notions of limits and colimits, as they are the best possible approximation of the colimit and limit functors that are compatible with the notion of quasi-isomorphisms. However, this is quite unsatisfactory as the category $D(R,I)$ depends on more than the category $D(R)$ alone (note that $D(R,I)$ is not equivalent to $D(R)^I$), and in general it is impossible to recontruct $D(R,I)$ from $D(R)$.

(c) To the ring R are associated several invariants such as its K-theory spectrum, its Hochschild (resp. cyclic) homology .... It is tempting to think that these invariants can be directly defined on the level of derived categories, but this is not the case (see [24]). However, it has been noticed that these invariants only depend on R up to some notion of equivalence that is much weaker than the notion of isomorphism. For instance, any functor $D(R) \longrightarrow D(R')$ which is induced by a complex of (R,R')-bi-modules induces a map on K-theory, Hochschild homology and cyclic homology. However, it is not clear that every functor $D(R) \longrightarrow D(R')$ comes from a complex of (R,R')-bi-modules (there are counter examples when R and R' are dg-algebras, see [7, Remarks 2.5 and 6.8]). Definitely, the derived category of complexes of (R,R')-bi-modules is not equivalent to the category of functors $D(R) \longrightarrow D(R')$. This is again an unsatisfactory situation and it is then quite difficult (if not impossible) to understand the true nature of these invariants (i.e. of which mathematical structures are they truly invariants?).

(d) Another important problem with the categories $D(R)$ is their non local nature. To explain this let $\mathbb{P}^1$ be the projective line (e.g. over $\mathbb{Z}$). As a scheme $\mathbb{P}^1$ is the push-out

$$\begin{array}{ccc} Spec\,\mathbb{Z}[X,X^{-1}] & \longrightarrow & Spec\,\mathbb{Z}[T] \\ \downarrow & & \downarrow \\ Spec\,\mathbb{Z}[U] & \longrightarrow & \mathbb{P}^1, \end{array}$$

where T is sent to X and U is sent to X^{-1}. According to the push-out square, the category of quasi-coherent sheaves on $\mathbb{P}^1$ can be described as the (2-categorical) pull-back

$$\begin{array}{ccc} QCoh(\mathbb{P}^1) & \longrightarrow & Mod(\mathbb{Z}[T]) \\ \downarrow & & \downarrow \\ Mod(\mathbb{Z}[U]) & \longrightarrow & Mod(\mathbb{Z}[X,X^{-1}]). \end{array}$$

In other words, a quasi-coherent module on $\mathbb{P}^1$ is the same thing as a triple (M,N,u), where M (resp. N) is a $\mathbb{Z}[T]$-module (resp. $\mathbb{Z}[U]$-module), and u is an isomorphism

$$u : M \otimes_{\mathbb{Z}[T]} \mathbb{Z}[X,X^{-1}] \simeq N \otimes_{\mathbb{Z}[U]} \mathbb{Z}[X,X^{-1}]$$

of $\mathbb{Z}[X,X^{-1}]$-modules. This property is extremely useful in order to reduce problems of quasi-coherent sheaves on schemes to problems of modules over rings. Unfortunately, this property is lost when passing to the derived categories. The square

$$\begin{array}{ccc} D_{qcoh}(\mathbb{P}^1) & \longrightarrow & D(\mathbb{Z}[T]) \\ \downarrow & & \downarrow \\ D(\mathbb{Z}[U]) & \longrightarrow & D(\mathbb{Z}[X,X^{-1}]), \end{array}$$

is not cartesian (in the 2-categorical sense) anymore (e.g. there exist non zero morphisms $\mathscr{O}_X \longrightarrow \mathscr{O}_X(-2)[1]$ that go to zero as a morphism in $D(\mathbb{Z}[U]) \times_{D(\mathbb{Z}[X,X^{-1}])} D(\mathbb{Z}[T])$). The derived categories of the affine pieces of $\mathbb{P}^1$ do not determine the derived category of quasi-coherent sheaves on $\mathbb{P}^1$.

The list of problems above suggests the existence of a some sort of categorical structure lying in between the category of complexes $C(R)$ and its derived category $D(R)$, which is rather close to $D(R)$ (i.e. in which the quasi-isomorphisms are inverted in some sense), but for which (1)–(4) above are no longer a problem. There exist several possible approaches, and my purpose is to present one of them using dg-categories.

Exercise 2.2.1 *Let $I = B\mathbb{N}$ be the category with a unique object $*$ and with the monoid $\mathbb{N}$ of natural numbers as endomorphism of this object. There is a bijection between the set of functors from I to a category C and the set of pairs (x,h), where x in an object in C and h is an endomorphism of x.*

Let R be a commutative ring.

(a) Show that there is a natural equivalence of categories

$$D(R,I) \simeq D(R[X]),$$

where $D(R,I)$ is the derived category of I-diagram of complexes of R-modules as described in example (2) above. Deduce from this that $D(R,I)$ is never an abelian category (unless $R = 0$).

(b) Prove that $D(R)$ is abelian when R is a field (show that $D(R)$ is equivalent to the category of $\mathbb{Z}$-graded R-vector spaces).

(c) Deduce that $D(R,I)$ and $D(R)^I$ can not be equivalent in general.

(d) Let now I be the category with two objects 0 *and* 1 *and a unique morphism from* 1 *to* 0*. Using a similar approach as above show that $D(R,I)$ and $D(R)^I$ are not equivalent in general.*

2.3 DG-Categories and DG-Functors

We now fix a base commutative ring k. Unless specified, all the modules and tensor products will be over k.

2.3.1 DG-Categories

We start by recalling that a *dg-category* T (over k) consists of the following data.

- A set of objects $Ob(T)$, also sometimes denoted by T itself
- For any pair of objects $(x,y) \in Ob(T)^2$ a complex $T(x,y) \in C(k)$

- For any triple $(x,y,z) \in Ob(T)^3$ a composition morphism $\mu_{x,y,z} : T(x,y) \otimes T(y,z) \longrightarrow T(x,z)$.
- For any object $x \in Ob(T)$, a morphism $e_x : k \longrightarrow T(x,x)$

These data are required to satisfy the following associativity and unit conditions.

(a) (Associativity) For any four objects (x,y,z,t) in T, the following diagram

$$\begin{array}{ccc} T(x,y) \otimes T(y,z) \otimes T(z,t) & \xrightarrow{id \otimes \mu_{y,z,t}} & T(x,y) \otimes T(y,t) \\ \downarrow{\scriptstyle \mu_{x,y,z} \otimes id} & & \downarrow{\scriptstyle \mu_{x,y,t}} \\ T(x,z) \otimes T(z,t) & \xrightarrow[\mu_{x,z,t}]{} & T(x,t) \end{array}$$

commutes.

(b) (Unit) For any $(x,y) \in Ob(T)^2$ the two morphisms

$$T(x,y) \simeq k \otimes T(x,y) \xrightarrow{e_x \otimes id} T(x,x) \otimes T(x,y) \xrightarrow{\mu_{x,x,y}} T(x,y)$$

$$T(x,y) \simeq T(x,y) \otimes k \xrightarrow{id \otimes e_y} T(x,y) \otimes T(y,y) \xrightarrow{\mu_{x,y,y}} T(x,y)$$

are equal to the identities.

In a more explicit way, a dg-category T can also be described as follows. It has a set of objects $Ob(T)$. For any two objects x and y, and any $n \in \mathbb{Z}$ it has a k-module $T(x,y)^n$, though as morphisms of degree n from x to y. For three objects x, y and z, and any integers n and m there is a composition map

$$T(x,y)^n \times T(y,z)^m \longrightarrow T(x,z)^{n+m}$$

which is bilinear and associative. For any object x, there is an element $e_x \in T(x,x)^0$, which is a unit for the composition. For any two objects x and y there is a differential $\mathrm{d} : T(x,y)^n \longrightarrow T(x,y)^{n+1}$, such that $\mathrm{d}^2 = 0$. And finally, we have the graded Leibnitz rule

$$\mathrm{d}(f \circ g) = \mathrm{d}(f) \circ g + (-1)^m f \circ \mathrm{d}(g),$$

for f and g two composable morphisms, with f of degree m. Note that this implies that $\mathrm{d}(e_x) = 0$, and thus that e_x is always a zero cycle in the complex $T(x,x)$.

On a more conceptual side, a dg-category is a $C(k)$-enriched category in the sense of [17], where $C(k)$ is the symmetric monoidal category of complexes of k-modules. All the basic notions of dg-categories presented in these notes can be expressed in terms of enriched category theory, but we will not use this point of view. We, however, encourage the reader to consult [17] and to (re)consider the definitions of dg-categories, dg-functors, tensor product of dg-categories ... in the light of enriched category theory.

Examples:

(a) A very simple example is the opposite dg-category T^{op} of a dg-category T. The set of objects of T^{op} is the same as the one of T, and we set

$$T^{op}(x,y) := T(y,x)$$

together with the obvious composition maps

$$T(y,x) \otimes T(z,y) \simeq T(z,y) \otimes T(y,x) \longrightarrow T(z,x),$$

where the first isomorphism is the symmetry isomorphism of the monoidal structure on the category of complexes (see [5, Sect. X.4.1] for the signs rule).

(b) A fundamental example of dg-category over k is the one given by considering the category of complexes over k itself. Indeed, we define a dg-category $\underline{C(k)}$ by setting its set of objects to be the set of complexes of k-modules. For two complexes E and F, we define $\underline{C(k)}(E,F)$ to be the complex $\underline{Hom}^*(E,F)$ of morphisms from E to F. Recall, that for any $n \in \mathbb{Z}$ the k-module of elements of degree n in $\underline{Hom}^*(E,F)$ is given by

$$\underline{Hom}^n(E,F) := \prod_{i\in\mathbb{Z}} Hom(E^i, F^{i+n}).$$

The differential

$$\mathrm{d} : \underline{Hom}^n(E,F) \longrightarrow \underline{Hom}^{n+1}(E,F)$$

sends a family $\{f^i\}_{i\in\mathbb{Z}}$ to the family $\{\mathrm{d} \circ f^i - (-1)^n f^{i+1} \circ \mathrm{d}\}_{i\in\mathbb{Z}}$. Note that the zero cycles in $\underline{Hom}^*(E,F)$ are precisely the morphisms of complexes from E to F. The composition of morphisms induces composition morphisms

$$\underline{Hom}^n(E,F) \times \underline{Hom}^m(E,F) \longrightarrow \underline{Hom}^{n+m}(E,F).$$

It is easy to check that these data defines a dg-category $\underline{C(k)}$.

(c) There is slight generalization of the previous example for the category $C(R)$ of complexes of (left) R-modules, where R is any associative and unital k-algebra. Indeed, for two complexes of R-modules E and F, there is a complex $\underline{Hom}^*(E,F)$ defined as in the previous example. The only difference is that now $\underline{Hom}^*(E,F)$ is only a complex of k-modules and not of R-modules in general (except when R is commutative). These complexes define a dg-category $\underline{C(R)}$ whose objects are complexes of R-modules.

(d) A far reaching generalization of the two previous examples is the case of complexes of objects in any k-linear Grothendieck category (i.e. an abelian cocomplete category with a small generator and for which filtered colimits are exact, or equivalently a localization of a modules category, see [10]). Indeed, for such a category $\mathscr{A}$ and two complexes E and F of objects in $\mathscr{A}$, we define a complex of k-modules $\underline{Hom}^*(E,F)$ as above

$$\underline{Hom}^n(E,F) := \prod_{i\in\mathbb{Z}} Hom(E^i, F^{i+n}),$$

with the differential given by the same formula as in example (2). The composition of morphisms induce morphisms

$$\underline{Hom}^n(E,F) \times \underline{Hom}^m(F,G) \longrightarrow \underline{Hom}^{n+m}(E,G).$$

It is easy to check that these data define a dg-category whose objects are complexes in $\mathscr{A}$. It will be denoted by $\underline{C(\mathscr{A})}$.

(e) From a dg-category T, we can construct a category $Z^0(T)$ of 0-cycles as follows. It has the same objects as T, and for two such objects x and y the set of morphisms between x and y in $Z^0(T)$ is defined to be the set of 0-cycles in $T(x,y)$ (i.e. degree zero morphisms $f \in T(x,y)^0$ such that $\mathrm{d}(f) = 0$. The Leibniz rule implies that the composition of two 0-cycles is again a 0-cycle, and thus we have induced composition maps

$$Z^0(T(x,y)) \times Z^0(T(y,z)) \longrightarrow Z^0(T(x,z)).$$

These composition maps define the category $Z^0(T)$. The category $Z^0(T)$ is often named the *underlying category of T*. We observe that $Z^0(T)$ is more precisely a k-linear category (i.e. that *Homs* sets are endowed with k-module structures such that the composition maps are bilinear).
For instance, let $\mathscr{A}$ be a Grothendieck category and $\underline{C(\mathscr{A})}$ its associated dg-category of complexes as defined in example (4) above. The underlying category of $\underline{C(\mathscr{A})}$ is then isomorphic to the usual category $C(\mathscr{A})$ of complexes and morphisms of complexes in $\mathscr{A}$.

(f) Conversely, if C is a k-linear category we view C as a dg-category in a rather obvious way. The set of objects is the same of the one of C, and the complex of morphisms from x to y is simply the complex $C(x,y)[0]$, which is $C(x,y)$ in degree 0 and 0 elsewhere. In the sequel, every k-linear category will be considered as a dg-category in this obvious way. Note that, this way the category of k-linear categories and k-linear functors form a full sub-category of dg-categories and dg-functors (see Sect. 1.3.2 below).

(g) A dg-category T with a unique object is essentially the same thing as a dg-algebra. Indeed, if x is the unique object the composition law on $T(x,x)$ induces a unital and associative dg-algebra structure on $T(x,x)$. Conversely, if B is a unital and associative dg-algebra we can construct a dg-category T with a unique object x and with $T(x,x) := B$. The multiplication in B is then used to define the composition on $T(x,x)$.

(h) Here is now a non-trivial example of a dg-category arising from geometry. In this example $k = \mathbb{R}$. Let X be a differential manifold (say $\mathscr{C}^\infty$). Recall that a flat vector bundle on X consists the data of a smooth (real) vector bundle V on X together with a connexion

$$\nabla : A^0(X,V) \longrightarrow A^1(X,V),$$

(where $A^n(X,V)$ is the space of smooth n-forms on X with coefficients in V) such that $\nabla^2 = 0$. For two such flat bundles (V,∇_V) and (W,∇_W) we define a complex $A^*_{DR}(V,W)$ by

$$A^*_{DR}(V,W)^n := A^n(X, \underline{Hom}(V,W)),$$

where $\underline{Hom}(V,W)$ is the vector bundle of morphisms from V to W. The differential

$$\mathrm{d} : A^n_{DR}(V,W) \longrightarrow A^{n+1}_{DR}(V,W)$$

is defined by sending $\omega \otimes f$ to $\mathrm{d}(\omega) \otimes f + (-1)^n \omega \wedge \nabla(f)$. Here, $\nabla(f)$ is the 1-form with coefficients in $\underline{Hom}(V,W)$ defined by

$$\nabla(f) := \nabla_W \circ f - (f \otimes id) \circ \nabla_V.$$

The fact that $\nabla^2_V = \nabla^2_W = 0$ implies that $A^*_{DR}(V,W)$ is a complex. Moreover, we define a composition

$$A^n_{DR}(U,V) \times A^m_{DR}(V,W) \longrightarrow A^{n+m}(U,W)$$

for three flat bundles U, V and W by

$$(\omega \otimes f).(\omega' \otimes g) := (\omega \wedge \omega') \otimes (f \circ g).$$

It is easy to check that these data defines a dg-category $T_{DR}(X)$ (over $\mathbb{R}$) whose objects are flat bundles on X, and whose complex of morphisms from (V,∇_V) to (W,∇_W) are the complexes $A^*_{DR}(V,W)$.
By construction the underlying category of $T_{DR}(X)$ is the category of flat bundles and flat maps. By the famous Riemann–Hilbert correspondence (see [6] for the analog statement in the complex analytic case) this category is thus equivalent to the category of finite dimensional linear representations of the fundamental group of X, or equivalently of finite dimensional local systems (i.e. of locally constant sheaves of finite dimensional $\mathbb{C}$-vector spaces). Moreover, for two flat bundles (V,∇_V) and (W,∇_W), corresponding to two local systems L_1 and L_2, the cohomology group $H^i(T_{DR}(X)(V,W)) = H^i(A^*_{DR}(V,W))$ is isomorphic to the Ext group $Ext^i(L_1,L_2)$, computed in the category of abelian sheaves over X. Therefore, we see that even when X is simply connected the dg-category $T_{DR}(X)$ contains interesting informations about the cohomology of X (even though the underlying category of $T_{DR}(X)$ is simply the category of finite dimensional vector spaces).

(i) The previous example has the following complex analog. Now we let $k = \mathbb{C}$, and X be a complex manifold. We define a dg-category $T_{Dol}(X)$ in the following way. The objects of $T_{Dol}(X)$ are the holomorphic complex vector bundles on X. For two such holomorphic bundles V and W we let

$$T_{Dol}(X)(V,W) := A^*_{Dol}(V,W),$$

where $A^*_{Dol}(V,W)$ is the Dolbeault complex with coefficients in the vector bundle of morphisms from V to W. Explicitely,

$$A^q_{Dol}(V,W) := A^{0,q}(X, \underline{Hom}(V,W))$$

is the space of $(0,q)$-forms on X with coefficients in the holomorphic bundle $\underline{Hom}(V,W)$ of morphisms from V to W. The differential

$$A^q_{Dol}(V,W) \longrightarrow A^{q+1}_{Dol}(V,W)$$

is the operator $\overline{\partial}$, sending $\omega \otimes f$ to

$$\overline{\partial}(\omega \otimes f) := \overline{\partial}(\omega) \otimes f + (-1)^q \omega \wedge \overline{\partial}(f),$$

where $\overline{\partial}(f)$ is defined by

$$\overline{\partial}(f) = \overline{\partial}_W \circ f - (f \otimes id) \circ \overline{\partial}_V,$$

with

$$\overline{\partial}_V : A^0(X,V) \longrightarrow A^{0,1}(X,V) \qquad \overline{\partial}_W : A^0(X,W) \longrightarrow A^{0,1}(X,W)$$

being the operators induced by the holomorphic structures on V and W (see [11, Chap 0 Sect. 5]). As in the previous example we can define a composition

$$A^*_{Dol}(U,V) \times A^*_{Dol}(V,W) \longrightarrow A^*_{Dol}(U,W)$$

for three holomorphic bundles U, V and W on X. These data defines a dg-category $T_{Dol}(X)$ (over $\mathbb{C}$).
By construction, the underlying category of $T_{Dol}(X)$ has objects the holomorphic vector bundles, and the morphisms in this category are the $\mathscr{C}^\infty$-morphisms of complex vector bundles $f : V \longrightarrow W$ satisfying $\overline{\partial}(f) = 0$, or equivalently the holomorphic morphisms. Moroever, for two holomorphic vector bundles V and W the cohomology group $H^i(T_{Dol}(X))$ is isomorphic to $Ext^i_{\mathscr{O}_X}(\mathscr{V},\mathscr{W})$, the i-th ext-group between the associated sheaves of holomorphic sections (or equivalently the ext-group in the category of holomorphic coherent sheaves). For instance, if $\mathbf{1}$ is the trivial vector bundle of rank 1 and V is any holomorphic vector bundle, we have

$$H^i(T_{Dol}(\mathbf{1},V)) \simeq H^i_{Dol}(X,V),$$

the i-th Dolbeault cohomology group of V.
The dg-category $T_{Dol}(X)$ is important as it provides a rather explicit model for the derived category of coherent sheaves on X. Indeed, the homotopy category $H^0(T_{Dol}(X))$ (see definition 2.3.1) is equivalent to the full sub-category of $D^b_{coh}(X)$, the bounded coherent derived category of X, whose objects are holomorphic vector bundles. Also, for two such holomorphic vector bundles V and W and all i we have

$$Hom_{D^b_{coh}(X)}(V,W[i]) \simeq H^i(T_{Dol}(X)(V,W)) \simeq Ext^i_{\mathscr{O}_X}(\mathscr{V},\mathscr{W}).$$

(j) Here is one last example of a dg-category in the topological context. We construct a dg-category $dgTop$, whose set of objects is the set of all topological spaces. For two topological spaces X and Y, we define a complex of morphisms $dgTop(X,Y)$ in the following way. We first consider $Hom^{\Delta}(X,Y)$, the simplicial set (see [13] for the notion of simplicial sets) of continuous maps between X and Y: by definition the set of n-simplicies in $Hom^{\Delta}(X,Y)$ is the set of continuous maps $X \times \Delta^n \longrightarrow Y$, where $\Delta^n := \{x \in [0,1]^{n+1} / \sum x_i = 1\}$ is the standard simplex of dimension n in $\mathbb{R}^{n+1}$. The face and degeneracy operators of $Hom^{\Delta}(X,Y)$ are defined using the face embeddings ($0 \leq i \leq n$)

$$\begin{array}{rcl} d_i : \Delta^n & \longrightarrow & \Delta^{n+1} \\ x & \mapsto & (x_0, \ldots, x_{i-1}, 0, x_i, \ldots x_n), \end{array}$$

and the natural projections ($0 \leq i \leq n$)

$$\begin{array}{rcl} s_i : \Delta^{n+1} & \longrightarrow & \Delta^n \\ x & \mapsto & (x_0, \ldots, x_i + x_{i+1}, x_{i+2}, \ldots, x_{n+1}). \end{array}$$

Now, for any two topological spaces X and Y we set

$$dgTop(X,Y) := C_*(Hom^{\Delta}(X,Y)),$$

the homology chain complex of $Hom^{\Delta}(X,Y)$ with coefficients in k. Explicitly, $C_n(Hom^{\Delta}(X,Y))$ is the free k-module generated by continuous maps $f : X \times \Delta^n \longrightarrow Y$. The differential of such a map is given by the formula

$$d(f) := \sum_{0 \leq i \leq n} (-1)^i d_i(f),$$

where $d_i(f)$ is the map $X \times \Delta^{n-1} \longrightarrow Y$ obtained by composition

$$X \times \Delta^{n-1} \xrightarrow{id \times d_i} X \times \Delta^n \xrightarrow{f} Y.$$

For three topological spaces X, Y and Z, there exists a composition morphism at the level of simplicial sets of continuous maps

$$Hom^{\Delta}(X,Y) \times Hom^{\Delta}(Y,Z) \longrightarrow Hom^{\Delta}(X,Z).$$

This induces a morphism on the level of chain complexes

$$C_*(Hom^{\Delta}(X,Y) \times Hom^{\Delta}(Y,Z)) \longrightarrow C_*(Hom^{\Delta}(X,Z)).$$

Composing this morphism with the famous Eilenberg–MacLane map (see [20, Sect. 29])

$$C_*(Hom^{\Delta}(X,Y)) \otimes C_*(Hom^{\Delta}(Y,Z)) \longrightarrow C_*(Hom^{\Delta}(X,Y) \times Hom^{\Delta}(Y,Z))$$

defines a composition

$$dgTop(X,Y) \otimes dgTop(Y,Z) \longrightarrow dgTop(X,Z).$$

The fact that the Eilenberg–MacLane morphisms are associative and unital (they are moreover commutative, see [20, Sect. 29]) implies that this defines a dg-category $dgTop$.

2.3.2 DG-Functors

For two dg-categories T and T', a *morphism of dg-categories* (or simply a *dg-functor*) $f : T \longrightarrow T'$ consists of the following data.

- A map of sets $f : Ob(T) \longrightarrow Ob(T')$.
- For any pair of objects $(x,y) \in Ob(T)^2$, a morphism in $C(k)$

$$f_{x,y} : T(x,y) \longrightarrow T'(f(x), f(y)).$$

These data are required to satisfy the following associativity and unit conditions.

(a) For any $(x,y,z) \in Ob(T)^3$ the following diagram

$$\begin{array}{ccc} T(x,y) \otimes T(y,z) & \xrightarrow{\mu_{x,y,z}} & T(x,z) \\ \downarrow{\scriptstyle f_{x,y} \otimes f_{y,z}} & & \downarrow{\scriptstyle f_{x,z}} \\ T'(f(x),f(y)) \otimes T'(f(y),f(z)) & \xrightarrow[\mu'_{f(x),f(y),f(z)}]{} & T'(f(x),f(z)) \end{array}$$

commutes.

(b) For any $x \in Ob(T)$, the following diagram

$$\begin{array}{ccc} k & \xrightarrow{e_x} & T(x,x) \\ & {\scriptstyle e'_{f(x)}} \searrow & \downarrow{\scriptstyle f_{x,x}} \\ & & T'(f(x),f(x)) \end{array}$$

commutes.

Examples:

(a) Let T be any dg-category and $x \in T$ be an object. We define a dg-functor

$$f = h^x : T \longrightarrow \underline{C(k)}$$

in the following way (recall that $\underline{C(k)}$ is the dg-category of complexes over k). The map on the set of objects sends an object $y \in T$ to the complex $T(x,y)$. For two objects y and z in T we define a morphism

$$f_{y,z} : T(y,z) \longrightarrow \underline{C(k)}(f(y), f(x)) = \underline{Hom}^*(T(x,y), T(x,z)),$$

which by definition is the adjoint to the composition morphism

$$m_{x,y,z} : T(x,y) \otimes T(y,z) \longrightarrow T(x,z).$$

The associativity and unit condition on composition of morphisms in T imply that this defines a morphism of dg-categories

$$h^x : T \longrightarrow \underline{C(k)}.$$

Dually, we can also define a morphism of dg-categories

$$h_x : T^{op} \longrightarrow \underline{C(k)}$$

by sending y to $T(y,x)$.

(b) For any dg-category T there exists a dg-functor

$$T \otimes T^{op} \longrightarrow \underline{C(k)},$$

sending a pair of objects (x,y) to the complex $T(y,x)$. Here, $T \otimes T^{op}$ denotes the dg-category whose set of objects is $Ob(T) \times Ob(T')$, and whose complex of morphisms are given by

$$(T \otimes T^{op})((x,y),(x',y')) := T(x,x') \otimes T(y',y).$$

We refer to Exercise 2.3.4 and Sect. 3.2 for more details about the tensor product of dg-categories.

(c) Let R and S be two associative and unital k-algebras, and $f : R \longrightarrow S$ be a k-morphism. The morphism f induces two functors

$$f^* : C(R) \longrightarrow C(S) \qquad C(R) \longleftarrow C(S) : f_*,$$

adjoint to each others. The functor f_* sends a complex of S-modules to the corresponding complex of R-modules obtained by restricting the scalars from S to R by the morphism f. Its left adjoint f^* sends a complex of R-modules E to the complex $S \otimes_R E$. It is not difficult to show that the functors f_* and f^* are compatible with the complex of morphisms $\underline{Hom}^*$ and thus define morphisms of dg-categories

$$f^* : \underline{C(R)} \longrightarrow \underline{C(S)} \qquad \underline{C(R)} \longleftarrow \underline{C(S)} : f_*.$$

More generally, if $f : \mathscr{A} \longrightarrow \mathscr{B}$ is any k-linear functor between Grothendieck categories, there is an induced morphism of dg-categories

$$f : \underline{C(\mathscr{A})} \longrightarrow \underline{C(\mathscr{B})}.$$

(d) Let $f : X \longrightarrow Y$ be a $\mathscr{C}^\infty$-morphism between two differential manifolds. Then, the pull-back for flat bundles and differential forms defines a morphism of dg-categories constructed in our example 8

$$f^* : T_{DR}(Y) \longrightarrow T_{DR}(X).$$

In the same way, if now f is a holomorphic morphism between two complex varieties, then there is a dg-functor

$$f^* : T_{Dol}(Y) \longrightarrow T_{Dol}(X)$$

ontained by pulling-back the holomorphic vector bundles and differential forms.

DG-functors can be composed in an obvious manner, and dg-categories together with dg-functors form a category denoted by $dg - cat_k$ (or $dg - cat$ if the base ring k is clear).

For a dg-category T, we define a category $H^0(T)$ in the following way. The set of objects of $H^0(T)$ is the same as the set of objects of T. For two objects x and y the set of morphisms in $H^0(T)$ is defined by

$$H^0(T)(x,y) := H^0(T(x,y)).$$

Finally, the composition of morphisms in $H^0(T)$ is defined using the natural morphisms

$$H^0(T(x,y)) \otimes H^0(T(y,z)) \longrightarrow H^0(T(x,y) \otimes T(y,z))$$

composed with the morphism

$$H^0(\mu_{x,y,z}) : H^0(T(x,y) \otimes T(y,z)) \longrightarrow H^0(T(x,z)).$$

Definition 2.3.1 *The category $H^0(T)$ is called the* homotopy category of T.

Examples:

(a) If C is a k-linear category considered as a dg-category as explained in our example 6 above, then $H^0(C)$ is naturally isomorphic to C itself.
(b) We have $H^0(T^{op}) = H^0(T)^{op}$ for any dg-category T.
(c) For a k-algebra R, the homotopy category $H^0(\underline{C(R)})$ is usually denoted by $K(R)$, and is called the *homotopy category of complexes of R-modules*. More generally, if $\mathscr{A}$ is a Grothendieck category, $H^0(\underline{C(\mathscr{A})})$ is denoted by $K(\mathscr{A})$, and is called the homotopy category of complexes in $\mathscr{A}$.
(d) If X is a differentiable manifold, then $H^0(T_{DR}(X))$ coincides with $Z^0(T_{DR}(X))$ and is isomorphic to the category of flat bundles and flat maps between them. As we already mentioned, this last category is equivalent by the Riemann–Hilbert correspondence to the category of local systems on X.
When X is a complex manifold, we also have that $H^0(T_{Dol}(X))$ coincides with $Z^0(T_{Dol}(X))$ and is isomorphic to the category of holomorphic vector bundles and holomorphic maps between them.
(e) The category $H^0(dgTop)$ is the category whose objects are topological spaces and whose set of morphisms between X and Y is the free k-module over the set of homotopy classes of maps from X to Y.

One of the most important notions in dg-category theory is the notion of quasi-equivalences, a mixture in between quasi-isomorphisms and categorical equivalences.

Definition 2.3.2 *Let $f : T \longrightarrow T'$ be a dg-functor between dg-categories*

(a) The morphism f is quasi-fully faithful *if for any two objects x and y in T the morphism $f_{x,y} : T(x,y) \longrightarrow T'(f(x), f(y))$ is a quasi-isomorphism of complexes.*
(b) The morphism f is quasi-essentially surjective *if the induced functor $H^0(f)$: $H^0(T) \longrightarrow H^0(T')$ is essentially surjective.*
(c) The morphism f is a quasi-equivalence *if it is quasi-fully faithful and quasi-essentially surjective.*

We will be mainly interested in dg-categories up to quasi-equivalences. We therefore introduce the following category.

Definition 2.3.3 *The* homotopy category of dg-categories *is the category $dg-cat$ localized along quasi-equivalences. It is denoted by $Ho(dg-cat)$. Morphisms in $Ho(dg-cat)$ between two dg-categories T and T' will often be denoted by*

$$[T,T'] := Hom_{Ho(dg-cat)}(T,T').$$

Note that the construction $T \mapsto H^0(T)$ provides a functor $H^0(-) : dg-cat \longrightarrow Cat$, which descends as a functor on homotoy categories

$$H^0(-) : Ho(dg-cat) \longrightarrow Ho(Cat).$$

Remark 1. In the last section we have seen that the localization construction is not well behaved, but in the definition above we consider $Ho(dg-cat)$ which is obtained by localization. Therefore, the category $Ho(dg-cat)$ will not be well behaved itself. In order to get the most powerful approach the category $dg-cat$ should have been itself localized in a more refined maner (e.g. as a higher category, see [29, Sect. 2]). We will not need such an evolved approach, and the category $Ho(dg-cat)$ will be enough for most of our purpose.

Examples:

(a) Let $f : T \longrightarrow T'$ be a quasi-fully faithful dg-functor. We let T'_0 be the full (i.e. with the same complexes of morphisms as T') sub-dg-category of T' consisting of all objects $x \in T'$ such that x is isomorphic in $H^0(T')$ to an object in the image of the induced functor $H^0(f) : H^0(T) \longrightarrow H^0(T')$. Then the induced dg-functor $T \longrightarrow T'_0$ is a quasi-equivalence.

(b) Let $f : R \longrightarrow S$ be a morphism of k-algebras. If the morphism of dg-categories

$$f^* : \underline{C(R)} \longrightarrow \underline{C(S)}$$

is quasi-fully faithful then the morphism f is an isomorphism. Indeed, if f^* is quasi-fully faithful we have that

$$\underline{Hom}^*(R,R) \longrightarrow \underline{Hom}^*(S,S)$$

is a quasi-isomorphism. Evaluating this morphism of complexes at H^0 we find that the induced morphism

$$R \simeq H^0(\underline{Hom}^*(R,R)) \longrightarrow H^0(\underline{Hom}^*(S,S)) \simeq S$$

is an isomorphism. This last morphism being f itself, we see that f is an isomorphism.

(c) Suppose that T is a dg-category such that for all objects x and y we have $H^i(T(x,y)) = 0$ for all $i \neq 0$. We are then going to show that T and $H^0(T)$ are isomorphic in $Ho(dg-cat)$. We first define a dg-category $T_{\leq 0}$ in the following way. The dg-category $T_{\leq 0}$ possesses the same set of objects as T itself. For two objects x and y we let

$$T_{\leq 0}(x,y)^n := T(x,y) \; if \; n<0 \qquad T_{\leq 0}(x,y)^n := 0 \; if \; n>0$$

and

$$T_{\leq 0}(x,y)^0 := Z^0(T(x,y)) = Ker(d: T(x,y)^0 \to T(x,y)^1).$$

The differential on $T_{\leq 0}(x,y)$ is simply induced by the one on $T(x,y)$. It is not hard to see that the composition morphisms of T induces composition morphisms

$$T_{\leq 0}(x,y)^n \times T_{\leq 0}(y,z)^m \longrightarrow T_{\leq 0}(x,z)^{n+m}$$

which makes these data into a dg-category $T_{\leq 0}$ (this is because the composition of two 0-cocycles is itself a 0-cocycle). Moreover, there is a natural dg-functor

$$T_{\leq 0} \longrightarrow T$$

which is the identity on the set of objects and the natural inclusions of complexes

$$T_{\leq 0}(x,y) \subset T(x,y)$$

on the level of morphisms. Now, we consider the natural dg-functor (here, as always, the k-linear category $H^0(T)$ is considered as a dg-category in the obvious way)

$$T_{\leq 0} \longrightarrow H^0(T)$$

which is the identity on the set of objects and the natural projection

$$T_{\leq 0}(x,y) \longrightarrow H^0(T(x,y)) = H^0(T_{\leq 0}(x,y)) = T_{\leq 0}(x,y)^0/Im(T(x,y)^{-1} \to T_{\leq 0}(x,y)^0)$$

on the level of morphisms. We thus have a diagram of dg-categories and dg-functors

$$H^0(T) \longleftarrow T_{\leq 0} \longrightarrow T,$$

which by assumptions on T are all quasi-equivalences. This implies that T and $H^0(T)$ becomes isomorphic as objects in $Ho(dg-cat)$.

(d) Suppose that $f: X \longrightarrow Y$ is a $\mathscr{C}^\infty$ morphism between differentiable manifolds, such that there exists another $\mathscr{C}^\infty$ morphism $g: Y \longrightarrow X$ and two $\mathscr{C}^\infty$ morphisms

$$h: X \times \mathbb{R} \longrightarrow X \qquad k: Y \times \mathbb{R} \longrightarrow Y$$

with

$$h_{X\times\{0\}} = gf, \; h_{X\times\{1\}} = id \qquad k_{Y\times\{0\}} = fg, \; k_{Y\times\{1\}} = id.$$

Then the dg-functor

$$f^*: T_{DR}(Y) \longrightarrow T_{DR}(X)$$

is a quasi-equivalence. Indeed, we know that $H^0(T_{DR}(X))$ is equivalent to the category of linear representations of the fundamental group of X. Therefore, as the morphism f is in particular a homotopy equivalence it induces an isomorphisms on the level of the fundamental groups, and thus the induced functor

$$f^* : H^0(T_{DR}(Y)) \longrightarrow H^0(T_{DR}(X))$$

is an equivalence of categories. The fact that the dg-functor f^* is also quasi-fully faithful follows from the homotopy invariance of de Rham cohomology, and more precisely from the fact that the projection $p : X \times \mathbb{R} \longrightarrow X$ induces a quasi-equivalence of dg-categories

$$p^* : T_{DR}(X) \longrightarrow T_{DR}(X \times \mathbb{R}).$$

We will not give more details in these notes.
As particular case of the above statement we see that the projection $\mathbb{R}^n \longrightarrow *$ induces a quasi-equivalence

$$T_{DR}(*) \longrightarrow T_{DR}(\mathbb{R}^n).$$

As $T_{DR}(*)$ is itself isomorphic to the category of finite dimensional real vector spaces, we see that $T_{DR}(\mathbb{R}^n)$ is quasi-equivalent to the category of finite dimensional vector spaces.

(e) Let now X be a connected complex manifold and $p : X \longrightarrow *$ be the natural projection. Then the induced dg-functor

$$p^* : T_{Dol}(*) \longrightarrow T_{Dol}(X)$$

is quasi-fully faithful if and only if $H^i(X, \mathscr{O}_X) = 0$ for all $i \neq 0$ (here $\mathscr{O}_X$ is the sheaf of holomorphic functions on X). Indeed, all the vector bundles are trivial on $*$. Moreover, for $\mathbf{1}^r$ and $\mathbf{1}^s$ two trivial vector bundles of rank r and s on $*$ we have

$$T_{Dol}(X)(p^*(\mathbf{1}^r), p^*(\mathbf{1}^s)) \simeq T_{Dol}(X)(\mathbf{1}, \mathbf{1})^{rs},$$

where 1 also denotes the trivial holomorphic bundle on X. Therefore, p^* is quasi-fully faithful if and only if $H^i(T_{Dol}(X))(\mathbf{1}, \mathbf{1}) = 0$ for all $i \neq 0$. As we have

$$H^i(T_{Dol}(X)(\mathbf{1}, \mathbf{1})) = H^i_{Dol}(X, \mathbf{1}) = H^i(X, \mathscr{O}_X)$$

this implies the statement. As an example, we see that

$$T_{Dol}(*) \longrightarrow T_{Dol}(\mathbb{P}^n)$$

is quasi-fully faithful (here $\mathbb{P}^n$ denotes the complex projective space), but

$$T_{Dol}(*) \longrightarrow T_{Dol}(E)$$

is not for any complex elliptic curve E.
More generally, if $f : X \longrightarrow Y$ is any proper holomorphic morphism between complex manifolds, then the dg-functor

$$f^* : T_{Dol}(Y) \longrightarrow T_{Dol}(X)$$

is quasi-fully faithful if and only if we have

$$\mathbb{R}^i f_*(\mathscr{O}_X) = 0 \,\forall\, i > 0,$$

where $\mathbb{R}^i f_*(\mathscr{O}_X)$ denotes the higher direct images of the coherent sheaf $\mathscr{O}_X$ of holomorphic functions on X (see e.g. [11]). We will not prove this statement in these notes. As a consequence we see that f^* is quasi-fully faithful if it is a blow-up along a smooth complex sub-manifold of Y, or if it is a bundle in complex projective spaces.

(f) For more quasi-equivalences between dg-categories in the context of non-abelian Hodge theory see [26].

Exercise 2.3.4 *(a) Let T and T' be two dg-categories. Show how to define a dg-category $T\otimes T'$ whose set of objects is the product of the sets of objects of T and T', and for any two pairs (x,y) and (x',y')*

$$(T\otimes T')((x,y),(x',y')) := T(x,y)\otimes T'(x',y').$$

(b) Show that the construction $(T,T')\mapsto T\otimes T'$ defines a symmetric monoidal structure on the category $dg-cat$.

(c) Show that the symmetric monoidal structure $\otimes$ on $dg-cat$ is closed (i.e. that for any two dg-categories T and T' there exists a dg-category $\underline{Hom}(T,T')$ together with functorial isomorphisms

$$Hom(T'',\underline{Hom}(T,T'))\simeq Hom(T''\otimes T,T').$$

Exercise 2.3.5 *Let $k\longrightarrow k'$ be a morphism of commutative rings, and $dg-cat_k$ (resp. $dg-cat_{k'}$) the categories of dg-categories over k (resp. over k').*

(a) Show that there exists a forgetful functor

$$dg-cat_{k'} \longrightarrow dg-cat_k$$

which consists of seing complexes over k' as complexes over k using the morphism $k\longrightarrow k'$.

(b) Show that this forgetful functor admits a left adjoint

$$-\otimes_k k' : dg-cat_k \longrightarrow dg-cat_{k'}.$$

(c) Let $\mathbf{1}_{k'}$ be the dg-category over k with a single object and with k' as k-algebra of endomorphisms of this object. Show that for any dg-category T over k, there exists a natural isomorphism of dg-categories over k

$$T\otimes_k k' \simeq T\otimes \mathbf{1}_{k'},$$

where the tensor product on the right is the one of dg-categories over k as defined in exercise 2.3.4, and the left hand side is considered as an object in $dg-cat_k$ throught the forgetful functor.

(d) *Show that the forgetful functor*

$$dg-cat_{k'} \longrightarrow dg-cat_k$$

also possesses a right adjoint

$$(-)^{k'/k} : dg-cat_k \longrightarrow dg-cat_{k'}$$

(show that for any $T \in dg-cat_k$ the dg-category $\underline{Hom}(\mathbf{1}_{k'},T)$ can be naturally endowed with a structure of dg-category over k').

Exercise 2.3.6 *Let T be a dg-category and $u \in Z^0(T(x,y))$ a morphism in its underlying category. Show that the following four conditions are equivalent.*

(a) *The image of u in $H^0(T(x,y))$ is an isomorphism in $H^0(T)$.*
(b) *There exists $v \in Z^0(T(y,x))$ and two elements $h \in T(x,x)^{-1}$, $k \in T(y,y)^{-1}$ such that*
$$d(h) = vu - e_x \qquad d(k) = uv - e_y.$$
(c) *For any object $z \in T$, the composition with u*
$$u \circ - : T(z,x) \longrightarrow T(z,y)$$
is a quasi-isomorphism of complexes.
(d) *For any object $z \in T$, the composition with u*
$$- \circ u : T(y,z) \longrightarrow T(x,z)$$
is a quasi-isomorphism of complexes.

Exercise 2.3.7 *We denote by B the commutative k-dg-algebra whose underlying graded k-algebra is a (graded commutative) polynomial algebra in two variables $k[X,Y]$, with X in degree 0, Y in degree -1 and $d(Y) = X^2$. We consider B as a dg-category with a unique object.*

(a) *Show that there exists a natural quasi-equivalence*
$$p : B \longrightarrow k[X]/(X^2) =: k[\varepsilon],$$
where $k[\varepsilon]$ is the commutative algebra of dual numbers, considered as a dg-category with a unique object.
(b) *Show that p does not admit a section in $dg-cat$. Deduce from this that unlike the case of categories, there are quasi-equivalences $T \longrightarrow T'$ in $dg-cat$ such that the inverse of f in $Ho(dg-cat)$ can not be represented by a dg-functor $T' \longrightarrow T$ in $dg-cat$ (i.e. quasi-inverses do not exist in general).*

Exercise 2.3.8 *Show that two k-linear categories are equivalent (as k-linear categories) if and only if they are isomorphic in $Ho(dg-cat)$.*

2.4 Localizations as a dg-Category

For a k-algebra R, the derived category $D(R)$ is defined as a localization of the category $C(R)$, and thus has a universal property in $Ho(Cat)$. The purpose of this series of lectures is to show that $C(R)$ can also be localized as a dg-category $\underline{C(R)}$ in order to get an object $L(R)$ satisfying a universal property in $Ho(dg-cat)$. The two objects $L(R)$ and $D(R)$ will be related by the formula

$$H^0(L(R)) \simeq D(R),$$

and we will see that the extra information encoded in $L(R)$ is enough in order to solve all the problems mentioned in Sect. 1.2.

Let T be any dg-category, S be a subset of morphisms in the category $H^0(T)$, and let us define a subfunctor $F_{T,S}$ of the functor $[T,-]$, corepresented by $T \in Ho(dg-cat)$. We define

$$F_{T,S} : Ho(dg-cat) \longrightarrow Ho(Cat)$$

by sending a dg-category T' to the subset of morphisms $[T,T']$ consisting of all morphism f whose induced functor $H^0(f) : H^0(T) \longrightarrow H^0(T')$ sends morphisms of S to isomorphisms in $H^0(T')$. Note that the functor $H^0(f)$ is only determined as a morphism in $Ho(Cat)$, or in other words up to isomorphism. However, the property that $H^0(f)$ sends elements of S to isomorphisms is preserved under isomorphisms of functors, and thus only depends on the class of $H^0(f)$ as a morphism in $Ho(Cat)$.

Definition 2.4.1 *For T and S as above, a* localization of T along *S is a dg-category $L_S T$ corepresenting the functor $F_{T,S}$.*

To state the previous definition in more concrete terms, a localization is the data of a dg-category $L_S T$ and a dg-functor $l : T \longrightarrow L_S T$, such that for any dg-category T' the induced map

$$l^* : [L_S T, T'] \longrightarrow [T,T']$$

is injective and identifies the left hand side with the subset $F_{T,S}(T') \subset [T,T']$.

An important first question is the existence of localization as above. We will see that like localizations of categories they always exist. This, of course, requires to know how to compute the set $[T,T']$ of morphisms in $Ho(dg-cat)$. As the category $Ho(dg-cat)$ is itself defined by localization this is not an easy problem. We will give a solution to this question in the next lectures, based on an approach using model category theory.

3 Lecture 2: Model Categories and dg-Categories

The purpose of this second lecture is to study in more details the category $Ho(dg-cat)$. Localizations of categories are very difficult to describe in general. The purpose of model category theory is precisely to provide a general tool to describe localized categories. By definition, a model category is a category together with three classes of morphisms, fibrations, cofibrations and (weak) equivalences satisfying some axioms mimicking the topological notions of Serre fibrations, cofibrations and weak homotopy equivalences. When M is a model category, with W as equivalences, then the localized category $W^{-1}M$ possesses a very nice description in terms of homotopy classes of morphisms between objects belonging to a certain class of nicer objects called fibrant and cofibrant. A typical example is when $M = Top$ is the category of topological spaces and W is the class of weak equivalences (see example 5 of Sect. 2.1). Then all objects are fibrant, but the cofibrant objects are the retracts of CW-complexes. It is well known that the category $W^{-1}Top$ is equivalent to the category of CW-complexes and homotopy classes of continuous maps between them.

In this lecture, I will start by some brief reminders on model categories. I will then explain how model category structures appear in the context of dg-categories by describing the model category of dg-categories (due to G. Tabuada, [27]) and the model category of dg-modules. We will also see how model categories can be used in order to construct interesting dg-categories. In the next lecture these model categories will be used in order to understand maps in $Ho(dg-cat)$, and to prove the existence of several important constructions such as localization and internal Homs.

3.1 Reminders on Model Categories

In this section we use the conventions of [13] for the notion of model category. We also refer the reader to this book for the proofs of the statements we will mention.

We let M be a category with arbitrary limits and colimits. Recall that a (closed) model category structure on M is the data of three classes of morphisms in M, the fibrations Fib, the cofibrations Cof and the equivalences W, satisfying the following axioms (see [13]).

(a) If $X \xrightarrow{f} Y \xrightarrow{g} Z$ are morphisms in M, then f, g and gf are all in W if and only if two of them are in W.
(b) The fibrations, cofibrations and equivalences are all stable by retracts.
(c) Let

$$\begin{array}{ccc} A & \xrightarrow{f} & X \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle p} \\ B & \xrightarrow[g]{} & Y \end{array}$$

be a commutative square in M with $i \in Cof$ and $p \in Fib$. If either i or p is also in W then there is a morphism $h : B \longrightarrow X$ such that $ph = g$ and $hi = f$.
(d) Any morphism $f : X \longrightarrow Y$ can be factorized in two ways as $f = pi$ and $f = qj$, with $p \in Fib$, $i \in Cof \cap W$, $q \in Fib \cap W$ and $j \in Cof$. Moreover, the existence of these factorizations are required to be functorial in f.

The morphisms in $Cof \cap W$ are usually called *trivial cofibrations* and the morphisms in $Fib \cap W$ *trivial fibrations*. Objects x such that $\emptyset \longrightarrow x$ is a cofibration are called *cofibrant*. Dually, objects y such that $y \longrightarrow *$ is a fibration are called *fibrant*. The factorization axiom (4) implies that for any object x there is a diagram

$$Qx \xrightarrow{\ i\ } x \xrightarrow{\ p\ } Rx,$$

where i is a trivial fibration, p is a trivial cofibration, Qx is a cofibrant object and Rx is a fibrant object. Moreover, the functorial character of the factorization states that the above diagram can be, and will always be, chosen to be functorial in x.

Exercise 3.1.1 *Let M be a model category and $i : A \longrightarrow B$ a morphism. We assume that for every commutative square*

$$\begin{array}{ccc} A & \xrightarrow{\ f\ } & X \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle p} \\ B & \xrightarrow[\ g\ ]{} & Y, \end{array}$$

with p a fibration (resp. a trivial fibration) there is a morphism $h : B \longrightarrow X$ such that $ph = g$ and $hi = f$. Then i is a trivial cofibration (resp. a cofibration). (Hint: factor i using axiom (4) and use the stability of Cof and W by retracts). As a consequence, the class Fib is determined by W and Cof, and similarly the class Cof is determined by W and Fib.

By definition, the homotopy category of a model category M is the localized category

$$Ho(M) := W^{-1}M.$$

A model category structure is a rather simple notion, but in practice it is never easy to check that three given classes Fib, Cof and W satisfy the four axioms above. This can be explained by the fact that the existence of a model category structure on M has a very important consequence on the localized category $W^{-1}M$. For this, we introduce the notion of homotopy between morphisms in M in the following way. Two morphisms $f, g : X \longrightarrow Y$ are called *homotopic* if there is a commutative diagram in M

$$\begin{array}{ccc} X & & \\ {\scriptstyle i}\downarrow & \searrow^{f} & \\ C(X) & \xrightarrow{\ h\ } & Y \\ {\scriptstyle j}\uparrow & \nearrow_{g} & \\ X & & \end{array}$$

satisfying the following two properties:

(a) There exists a morphism $p : C(X) \longrightarrow X$, which belongs to $Fib \cap W$, such that $pi = pj = id$.
(b) The induced morphism

$$i \coprod j : X \coprod X \longrightarrow C(X)$$

is a cofibration.

When X is cofibrant and Y is fibrant in M (i.e. $\emptyset \longrightarrow X$ is a cofibration and $Y \longrightarrow *$ is a fibration), it can be shown that being homotopic as defined above is an equivalence relation on the set of morphisms from X to Y. This equivalence relation is shown to be compatible with composition, which implies the existence of a category $M^{cf}/\sim$, whose objects are cofibrant and fibrant objects and morphisms are homotopy classes of morphisms.

It is easy to see that if two morphisms f and g are homotopic in M then they are equal in $W^{-1}M$. Indeed, in the diagram above defining the notion of being homotopic, the image of p in $Ho(M)$ is an isomorphism. Therefore, so are the images of i and j. Moreover, the inverses of the images of i and j in $Ho(M)$ are equal (because equal to the image of p), which implies that i and j have the same image in $Ho(M)$. This implies that the image of f and of g are also equal. From this, we deduce that the localization functor

$$M \longrightarrow Ho(M)$$

restricted to the sub-category of cofibrant and fibrant objects M^{cf} induces a well defined functor

$$M^{cf}/\sim \longrightarrow Ho(M).$$

The main statement of model category theory is that this last functor is an equivalence of categories.

Our first main example of a model category will be $C(k)$, the category of complexes over some base commutative ring k. The fibrations are taken to be the degree-wise surjective morphisms, and the equivalences are taken to be the quasi-isomorphisms. This determines the class of cofibrations as the morphisms having the correct lifting property. It is an important theorem that this defines a model category structure on $C(k)$ (see [13]). The homotopy category of this model category is by definition $D(k)$ the derived category of k. Therefore, maps in $D(k)$ can be described as homotopy classes of morphisms between fibrant and cofibrant complexes. As the cofibrant objects in $C(k)$ are essentially the complexes of projective modules (see [13] or Exercise 3.1.2 below) and that every object is fibrant, this gives back essentially the usual way of describing maps in derived categories.

Exercise 3.1.2 *(a) Prove that if E is a cofibrant object in $C(k)$ then for any $n \in \mathbb{Z}$ the k-module E^n is projective.*

(b) Prove that if E is a complex which is bounded above (i.e. there is an n_0 such that $E^n = 0$ for all $n \geq n_0$), and such that E^n is projective for all n, then E is cofibrant.

(c) Contemplate the example in [13, Rem. 2.3.7] of a complex of projective modules which is not a cofibrant object in $C(k)$.

Here are few more examples of model categories.

Examples:

(a) The category *Top* of topological spaces is a model category whose equivalences are the weak equivalences (i.e. continuous maps inducing isomorphisms on all homotopy groups) and whose fibrations are the Serre fibrations (see [13, Def. 2.3.4]). All objects are fibrant for this model category, and the typical cofibrant objects are the CW-complexes. Its homotopy category $Ho(Top)$ is also equivalent to the category of CW-complexes and homotopy classes of continuous maps between them.

(b) For any model category M and any (small) category I we consider M^I the category of I-diagrams in M (i.e. of functors from I to M). We define a morphism $f : F \longrightarrow G$ in M^I to be a fibration (resp. an equivalence) if for all $i \in I$ the induced morphism $f_i : F(i) \longrightarrow G(i)$ is a fibration (resp. an equivalence) in M. When M satisfies a technical extra condition, precisely when M is *cofibrantly generated* (see [13, Sect. 2.1]), then these notions define a model category structure on M^I. The construction $M \mapsto M^I$ is very useful as it allows to construct new model categories from old ones.

(c) Let *Cat* be the category of categories. We define a morphism in *Cat* to be an equivalence if it is a categorical equivalence, and a cofibration if it is injective on the set of objects. This defines a model category structure on *Cat* (see [15]).

(d) Let $\mathscr{A}$ be any Grothendieck category and $M = C(\mathscr{A})$ be its category of complexes. Then it can be shown that there exists a model category structure on M whose equivalences are the quasi-isomorphisms and the cofibrations are the monomorphisms (see [14]).

Exercise 3.1.3 *Let M be a model category and $Mor(M)$ be the category of morphisms in M (objects are morphisms and morphisms are commutative squares in M). We define a morphism $(f,g) : u \longrightarrow v$*

$$\begin{array}{ccc} A & \xrightarrow{f} & A' \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle v} \\ B & \xrightarrow[g]{} & B', \end{array}$$

to be an equivalence (resp. a fibration) if both f and g are equivalences (resp. fibrations) in M. Show that this defines a model category structure on $Mor(M)$. Show *moreover that a morphism (f,g) a cofibration if and only if f is cofibration and the induced morphism*

$$B\coprod_A A' \longrightarrow B'$$

are cofibrations in M.

Before going back to dg-catgeories we will need a more structured notion of a model category structure, the notion of a $C(k)$*-model category structure*. Suppose that M is a model category. A $C(k)$-model category structure on M is the data of a functor

$$-\otimes - : C(k)\times M \longrightarrow M$$

satisfying the following two conditions.

(a) The functor $\otimes$ above defines a closed $C(k)$-module structure on M (see [13, Sect. 4]). In other words, we are given functorial isomorphisms in M

$$E\otimes(E'\otimes X)\simeq(E\otimes E')\otimes X \qquad k\otimes X\simeq X$$

for any $E,E'\in C(k)$ and $X\in M$ (satisfying the usual associativity and unit conditions, see [13, Sect. 4]). We are also given for two objects X and Y in M a complex $\underline{Hom}(X,Y)\in C(k)$, together with functorial isomorphisms of complexes

$$Hom(E,\underline{Hom}(X,Y))\simeq Hom(E\otimes X,Y)$$

for $E\in C(k)$, and $X,Y\in M$.

(b) For any cofibration $i:E\longrightarrow E'$ in $C(k)$, and any cofibration $j:A\longrightarrow B$ in M, the induced morphism

$$E\otimes B\coprod_{E\otimes A} E'\otimes A\longrightarrow E'\otimes B$$

is a cofibration in M, which is moreover an equivalence if i or j is so.

Condition (1) above is a purely categorical stucture, and simply asserts the existence of an enrichement of M into $C(k)$ in a rather strong sense. The second condition is a compatibility condition between this enrichement and the model structures on $C(k)$ and M (which is the non trivial part to check in practice).

Examples:

(a) The category $C(k)$ can be considered as enriched over itself by using the tensor product of complexes $-\otimes - : C(k)\times C(k)\longrightarrow C(k)$. For this tensoring it is a $C(k)$-model category (this is another way to state that $C(k)$ is a monoidal model category in the sense of [13, Def. 4.2.6]).

(b) Let X be a topological space. We let $Sh(X,k)$ be the category of sheaves of k-modules and $C(Sh(X,k))$ be the category of complexes in $Sh(X,k)$. As $Sh(X,k)$ is a Grothendieck category, the category $C(Sh(X,k))$ can be endowed with a model category structure for which the equivalences are the quasi-isomorphisms and the cofibrations are the monomorphisms of complexes. The category $Sh(X,k)$ has a natural tensoring over the category of k-modules, and this structure extends to a tensoring of $C(Sh(X,k))$ over the category $C(k)$. Explicitely, if $\mathscr{F}$ is any sheaf of complexes of k-modules over X and $E \in C(k)$, we let $E \otimes \mathscr{F}$ to be the sheaf associated with the presheaf $U \mapsto E \otimes \mathscr{F}(U) \in C(k)$. It can be shown that this tensoring makes $C(Sh(X,k))$ into a $C(k)$-model category.

One main consequence for a model category M to be a $C(k)$-model category is that its homotopy category $Ho(M)$ comes equipped with a natural enrichment over $D(k) = Ho(C(k))$. Explicitely, for two objects x and y in M we set

$$\mathbb{R}\underline{Hom}(x,y) := \underline{Hom}(Qx,Ry),$$

where Qx is a cofibrant replacement of x and Ry is a fibrant replacement of y. The object $\mathbb{R}\underline{Hom}(x,y) \in D(k)$ can be seen to define an enrichment of $Ho(M)$ into $D(k)$ (see [13, Thm. 4.3.4] for details). A direct consequence of this is the important formula

$$H^0(\mathbb{R}\underline{Hom}(x,y)) \simeq Hom_{Ho(M)}(x,y).$$

Therefore, we see that if x and y are cofibrant and fibrant, then set of morphisms between x and y in $Ho(M)$ can be identified with $H^0(\mathbb{R}\underline{Hom}(x,y))$.

Exercise 3.1.4 *Let $f : M \longrightarrow N$ be a functor between two model categories.*

(a) Show that if f preserves cofibrations and trivial cofibrations then it also preserves equivalences between cofibrant objects.
(b) Assume that f preserves cofibrations and trivial cofibrations and that it does admit a right adjoint $g : N \longrightarrow M$. Show that g preserves fibrations and trivial fibrations.
(c) Under the same conditions as in (2), define

$$\mathbb{L}f : Ho(M) \longrightarrow Ho(N)$$

by sending an object x to $f(Qx)$ where Qx is a cofibrant replacement of x. In the same way, define

$$\mathbb{R}g : Ho(M) \longrightarrow Ho(N)$$

by sending an object y to $g(Ry)$ where Ry is a fibrant replacement of y. Show that $\mathbb{L}f$ and $\mathbb{R}g$ are adjoint functors.

3.2 Model Categories and dg-Categories

We start by the model category of dg-categories itself. The equivalences for this model structure are the quasi-equivalences. The fibrations are defined to be the morphisms $f : T \longrightarrow T'$ satisfying the following two properties. The cofibrations are then defined to be the morphisms with the correct lifting property.

(a) For any two objects x and y in T, the induced morphism

$$f_{x,y} : T(x,y) \longrightarrow T'((f(x),f(y))$$

is a fibration in $C(k)$ (i.e. is surjective).

(b) For any isomorphism $u' : x' \to y'$ in $H^0(T')$, and any $y \in H^0(T)$ such that $f(y) = y'$, there is an isomorphism $u : x \to y$ in $H^0(T)$ such that $H^0(f)(u) = u'$.

Theorem 3.2.1 *(see [27]) The above definitions define a model category structure on $dg-cat$.*

This is a key statement in the homotopy theory of dg-categories, and many results in the sequel will depend in an essential way from the existence of this model structure. We will not try to describe its proof in these notes, this would lead us too far.

The theorem 3.2.1 is of course very useful, even though it is not very easy to find cofibrant dg-categories and also to describe the homotopy equivalence relation in general. However, we will see in the next lecture that this theorem implies another statement which provide a very useful way to described maps in $Ho(dg-cat)$. It is this last description that will be used in order to check that localizations in the sense of dg-categories (see definition 2.4.1) always exist.

Exercise 3.2.2 *(a) Let* **1** *be the dg-category with a unique object and k as endomorphism of this object (this is also the unit for the monoidal structure on $dg-cat$). Show that* **1** *is a cofibrant object.*

(b) Let Δ^1_k be the k-linear category with two objects 0 and 1 and with (all k's are here placed in degree 0)

$$\Delta^1_k(0,0) = k \qquad \Delta^1_k(0,1) = k \qquad \Delta^1_k(1,1) = k \qquad \Delta^1_k(1,0) = 0$$

and obvious compositions (Δ^1_k is the k-linearization of the category with two objects and a unique non trivial morphism between them). Show that Δ^1_k is a cofibrant object.

(c) Use exercice 2.3.7 in order to show that $k[\varepsilon]$ is not a cofibrant dg-category (when considered as a dg-category with a unique object).

(d) Let T be the dg-category with four objects x, x′, y and y′ and with the following non trivial complex of morphisms (here we denote by $k < x >$ the rank 1 free k-module with basis x)

$$T(x,x')^0 = k<f> \quad T(x,y)^0 = k<u> \quad T(x',y')^0 = k<u'> \quad T(y,y')^0 = k<g>$$

$$T(x,y')^0 = k<u'f> \oplus k<gu> \quad T(x,y')^{-1} = k<h> \quad T(x,y')^i = 0 \; for \; i \neq 0,-1$$

such that $d(h) = u'f - gu$. In other words, T is freely generated by four morphisms of degree 0, u, u′, f and g, one morphism of degre −1, h, and has a unique relation $d(h) = u'f - gu$. Show that there exists a trivial fibration

$$T \longrightarrow \Delta^1_k \otimes \Delta^1_k.$$

Show moreover that this trivial fibration possesses no section, and conclude that $\Delta^1_k \otimes \Delta^1_k$ is not a cofibrant dg-category.

Let now T be a dg-category. A T-dg-module is the data of a dg-functor $F : T \longrightarrow \underline{C(k)}$. In other words a T-dg-module F consists of the data of complexes $F_x \in C(k)$ for each object x of T, together with morphisms

$$F_x \otimes T(x,y) \longrightarrow F_y$$

for each objects x and y, satisfying the usual associativity and unit conditions. A morphism of T-dg-module consists of a natural transformation between dg-functors (i.e. families of morphisms $F_x \longrightarrow F'_x$ commuting with the maps $F_x \otimes T(x,y) \longrightarrow F_y$ and $F'_x \otimes T(x,y) \longrightarrow F'_y$).

We let $T-Mod$ be the category of T-dg-modules. We define a model category structure on $T-Mod$ by defining equivalences (resp. fibrations) to be the morphisms $f : F \longrightarrow F'$ such that for all $x \in T$ the induced morphism $f_x : F_x \longrightarrow F'_x$ is an equivalence (resp. a fibration) in $C(k)$. It is known that this defines a model category structure (see [28]). This model category is in a natural way a $C(k)$-model category, for which the $C(k)$-enrichment is defined by the formula $(E \otimes F)_x := E \otimes F_x$.

Definition 3.2.3 *The* derived category *of a dg-category T is*

$$D(T) := Ho(T-Mod).$$

The previous definition generalizes the derived categories of rings. Indeed, if R is a k-algebra it can also be considered as a dg-category, sometimes denoted by BR, with a unique object and R as endomorphism of this object (considered as a complex of k-modules concentrated in degree 0). Then $D(BR) \simeq D(R)$. Indeed, a BR-dg-module is simply a complex of R-modules.

Exercise 3.2.4 *Let T be a dg-category.*

(a) Let $x \in T$ be an object in T and $\underline{h}_x : T^{op} \longrightarrow \underline{C(k)}$ the T-dg-module represented by x (the one sending y to $T(y,x)$). Prove that $\underline{h}_x$ is cofibrant and fibrant as an object in $T^{op}-Mod$.

(b) Prove that $x \mapsto \underline{h}_x$ defines a functor

$$H^0(T) \longrightarrow D(T^{op}).$$

(c) Show that for any $F \in D(T^{op})$ there is a functorial bijection

$$Hom_{D(T^{op})}(\underline{h}_x, F) \simeq H^0(F_x).$$

(d) Show that the above functor $H^0(T) \longrightarrow D(T^{op})$ is fully faithful.

Any morphism of dg-categories $f : T \longrightarrow T'$ induces an adjunction on the corresponding model categories of dg-modules

$$f_! : T-Mod \longrightarrow T'-Mod \qquad T-Mod \longleftarrow T'-Mod : f^*,$$

for which the functor f^* is defined by composition with f, and $f_!$ is its left adjoint. This adjunction is a *Quillen adjunction*, i.e. f^* preserves fibrations and trivial fibrations, and therefore can be derived into an adjunction on the level of homotopy categories (see exercice 3.1.4 and [13, Lem. 1.3.10])

$$\mathbb{L}f_! : D(T) \longrightarrow D(T') \qquad D(T) \longleftarrow D(T') : f^* = \mathbb{R}f^*.$$

It can be proved that when f is a quasi-equivalence then f^* and $\mathbb{L}f_!$ are equivalences of categories inverse to each others (see [28, Prop. 3.2]).

Exercise 3.2.5 *Let $f : T \longrightarrow T'$ be a dg-functor. Prove that for any $x \in T$ we have*

$$\mathbb{L}f_!(\underline{h}^x) \simeq \underline{h}^{f(x)}$$

in $D(T')$ (recall that $\underline{h}^x$ is the T-dg-module corepresented by x, sending y to $T(x,y)$).

For a $C(k)$-model category M we can also define a notion of *T-dg-modules with coefficients in M* as being dg-functors $T \longrightarrow M$ (where M is considered as a dg-category using its $C(k)$-enrichment). This category is denoted by M^T (so that $T - Mod = C(k)^T$). When M satisfies some mild assumptions (e.g. being cofibrantly generated, see [13, Sect. 2.1]) we can endow M^T with a model category structure similar to $T - Mod$, for which equivalences and fibrations are defined levelwise in M. The existence of model categories as M^T will be used in the sequel to describe morphisms in $Ho - (dg - cat)$.

Exercise 3.2.6 *Let T and T' be two dg-categories. Prove that there is an equivalence of categories*

$$M^{(T\otimes T')} \simeq (M^T)^{T'}.$$

Show moreover that this equivalence of categories is compatible with the two model category structures on both sides.

We finish this second lecture by describing a way to construct many examples of dg-categories using model categories. For this, let M be a $C(k)$-enriched model category. Using the $C(k)$-enrichment M can also be considered as a dg-category whose set of objects is the same as the set of objects of M and whose complexes of morphisms are $\underline{Hom}(x,y)$. This dg-category will sometimes be denoted by $\underline{M}$, but it turns out not to be the right dg-category associated to the $C(k)$-model category M (at least it is not the one we will be interested in the sequel). Instead, we let $Int(M)$ be the full sub-dg-category of $\underline{M}$ consisting of fibrant and cofibrant objects in M. From the general theory of model categories it can be easily seen that the category $H^0(Int(M))$ is naturally isomorphic to the category of fibrant and cofibrant objects in M and homotopy classes of morphisms between them. In particular there exists a natural equivalence of categories

$$H^0(Int(M)) \simeq Ho(M).$$

The dg-category $Int(M)$ is therefore a dg-enhancement of the homotopy category $Ho(M)$. Of course, not every dg-category is of form $Int(M)$. However, we will see that any dg-catgeory can be, up to a quasi-equivalence, fully embedded into some dg-category of the form $Int(M)$. This explains the importance of $C(k)$-model categories in the study of dg-categories.

Remark 2. The construction $M \mapsto Int(M)$ is an ad-hoc construction, and does not seem very intrinsic (e.g. as it is defined it depends on the choice of fibrations and cofibrations in M, and not only on equivalences). However, we will see in the next lecture that $Int(M)$ can also be characterized by as the localization of the dg-category $\underline{M}$ along the equivalences in M, showing that it only depends on the dg-category $\underline{M}$ and the subset W (and not of the classes Fib and Cof)

Let T be a dg-category. We can consider the $C(k)$-enriched Yoneda embedding

$$\underline{h}_{-} : T \longrightarrow T^{op} - Mod,$$

which is a dg-functor when $T^{op} - Mod$ is considered as a dg-category using its natural $C(k)$-enrichment. It turns out that for any $x \in T$, the T^{op}-dg-module $\underline{h}_x$ is cofibrant (see exercice 3.2.4) and fibrant (avery T^{op}-dg-module if fibrant by definition). We therefore get a natural dg-functor

$$\underline{h} : T \longrightarrow Int(T^{op} - Mod).$$

It is easy to check that $\underline{h}$ is quasi-fully faithful (it even induces isomorphisms on complexes of morphisms).

Definition 3.2.7 *For a dg-category T the morphism*

$$\underline{h} : T \longrightarrow Int(T^{op} - Mod)$$

is called the Yoneda embedding of the dg-category T.

4 Lecture 3: Structure of the Homotopy Category of dg-Categories

In this lecture we will truly start to go into the heart of the subject and describe the category $Ho(dg-cat)$. I will start by a theorem describing the set of maps between two objects in $Ho(dg-cat)$. This fundamental result has two important consequences: the existence of localizations of dg-categories, and the existence of dg-categories of morphisms between two dg-categories, both characterized by universal properties in $Ho(dg-cat)$. At the end of this lecture, I will introduce the notion of *Morita equivalences and triangulated dg-categories*, and present a refine version of the category $Ho(dg-cat)$, better suited for many purposes.

4.1 Maps in the Homotopy Category of dg-Categories

We start by computing the set of maps in $Ho(dg-cat)$ from a dg-category T to a dg-category of the form $Int(M)$. As any dg-category can be fully embedded into some $Int(M)$ this will be enough to compute maps in $Ho(dg-cat)$ between any two objects.

Let M be a $C(k)$-model category. We assume that M satisfies the following three technical conditions (they will always be satisfied for the applications we have in mind).

(a) M is cofibrantly generated, and the domain and codomain of the generating cofibrations are cofibrant objects in M.
(b) For any cofibrant object X in M, and any quasi-isomorphism $E \longrightarrow E'$ in $C(k)$, the induced morphism $E \otimes X \longrightarrow E' \otimes X$ is an equivalence.
(c) Infinite sums preserve weak equivalences in M.

Exercise 4.1.1 *Let R be a k-algebra considered as a dg-category. Show that the $C(k)$-model category $R-Mod = C(R)$ does not satisfy condition (2) above if R is not flat over k.*

Condition (1) this is a very mild condition, as almost all model categories encountered in real life are cofibrantly generated. Condition (2) is more serious, as it states that cofibrant objects of M are flat in some sense, which is not always the case. For example, to be sure that the model category $T-Mod$ satisfies (2) we need to impose the condition that all the complexes $T(x,y)$ are flat (e.g. cofibrant in $C(k)$). Conditions (3) is also rather mild and is often satisfied for model categories of algebraic nature. The following proposition is the main result concerning the description of the set of maps in $Ho(dg-cat)$, and almost all the further results are consequences of it. Note that it is wrong if condition (2) above is not satisfied.

Proposition 4.1.2 *Let T be any dg-category and M be a $C(k)$-model category satisfying conditions (1), (2) and (3) above. Then, there exists a natural bijection*

$$[T, Int(M)] \simeq Iso(Ho(M^T))$$

between the set of morphisms from T to $Int(M)$ in $Ho(dg-cat)$ and the set of isomorphism classes of objects in $Ho(M^T)$.

Ideas of proof (see [28] for details): Let $Q(T) \longrightarrow T$ be a cofibrant model for T. The pull-back functor on dg-modules with coefficients in M induces a functor

$$Ho(M^T) \longrightarrow Ho(M^{Q(T)}).$$

Condition (2) on M insures that this is an equivalence of categories, as shown by the following lemma.

Lemma 4.1.3 *Let $f : T' \longrightarrow T$ be a quasi-equivalence between dg-categories and M be a $C(k)$-model category satisfying conditions* (1)*,* (2) *and* (3) *as above. Then the Quillen adjunction*

$$f_! : Ho(M^{T'}) \longrightarrow Ho(M^T) \qquad Ho(M^{T'}) \longleftarrow Ho(M^T) : f^*$$

is a Quillen equivalence.

Idea of a proof of the lemma: We need to show that the two natural transformations

$$\mathbb{L}f_! f^* \Rightarrow id \qquad id \Rightarrow f^*\mathbb{L}f_!$$

are isomorphism. For this, we first check that this is the case when evaluated at a certain kind of objects. Let $x \in T$ and $X \in M$ be a cofibrant object. We consider the object $\underline{h}^x \otimes X \in Ho(M^T)$, sending $y \in T$ to $T(x,y) \otimes X \in M$. Let $x' \in T'$ be an object such that $f(x')$ and x are isomorphic in $H^0(T')$. Because of our condition (2) on M it is not hard to show that $\underline{h}^x \otimes X$ and $\underline{h}^{f(x')} \otimes X$ are isomorphic in $Ho(M^T)$. Therefore, we have

$$f^*(\underline{h}^x \otimes X) \simeq f^*(\underline{h}^{f(x')} \otimes X).$$

Moreover, $f^*(\underline{h}^{f(x')} \otimes X) \in Ho(M^{T'})$ sends an object $y' \in T'$ to $T(f(x'), f(y')) \otimes X$. Because f is quasi-fully faithful (and because of our assumption (2) on M) we see that $f^*(\underline{h}^{f(x')} \otimes X)$ is isomorphic in $Ho(M^{T'})$ to $\underline{h}^{x'} \otimes X$ which sends y' to $T'(x',y') \otimes X$. Finally, it is not hard to see that $\underline{h}^{x'} \otimes X$ is a cofibrant object and that

$$\mathbb{L}f_!(\underline{h}^{x'} \otimes X) \simeq f_!(\underline{h}^{x'} \otimes X) \simeq \underline{h}^{f(x')} \otimes X.$$

Thus, we have

$$\mathbb{L}f_! f^*(\underline{h}^x \otimes X) \simeq \mathbb{L}f_!(\underline{h}^{x'} \otimes X) \simeq \underline{h}^{f(x')} \otimes X \simeq \underline{h}^x \otimes X,$$

or in other words the adjunction morphism

$$\mathbb{L}f_! f^*(\underline{h}^x \otimes X) \longrightarrow \underline{h}^x \otimes X$$

is an isomorphism. In the same way, we can see that for any $x' \in T'$ the adjunction morphism

$$\underline{h}^{x'} \otimes X \longrightarrow f^*\mathbb{L}f_!(\underline{h}^{x'} \otimes X)$$

is an isomorphism.

To conclude the proof of the lemma we use that the objects $\underline{h}^x \otimes X$ generate the category $Ho(M^T)$ be homotopy colimits and that f^* and $\mathbb{L}f_!$ both commute with homotopy colimits. To see that $\underline{h}^x \otimes X$ generates $Ho(M^T)$ by homotopy colimits we use the condition (1), and the small object argument, which shows that any object is equivalent to a transfinite composition of push-outs along morphisms $\underline{h}^x \otimes X \longrightarrow \underline{h}^x \otimes Y$ for $X \to Y$ a cofibration between cofibrant objects in M. The fact that $\mathbb{L}f_!$ preserves homotopy colimits is formal and follows from the general fact that the left derived functor of a left Quillen functor always preserves homotopy colimits. Finally, the fact that f^* preserves homotopy colimits uses the condition (3) (which up to now has not been used). Indeed, we need to show that f^* preserves infinite homotopy sums and homotopy push-outs. As infinite sums are also infinite homotopy sums in M^T (because of conditions (3)), the fact that f^* preserves infinite homotopy sums follows from the fact that the functor f^* commutes with infinite sums. To show that f^* commutes with homotopy push-outs we use that M^T is a $C(k)$-model category, and thus a stable model category in the sense of [13, Sect. 7]. This implies that homotopy push-outs squares are exactly the homotopy pull-backs squares. As f^* is right Quillen it preserves homotopy pull-backs squares, and thus homotopy push-outs.

Therefore, we deduce from what we have seen that the adjunction morphism

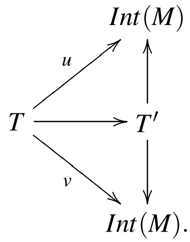
$$\mathbb{L}f_! f^*(E) \longrightarrow E$$

is an isomorphism for any $E \in Ho(M^T)$. In the same we way we see that for any $E' \in Ho(M^{T'})$ the adjunction morphism

$$E' \longrightarrow f^* \mathbb{L}f_!(E')$$

is an isomorphism. This finishes the proof of the lemma. □

The above lemma imply that we can assume that T is a cofibrant dg-catgeory. As all objects in $dg-cat$ are fibrant $[T, Int(M)]$ is then the quotient of the set of morphisms in $dg-cat$ by the homotopy relations. In particular, the natural map $[T, Int(M)] \longrightarrow Iso(Ho(M^T))$ is surjective (this uses that a cofibrant and fibrant object in M^T factors as $T \to Int(M) \to M$, i.e. is levelwise fibrant and cofibrant). To prove injectivity, we start with two morphisms $u, v : T \longrightarrow Int(M)$ in $dg-cat$, and we assume that the corresponding objects F_u and F_v in M^T are equivalent. Using that any equivalences can be factorized as a composition of trivial cofibrations and trivial fibrations, we easily reduce the problem to the case where there exists a trivial fibration $F_u \longrightarrow F_v$ (the case of cofibration is somehow dual). This morphism can be considered as a dg-functor $T \longrightarrow Int(Mor(M))$, where $Mor(M)$ is the model category of morphisms in M (note that fibrant objects in $Mor(M)$ are fibrations between fibrant objects in M). Moreover, this dg-functor factors throught $T' \subset Int(Mor(M))$, the full sub-dg-category corresponding to equivalences in M. We therefore have a commutative diagram in $dg-cat$



The two morphisms $T' \longrightarrow Int(M)$ are easily seen to be quasi-equivalences, and to possess a common section $Int(M) \longrightarrow T'$ sending an object of M to the its identity morphism. Projecting this diagram in $Ho(dg-cat)$, we see that $[u] = [v]$ in $Ho(dg-cat)$. □

We will now deduce from proposition 4.1.2 a description of the set of maps $[T,T']$ between two objects in $Ho(dg-cat)$. For this we use the $C(k)$-enriched Yoneda embedding

$$\underline{h} : T' \longrightarrow Int((T')^{op} - Mod),$$

sending an object $x \in T'$ to the $(T')^{op}$-dg-module defined by

$$\begin{array}{rccc} \underline{h}_x : & (T')^{op} & \longrightarrow & \underline{C(k)} \\ & y & \mapsto & T'(y,x). \end{array}$$

The dg-module $\underline{h}$ is easily seen to be cofibrant and fibrant in $(T')^{op} - Mod$, and thus we have $\underline{h}_x \in Int((T')^{op} - Mod)$ as required. The enriched version of the Yoneda lemma implies that $\underline{h}$ is a quasi-fully faithful dg-functor. More precisely, we can show that the induced morphism of complexes

$$T'(x,y) \longrightarrow \underline{Hom}(\underline{h}_x,\underline{h}_y) = Int((T')^{op} - Mod)((\underline{h}_x,\underline{h}_y)$$

is an isomorphims of complexes.

Using the description of maps in $Ho(dg-cat)$ as being homotopy classes of morphisms between cofibrant objects, we see that the morphism $\underline{h}$ induces a injective map

$$[T,T'] \hookrightarrow [T,Int((T')^{op} - Mod)]$$

whose image consists of morphisms $T \longrightarrow Int((T')^{op} - Mod)$ factorizing in $Ho(dg-cat)$ throught the quasi-essential image of $\underline{h}$. We easily get this way the following corollary (see Sect. 3.2 and exercice 2.3.4 for the definition of the tensor product of two dg-categories).

Corollary 1. *Let T and T' be two dg-categories, one of them having cofibrant complexes of morphisms. Then, there exists a natural bijection between $[T,T']$ and the subset of $Iso(Ho(T \otimes (T')^{op} - Mod))$ consisting of $T \otimes (T')^{op}$-dg-modules F such that for any $x \in T$, there is $y \in T'$ such that $F_{x,-}$ and $\underline{h}_y$ are isomorphic in $Ho((T')^{op} - Mod)$.*

Exercise 4.1.4 *Let T be a dg-category.*

(a) Show that $[\mathbf{1},T]$ is in bijection with the set of isomorphism classes of objects in the category $H^0(T)$ (recall that $\mathbf{1}$ is the unit dg-category, with a unique object and k as algebra of endormorphisms).
(b) Show that $[\Delta^1_k,T]$ is in bijection with the set of isomorphism classes of morphisms in the category $H^0(T)$ (recall that Δ^1_k is the dg-category with two object and freely generated by a unique non trivial morphism).

Exercise 4.1.5 *Let C and D be two k-linear categories, also considered as dg-categories over k. Show that there exists a natural bijection between $[C,D]$ and the set of isomorphism classes of k-linear functors from C to D. Deduce from this that there exists a fully faithful functor*

$$Ho(k-cat) \longrightarrow Ho(dg-cat_k),$$

from the homotopy category of k-linear categories (k-linear categories and isomorphism classes of k-linear functors) and the homotopy category of dg-categories.

Exercise 4.1.6 *Let R be an associative and unital k-algebra, which is also considered as dg-category with a unique object and R as endomorphisms of this object. Show that there is a natural bijection between $[R, Int(C(k))]$ and the set of isomorphism classes of the derived category $D(R)$.*

4.2 Existence of Internal Homs

For two dg-catgeories T and T' we can construct their tensor product $T \otimes T'$ in the following way. The set of objects of $T \otimes T'$ is the product $Ob(T) \times Ob(T')$. For $(x,y) \in Ob(T)^2$ and $(x',y') \in Ob(T')^2$ we set

$$(T \otimes T')((x,x'),(y,y')) := T(x,y) \otimes T(x',y'),$$

with the obvious compositions and units. When k is not a field the functor $\otimes$ does not preserves quasi-equivalences. However, it can be derived by the following formula

$$T \otimes^{\mathbb{L}} T' := Q(T) \otimes Q(T'),$$

where Q is a cofibrant replacement functor on $dg-cat$. This defines a symmetric monoidal structure

$$-\otimes^{\mathbb{L}} - : Ho(dg-cat) \times Ho(dg-cat) \longrightarrow Ho(dg-cat).$$

Proposition 4.2.1 *The monoidal structure $-\otimes^{\mathbb{L}}-$ is closed. In other words, for two dg-categories T and T' there is third dg-category $\mathbb{R}\underline{Hom}(T,T') \in Ho(dg-cat)$, such that for any third dg-category U there exists a bijection*

$$[U, \mathbb{R}\underline{Hom}(T,T')] \simeq [U \otimes^{\mathbb{L}} T, T'],$$

functorial in $U \in Ho(dg-cat)$.

Idea of proof: As for the corollary 1 we can reduce the problem of showing that $\mathbb{R}\underline{Hom}(T, Int(M))$ exists for a $C(k)$-model category M satisfying the same conditions as in proposition 4.1.2. Under the same hypothesis than corollary 1 it can be checked (using proposition 4.1.2) that $\mathbb{R}\underline{Hom}(T, Int(M))$ exists and is given by $Int(M^T)$. □

For two dg-categories T and T', one of them having cofibrant complexes of morphisms it is possible to show that $\mathbb{R}\underline{Hom}(T,T')$ is given by the full sub-dg-category of $Int(T \otimes (T')^{op} - Mod)$ consisting of dg-modules satifying the condition of corollary 1.

Finally, note that when $M = C(k)$ we have

$$\mathbb{R}\underline{Hom}(T, Int(C(k))) \simeq Int(T - Mod).$$

In particular, we find a natural equivalence of categories

$$D(T) \simeq H^0(\mathbb{R}\underline{Hom}(T, Int(C(k)))),$$

which is an important formula.

In the sequel we will use the following notations for a dg-category T

$$L(T) := Int(T - Mod) \simeq \mathbb{R}\underline{Hom}(T, Int(C(k)))$$

$$\widehat{T} := Int(T^{op} - Mod) \simeq \mathbb{R}\underline{Hom}(T^{op}, Int(C(k))).$$

Note that we have natural equivalences

$$H^0(L(T)) \simeq D(T) \qquad H^0(\widehat{T}) \simeq D(T^{op}).$$

Therefore, $L(T)$ and $\widehat{T}$ are dg-enhancement of the derived categories $D(T)$ and $D(T^{op})$. Note also that the Yoneda embedding of definition 3.2.7 is now a dg-functor

$$\underline{h} : T \longrightarrow \widehat{T}.$$

Exercise 4.2.2 *(a) Let R be an associative and unital k-algebra which is considered as a dg-category with a unique object. Show that there is an isomorphism in* $Ho(dg - cat)$

$$\mathbb{R}\underline{Hom}(R, Int(C(k))) \simeq L(R).$$

(b) Show that for any two k-algebras R and R'*, one of them being flat over k we have*

$$\mathbb{R}\underline{Hom}(R, L(R')) \simeq L(R \otimes R').$$

Exercise 4.2.3 *Let T be a dg-category. We define the Hochschild cohomology of T by*

$$HH^*(T) := H^*(\mathbb{R}\underline{Hom}(T, T)(id, id)).$$

Let R be an associative k-algebra, flat over k, and considered as a dg-category with a unique object. Show that we have

$$HH^*(R) := Ext^*_{R \otimes R^{op}}(R, R),$$

where the right hand side are the ext-groups computed in the derived category of $R \otimes R^{op}$*-modules.*

4.3 Existence of Localizations

Let T be a dg-category and let S be subset of morphisms in $H^0(T)$ we would like to invert in $Ho(dg - cat)$. For this, we will say that a morphism $l : T \longrightarrow L_S T$ in $Ho(dg - cat)$ is a *localization of T along S* if for any $T' \in Ho(dg - cat)$ the induced morphism

$$l^* : [L_S T, T'] \longrightarrow [T, T']$$

is injective and its image consists of all morphisms $T \longrightarrow T'$ in $Ho(dg-cat)$ whose induced functor $H^0(T) \longrightarrow H^0(T')$ sends all morphisms in S to isomorphisms in $H^0(T)$. Note that the functor $H^0(T) \longrightarrow H^0(T')$ is only well defined in $Ho(Cat)$ (i.e. up to isomorphism), but this is enough for the definition to makes sense as the condition of sending S to isomorphisms is stable by isomorphism between functors.

Proposition 4.3.1 *For any dg-category T and any set of maps S in $H^0(T)$, a localization $T \longrightarrow L_S T$ exists in $Ho(dg-cat)$.*

Idea of proof (see [28] for details): We start by the most simple example of a localization. We first suppose that $T := \Delta^1_k$ is the dg-category freely generated by two objects, 0 and 1, and a unique morphism $u : 0 \to 1$. More concretely, $T(0,1) = T(0,0) = T(1,1) = k$ and $T(1,0) = 0$, together with the obvious compositions and units. We let $\mathbf{1}$ be the dg-category with a unique object $*$ and $\mathbf{1}(*,*) = k$ (with the obvious composition). We consider the dg-foncteur $T \longrightarrow \mathbf{1}$ sending the non trivial morphism of T to the identity of $*$ (i.e. $k = T(0,1) \to \mathbf{1}(*,*) = k$ is the identity). We claim that this morphism $T \longrightarrow \mathbf{1}$ is a localization of T along S consisting of the morphism $u : 0 \to 1$ of $T = H^0(T)$. This in fact follows easily from our Proposition 4.1.2. Indeed, for a $C(k)$-model category M the model category M^T is the model category of morphisms in M. It is then easy to check that the functor $Ho(M) \longrightarrow Ho(M^T)$ sending an object of M to the identity morphism in M is fully faithful and that its essential image consists of all equivalences in M.

In the general case, let S be a subset of morphisms in $H^0(T)$ for some dg-category T. We can represent the morphisms S by a dg-functor

$$\coprod_S \Delta^1_k \longrightarrow T,$$

sending the non trivial morphism of the component s to a representative of s in T. We define $L_S T$ as being the homotopy push-out (see [13] for this notion)

$$L_S T := (\coprod_S \mathbf{1}) \coprod^{\mathbb{L}}_{\coprod_S \Delta^1_k} T.$$

The fact that each morphism $\Delta^1_k \longrightarrow \mathbf{1}$ is a localization and the universal properties of homotopy push-outs imply that the induced morphism $T \longrightarrow L_S T$ defined as above is a localization of T along S. □

Exercise 4.3.2 *Let Δ^1_k be the dg-category with two objects and freely generated by a non trivial morphism u between these two objects. We let $S := \{u\}$ be the image of u in $H^0(\Delta^1_k)$. Show that $L_S \Delta^1_k \simeq \mathbf{1}$.*

Exercise 4.3.3 *Let T and T' be two dg-categories and S and S' be two sets of morphisms in $H^0(T)$ and $H^0(T')$ that contain all the identities. Prove that there is a natural isomorphism in $Ho(dg-cat)$*

$$L_S T \otimes^{\mathbb{L}} L_{S'} T' \simeq L_{S \otimes^{\mathbb{L}} S'} T \otimes^{\mathbb{L}} T'.$$

The following proposition describes $Int(M)$ as a dg-localization of M.

Proposition 4.3.4 *Let M be a cofibrantly generated $C(k)$-model category, considered also as dg-category $\underline{M}$. There exists a natural isomorphism in $Ho(dg-cat)$*

$$Int(M) \simeq L_W \underline{M}.$$

Idea of proof: We consider the natural inclusion dg-functor $i : Int(M) \longrightarrow M$. This inclusion factors as

$$Int(M) \xrightarrow{j} \underline{M}^f \xrightarrow{k} \underline{M},$$

where $\underline{M}^f$ is the full sub-dg-category of $\underline{M}$ consisting of fibrant objects. Using that M is cofibrantly generated we can construct dg-functors

$$r : \underline{M} \longrightarrow \underline{M}^f \qquad q : \underline{M}^f \longrightarrow Int(M)$$

together with morphisms

$$jq \to id \qquad qj \to id \qquad id \to ri \qquad id \to ir.$$

Moreover, these morphisms between dg-functors are levelwise in W. This can be seen to imply that the induced morphisms on localizations

$$L_W Int(M) \longrightarrow L_W \underline{M}^f \longrightarrow L_W \underline{M}$$

are isomorphisms in $Ho(dg-cat)$. Finally, as morphisms in W are already invertible in $H^0(Int(M)) \simeq Ho(M)$, we have $L_W Int(M) \simeq Int(M)$. □

Finally, one possible way to understand localizations of dg-categories is by the following proposition.

Proposition 4.3.5 *Let T be a dg-category and S be a subset of morphisms in $H^0(T)$. Then, the localization morphism $l : T \longrightarrow L_S T$ induces a fully faithful functor*

$$l^* : D(L_S T) \longrightarrow D(T)$$

whose image consists of all T-dg-modules $F : T \longrightarrow \underline{C(k)}$ sending all morphisms of S to quasi-isomorphisms in $C(k)$.

Idea of proof: This follows from the existence of internal Homs and localizations, as well as the formula

$$D(T) \simeq H^0(\mathbb{R}\underline{Hom}(T, Int(C(k)))) \qquad D(L_S T) \simeq H^0(\mathbb{R}\underline{Hom}(L_S T, Int(C(k)))).$$

Indeed, the universal properties of localizations and internal Homs implies that $\mathbb{R}\underline{Hom}(L_S T, Int(C(k)))$ can be identified full the full sub-dg-category of $\mathbb{R}\underline{Hom}(T, Int(C(k)))$ consisting of dg-functors sending S to quasi-isomorphisms in $C(k)$. □

Exercise 4.3.6 *Let $l : T \longrightarrow L_S T$ be a localization of a dg-category with respect to set of morphisms S in $H^0(T)$, and let*

$$\mathbb{L}l_! : D(T^{op}) \longrightarrow D(L_S T^{op})$$

be the induced functor in the corresponding derived categories of modules. Let W_S be the subset of morphisms u in $D(T^{op})$ such that $\mathbb{L}l_!(u)$ is an isomorphism in $D(L_S T^{op})$.

(a) Show that a morphism $u : E \longrightarrow F$ of $D(T^{op})$ is in W_S if and only if for any $G \in D(T^{op})$ such that $G_x \longrightarrow G_y$ is a quasi-isomorphism for all $x \to y$ in S, the induced map

$$u^* : Hom_{D(T^{op})}(F,G) \longrightarrow Hom_{D(T^{op})}(E,G)$$

is bijective.

(b) Show that the induced functor

$$W_S^{-1} D(T^{op}) \longrightarrow D(L_S T^{op})$$

is an equivalence of categories.

4.4 Triangulated dg-Categories

In this section we will introduce a class of dg-categories called *triangulated*. The notion of being triangulated is the dg-analog of the notion of being Karoubian for linear categories. We will see that any dg-category has a triangulated hull, and this will allow us to introduce a notion of Morita equivalences which is a dg-analog of the usual notion of Morita equivalences between linear categories. The homotopy category of dg-categories up to Morita equivalences will then be introduced and shown to have better properties than the category $Ho(dg-cat)$. We will see in the next lecture that many invariants of dg-categories (K-theory, Hochschild homology ...) factor throught Morita equivalences.

Let T be a dg-category. We recall the existence of the Yoneda embedding (see definition 3.2.7)

$$T \longrightarrow \widehat{T} = Int(T^{op} - Mod),$$

which is quasi-fully faithful. Passing to homotopy categories we get a fully faithful morphism

$$\underline{h} : H^0(T) \longrightarrow D(T^{op}).$$

An object in the essential image of this functor will be called *quasi-representable*.

Recall that an object $x \in D(T^{op})$ is *compact* if the functor

$$[x,-] : D(T^{op}) \longrightarrow k - Mod$$

commutes with arbitrary direct sums. It is easy to see that any quasi-representable object is compact (see exercice 3.2.4). The converse is not true and we set the following definition.

Definition 4.4.1 *A dg-category T is* triangulated *if and only if every compact object in $D(T^{op})$ is quasi-representable.*

Remark 3. When T is triangulated we have an equivalence of categories $H^0(T) \simeq D(T^{op})_c$, where $D(T^{op})_c$ is the full sub-category of $D(T)$ of compact objects. The category $D(T)$ has a natural triangulated structure which restricts to a triangulated structure on compact objects (see [22] for more details on the notion of triangulated categories). Therefore, when T is triangulated dg-category its homotopy category $H^0(T)$ comes equiped with a natural triangulated structure. This explains the terminology of triangulated dg-category. For more about the relations between the notions of triangulated dg-categories and the notions of triangulated categories we refer to [4]. *However, it is not necessary to know the theory of triangulated categories in order to understand triangulated dg-categories, and thus we will not study in details the precise relations between triangulated dg-categories and triangulated categories.*

We let $Ho(dg-cat^{tr}) \subset Ho(dg-cat)$ be the full sub-category of triangulated dg-categories. Note that the notation $Ho(dg-cat^{tr})$ suggests that this category is the homotopy category of some model category. We will see that it is, equivalent to, the localization of the category $dg-cat$ along the class of *Morita equivalences*, that will be introduced later on in this section.

Proposition 4.4.2 *The natural inclusion*

$$Ho(dg-cat^{tr}) \longrightarrow Ho(dg-cat)$$

has a left adjoint. In other words, any dg-category has a triangulated hull.

Idea of proof: Let T be a dg-category. We consider the Yoneda embedding (see definition 3.2.7)

$$\underline{h} : T \longrightarrow \widehat{T}.$$

This is a quasi-fully faithful dg-functor. We consider $\widehat{T}_{pe} \subset \widehat{T}$, the full sub-dg-category consisting of all objects which are compact in $D(T^{op})$ (these objects will simply be called *compact*). The dg-category $\widehat{T}_{pe}$ will be called the dg-category of *perfect T^{op}-dg-modules*, or equivalently of *compact T^{op}-dg-modules*. As any quasi-representable object is compact, the Yoneda embedding factors as a full embedding

$$\underline{h} : T \longrightarrow \widehat{T}_{pe}.$$

Let now T' be a triangulated dg-category. By definition, the natural morphism

$$T' \longrightarrow \widehat{T'}_{pe}$$

is an isomorphism in $Ho(dg-cat)$. We can then consider the induced morphism

$$[\widehat{T}_{pe}, \widehat{T'}_{pe}] \longrightarrow [T, \widehat{T'}_{pe}],$$

induced by the resytiction along the morphism $T \longrightarrow \widehat{T}_{pe}$. The hard point is to show that this map in bijective and that $\widehat{T}_{pe}$ is a triangulated dg-category. These two facts can be deduced from the following lemma and the Proposition 4.1.2.

Lemma 4.4.3 *Let T be a dg-category, and $\underline{h}: T \longrightarrow \widehat{T}_{pe}$ be the natural inclusion. Let M be a $C(k)$-model category which satisfies the conditions* (1), (2) *and* (3) *of Sect. 3.1. Then the Quillen adjunction*

$$h_! : M^T \longrightarrow M^{\widehat{T}_{pe}} \qquad M^T \longleftarrow M^{\widehat{T}_{pe}} : h^*$$

is a Quillen equivalence.

The proof of the above lemma can be found in [28, Lem. 7.5]. It is based on the fundamental fact that the quasi-representable objects in $D(T^{op})$ generate the subcategory of compact objects by taking a finite number of finite homotopy colimits, shifts and retracts, together with the fact that $\mathbb{L}h_!$ and h^* both preserve these finite homotopy colimits, shifts and retracts. □

The proof of the proposition shows that the left adjoint to the inclusion is given by

$$\widehat{(-)}_{pe} : Ho(dg-cat) \longrightarrow Ho(dg-cat^{tr}),$$

sending a dg-category T to the full sub-dg-category $\widehat{T}_{pe}$ of $\widehat{T}$ consisting of all compact objects.

For example, if R is a k-algebra, considered as a dg-category with a unique object BR, $\widehat{BR}_{pe}$ is the dg-category of cofibrant and perfect complexes of R-modules. In particular

$$H^0(\widehat{BR}_{pe}) \simeq D_{parf}(R)$$

is the perfect derived category of R. This follows from the fact that compact objects in $D(R)$ are precisely the perfect complexes (this is a well known fact which can also be deduced from the general result [32, Prop. 2.2]). Therefore, we see that the dg-category of perfect complexes over some ring R is the triangulated hull of R.

Definition 4.4.4 *A morphism $T \longrightarrow T'$ in $Ho(dg-cat)$ is called a* Morita equivalence *if the induced morphism in the triangulated hull*

$$\widehat{T}_{pe} \longrightarrow \widehat{T'}_{pe}$$

is an isomorphism in $Ho(dg-cat)$.

It follows formally from the existence of the left adjoint $T \mapsto \widehat{T}_{pe}$ that $Ho(dg-cat^{tr})$ is equivalent to the localized category $W_{mor}^{-1}dg-cat$, where W_{mor} is the subset of Morita equivalences in $dg-cat$ as defined above.

Exercise 4.4.5 *Prove the above assertion: the functor*

$$\widehat{(-)}_{pe} : Ho(dg-cat) \longrightarrow Ho(dg-cat^{tr})$$

induces an equivalence of categories

$$W_{mor}^{-1}Ho(dg-cat) \simeq Ho(dg-cat^{tr}).$$

We can characterize the Morita equivalences in the following equivalent ways.

Proposition 4.4.6 *Let $f : T \longrightarrow T'$ be a morphism of dg-categories. The following are equivalent.*

(a) The morphism f is a Morita equivalence.
(b) For any triangulated dg-category T_0, the induced map

$$[T', T_0] \longrightarrow [T, T_0]$$

is bijective.
(c) The induced functor

$$f^* : D(T') \longrightarrow D(T)$$

is an equivalence of categories.
(d) The functor

$$\mathbb{L}f_! : D(T) \longrightarrow D(T')$$

induces an equivalence between the full sub-category of compact objects.

Exercise 4.4.7 *Prove the proposition 4.4.6.*

We finish this section by a description of morphisms in $Ho(dg-cat^{tr})$ in terms of derived categories of bi-dg-modules.

Proposition 4.4.8 *Let T and T' be two dg-categories. Then, there exists a natural bijection between $[\widehat{T}_{pe}, \widehat{T'}_{pe}]$ and the subset of $Iso(D(T \otimes^{\mathbb{L}} (T')^{op}))$ consisting of $T \otimes^{\mathbb{L}} (T')^{op}$-dg-modules F such that for any $x \in T$, the $(T')^{op}$-dg-module $F_{x,-}$ is compact.*

Exercise 4.4.9 *Give a proof of proposition 4.4.8.*

Exercise 4.4.10 *(a) Show that the full sub-category $Ho(dg-cat^{tr}) \subset Ho(dg-cat)$ is not stable by finite coproducts (taken inside $Ho(dg-cat)$).*
(b) Show that the category $Ho(dg-cat^{tr})$ has finite sums and finite products.
(c) Show that in the category $Ho(dg-cat^{tr})$, the natural morphism

$$T \coprod T' \longrightarrow T \times T',$$

for any T and T' objects in $Ho(dg-cat^{tr})$. Note that the symbols $\coprod$ and $\times$ refer here to the categorical sum and product in the category $Ho(dg-cat^{tr})$.
(d) Deduce from this that the set of morphisms $Ho(dg-cat^{tr})$ are endowed with natural structure of commutative monoids such that the composition is bilinear. Identify this monoid structure with the direct sum on the level of bi-dg-modules throught the bijection of corollary 1.

Exercise 4.4.11 *Let $T \longrightarrow T'$ be a Morita equivalence and T_0 be a dg-category. Show that the induced morphism*

$$T \otimes^{\mathbb{L}} T_0 \longrightarrow T' \otimes^{\mathbb{L}} T_0$$

is again a Morita equivalence (use the lemma 4.4.3).

Exercise 4.4.12 *Let T and T' be two triangulated dg-catgeory, and define*

$$T\widehat{\otimes}^{\mathbb{L}}T' := \widehat{T\otimes^{\mathbb{L}}T'}_{pe}.$$

(a) Show that $(T,T')\mapsto T\widehat{\otimes}^{\mathbb{L}}T'$ defines a symmetric monoidal structure on $Ho(dg-cat^{tr})$ in such a way that the functor

$$\widehat{(-)}_{pe} : Ho(dg-cat) \longrightarrow Ho(dg-cat^{tr})$$

is a symmetric monoidal functor.

(b) Show that the monoidal structure $\widehat{\otimes}^{\mathbb{L}}$ is closed on $Ho(dg-cat^{tr})$.

Exercise 4.4.13 *(a) Let T and T' be two dg-categories. Prove that the Yoneda embedding $\underline{h} : T \hookrightarrow \widehat{T}_{pe}$ induces an isomorphism in $Ho(dg-cat)$*

$$\mathbb{R}\underline{Hom}(\widehat{T}_{pe},\widehat{T'}_{pe}) \longrightarrow \mathbb{R}\underline{Hom}(T,\widehat{T'}_{pe}).$$

(b) Deduce from this that for any dg-category T there is a morphism in $Ho(dg-cat)$

$$\mathbb{R}\underline{Hom}(T,T) \longrightarrow \mathbb{R}\underline{Hom}(\widehat{T}_{pe},\widehat{T}_{pe})$$

which is quasi-fully faithful.

(c) Deduce from this that for any dg-category T there exist isomorphisms

$$HH^*(T) \simeq HH^*(\widehat{T}_{pe}).$$

5 Lecture 4: Some Applications

In this last lecture I will present some applications of the homotopy theory of dg-categories. We will see in particular how the problems mentioned in *Sect.* 1.2 can be solved using dg-categories. The very last section will be some discussions on the notion of saturated dg-categories and their use in the definition of a *secondary K-theory* functor.

5.1 Functorial Cones

One of the problem encountered with derived categories is the non existence of functorial cones. In the context of dg-categories this problem can be solved as follows.

Let T be a triangulated dg-category. We let Δ^1_k be the dg-category freely generated by two objects 0 and 1 and freely generated by one non trivial morphism $0 \to y$, and $\mathbf{1}$ be the unit dg-category (with a unique object and k for its endomorphism). There is a morphism

$$\Delta^1_k \longrightarrow \widehat{\mathbf{1}}_{pe}$$

sending 0 to 0 and 1 to k. We get an induced morphism in $Ho(dg-cat)$

$$\mathbb{R}\underline{Hom}(\widehat{\mathbf{1}}_{pe},T) \longrightarrow \mathbb{R}\underline{Hom}(\Delta_k^1,T).$$

As T is triangulated we have

$$\mathbb{R}\underline{Hom}(\widehat{\mathbf{1}}_{pe},T) \simeq \mathbb{R}\underline{Hom}(\mathbf{1},T) \simeq T.$$

Therefore, we have defined a morphism in $Ho(dg-cat)$

$$f : T \longrightarrow \mathbb{R}\underline{Hom}(\Delta_k^1,T) =: Mor(T).$$

The dg-category $Mor(T)$ is also the full sub-dg-category of $Int(Mor(T^{op}-Mod))$ corresponding to quasi-representable dg-modules, and is called the dg-category of morphisms in T. The morphism f defined above intuitively sends an object $x \in T$ to $0 \to x$ in $Mor(T)$ (note that 0 is an object in T because T is triangulated).

Proposition 5.1.1 *There exists a unique morphism in* $Ho(dg-cat)$

$$c : Mor(T) \longrightarrow T$$

such that the following two $(T \otimes^{\mathbb{L}} Mor(T)^{op})$*-dg-modules*

$$(z,u) \mapsto Mor(T)(u,f(z)) \qquad (z,u) \mapsto T(c(u),z)$$

are isomorphic in $D(T \otimes^{\mathbb{L}} Mor(T)^{op})$ *(In other words, the morphism* f *admits a* left adjoint*).*

Idea of proof: We consider the following explicit models for T, $Mor(T)$ and f. We let T' be the full sub-dg-category of $\widehat{T}$ consisting of quasi-representable objects (or equivalentely of compact objects as T is triangulated). We let $Mor(T)'$ be the full sub-dg-category of $Int(Mor(T^{op}-Mod))$ consisting of morphisms between quasi-representable objects (these are also the compact objects in $Ho(Mor(T^{op}-Mod))$ because T is triangulated). We note that $Mor(T)'$ is the dg-category whose objects are cofibrations between cofibrant and quasi-representable T^{op}-dg-modules. To each compact and cofibrant T^{op}-dg-module z we consider $0 \to z$ as an object in T'. This defines a dg-functor $T' \longrightarrow Mor(T)'$ which is a model for f. We define c as being a $C(k)$-enriched left adjoint to c (in the most naive sense), sending an object $c : x \longrightarrow y$ of $Mor(T)'$ to $c(u)$ defined by the push-out in $T^{op}-Mod$

$$\begin{array}{ccc} x & \longrightarrow & y \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & c(u). \end{array}$$

We note that the T^{op}-module $c(u)$ is compact and thus belongs to T'. It is easy to check that c, as a morphism in $Ho(dg-cat)$ satisfies the property of the proposition.

The unicity of c is proved formally, in the same way that one proves the unicity of adjoints in usual category theory. $\square$

The morphism $c : Mor(T) \longrightarrow T$ is a functorial cone construction for the triangulated dg-category T. The important fact here is that there is a natural functor

$$H^0(Mor(T)) \longrightarrow Mor(H^0(T)),$$

which is essentially surjective, full but not faithful in general. The functor

$$H^0(c) : H^0(Mor(T)) \longrightarrow H^0(T)$$

does not factor in general throught $Mor(H^0(T))$.

To finish, proposition 5.1.1 becomes really powerful when combined with the following fact.

Proposition 5.1.2 *Let T be a triangulated dg-category and T' be any dg-category. Then $\mathbb{R}\underline{Hom}(T',T)$ is triangulated.*

Exercise 5.1.3 *Deduce proposition 5.1.2 from exercice 4.4.11.*

One important feature of triangulated dg-categories is that any dg-functor $f : T \longrightarrow T'$ between triangulated dg-categories commutes with cones. In other words, the diagram

$$\begin{array}{ccc} Mor(T) & \xrightarrow{c} & T \\ {\scriptstyle c(f)}\downarrow & & \downarrow{\scriptstyle f} \\ Mor(T') & \xrightarrow[c]{} & T' \end{array}$$

commutes in $Ho(dg-cat)$. This has to be understood as a generalization of the fact that any linear functor between additive categories commutes with finite direct sums. This property of triangulated dg-categories is very useful in practice, as then any dg-functor $T \longrightarrow T'$ automatically induces a triangulated functor $H^0(T) \longrightarrow H^0(T')$.

Exercise 5.1.4 *Prove the above assertion, that*

$$\begin{array}{ccc} Mor(T) & \xrightarrow{c} & T \\ {\scriptstyle c(f)}\downarrow & & \downarrow{\scriptstyle f} \\ Mor(T') & \xrightarrow[c]{} & T' \end{array}$$

commutes in $Ho(dg-cat)$ (here T and T' are triangulated dg-categories).

5.2 Some Invariants

Another problem mentioned in *Sect.* 1.2 is the fact that the usual invariants, (K-theory, Hochschild homology and cohomology ...), are not invariants of derived categories. We will see here that these invariants can be defined on the level of $Ho(dg-cat^{tr})$. We will treat the examples of K-theory and Hochschild cohomology.

(a) Let T be a dg-category. We consider $T^{op}-Mod^{cc}$ the full sub-category of compact and cofibrant T^{op}-dg-modules. We can endow $T^{op}-Mod^{cc}$ with a structure of an exact complicial category (see [25]) whose equivalences are quasi-isomorphisms and cofibrations are the cofibrations of the model category structure on $T^{op}-Mod$. This Waldhausen category defines a K-theory space $K(T)$ (see [25]). We note that if T is triangulated we have

$$K_0(T) := \pi_0(K(T)) \simeq K_0(H^0(T)),$$

where the last K-group is the Grothendieck group of the triangulated category $H^0(T)$.
Now, let $f : T \longrightarrow T'$ be a morphism between dg-categories. It induces a functor

$$f_! : T^{op}-Mod \longrightarrow (T')^{op}-Mod.$$

This functor preserves cofibrations, compact cofibrant objects and push-outs. Therefore, it induces a functor between Waldhausen categories

$$f_! : T^{op}-Mod^{cc} \longrightarrow (T')^{op}-Mod^{cc}$$

and a morphism on the corresponding spaces

$$f_! : K(T) \longrightarrow K(T').$$

This defines a functor

$$K : dg-cat \longrightarrow Sp$$

from dg-categories to spectra. It is possible to show that this functor sends Morita equivalences to stable equivalences, and thus defines a functor

$$K : Ho(dg-cat^{tr}) \longrightarrow Ho(Sp).$$

We see it particular that two dg-categories which are Morita equivalent have the same K-theory.

(b) (See also exercice 4.4.13) Let T be a dg-category. We consider $\mathbb{R}\underline{Hom}(T,T)$, the dg-category of (derived) endomorphisms of T. The identity gives an object $id \in \mathbb{R}\underline{Hom}(T,T)$, and we can set

$$HH^{\cdot}(T) := \mathbb{R}\underline{Hom}(T,T)(id,id),$$

the Hochschild complex of T. This is a well defined object in $D(k)$, the derived category of complexes of k-modules, and the construction $T \mapsto HH^{\cdot}(T)$ provides a functor of groupoids

$$Ho(dg-cat)^{iso} \longrightarrow D(k)^{iso}.$$

Using the results of *Sect.* 3.2 we can see that

$$HH^*(T) \simeq Ext^*(T,T),$$

where the Ext-group is computed in the derived category of $T \otimes^{\mathbb{L}} T^{op}$-dg-modules. In particular, when T is given by an associative flat k-algebra R we find

$$HH^*(T) \simeq Ext^*_{R\otimes R}(R,R),$$

which is usual Hochschild cohomology. The Yoneda embedding $T \longrightarrow \widehat{T}_{pe}$, provides an isomorphism in $Ho(dg-cat)$

$$\mathbb{R}\underline{Hom}(\widehat{T}_{pe},\widehat{T}_{pe}) \simeq \mathbb{R}\underline{Hom}(T,\widehat{T}_{pe}),$$

and a quasi-fully faithful morphism

$$\mathbb{R}\underline{Hom}(T,T) \longrightarrow \mathbb{R}\underline{Hom}(T,\widehat{T}_{pe}).$$

Therefore, we get a quasi-fully faithful morphism in $Ho(dg-cat)$

$$\mathbb{R}\underline{Hom}(T,T) \longrightarrow \mathbb{R}\underline{Hom}(\widehat{T}_{pe},\widehat{T}_{pe})$$

sending the identity to the indentity. Therefore, we obtain a natural isomorphism

$$HH^{\cdot}(T) \simeq HH^{\cdot}(\widehat{T}_{pe}).$$

We get that way that Hochschild cohomology is a Morita invariant.

(c) There also exists an interpretation of Hochschild homology purely in terms of dg-categories in the following way. We consider two dg-categories T and T', and the Yoneda embedding $\underline{h} : T \hookrightarrow \widehat{T}$. We obtain an induced functor

$$\underline{h}_! : \mathbb{R}\underline{Hom}(T,\widehat{T'}) \longrightarrow \mathbb{R}\underline{Hom}(\widehat{T},\widehat{T'}).$$

It is possible to show that this morphism is quasi-fully faithful and that its quasi-essential image consists of all morphisms $\widehat{T} \longrightarrow \widehat{T'}$ which are continuous (i.e. commute with direct sums). We refer to [28, Thm. 7.2] for more details about this statement. This implies that there is an isomorphism in $Ho(dg-cat)$

$$\mathbb{R}\underline{Hom}(T,\widehat{T'}) \simeq \mathbb{R}\underline{Hom}_c(\widehat{T},\widehat{T'}),$$

where $\mathbb{R}\underline{Hom}_c$ denotes the full sub-dg-category of continuous dg-functors.
Let now T be a dg-category and consider the $T \otimes^{\mathbb{L}} T^{op}$-dg-module sending (x,y) to $T(y,x)$. This dg-module can be represented by an object in the dg-category

$$L(T \otimes^{\mathbb{L}} T^{op}) \simeq \mathbb{R}\underline{Hom}(T \otimes^{\mathbb{L}} T^{op},\widehat{\mathbf{1}}) \simeq \mathbb{R}\underline{Hom}_c(\widehat{T \otimes^{\mathbb{L}} T^{op}},\widehat{\mathbf{1}}),$$

and thus by a continuous in $Ho(dg-cat)$

$$L(T \otimes^{\mathbb{L}} T^{op}) \longrightarrow \widehat{1}.$$

The image of T, considered as a bi-module sending (x,y) to $T(y,x)$, by this morphism is denoted by $HH_{.}(T) \in D(k) \simeq H^0(\widehat{1})$, and is called the Hochschild homology complex of T. When T is a flat k-algebra R then we have

$$HH_{.}(T) \simeq R \otimes^{\mathbb{L}}_{R\otimes R^{op}} R \in D(k).$$

From its definition, it is not hard to show that $T \mapsto HH_{.}(T)$ is invariant by Morita equivalences.

5.3 Descent

In this section we will see how to solve the non-local nature of derived categories explained in *Sect.* 1.2. For this, let X be a scheme. We have the Grothendieck category $C(\mathscr{O}_X)$ of (unbounded) complexes of sheaves of $\mathscr{O}_X$-modules. This category can be endowed with a model category structure for which the equivalences are the quasi-isomorphisms (of complexes of sheaves) and the cofibrations are the monomorphisms (see e.g. [14]). Moreover, when X is a k-scheme then the natural $C(k)$-enrichment of $C(\mathscr{O}_X)$ makes it into a $C(k)$-model category. We let

$$L(\mathscr{O}_X) := Int(C(\mathscr{O}_X)),$$

and we let $L_{pe}(X)$ be the full sub-dg-category consisting of perfect complexes on X. The K-theory of X can be defined as

$$K(X) := K(L_{pe}(X)),$$

using the definition of K-theory of dg-categories we saw in the last section.

When $f : X \longrightarrow Y$ is a morphism of schemes, it is possible to define two morphisms in $Ho(dg-cat)$

$$\mathbb{L}f^* : L(\mathscr{O}_Y) \longrightarrow L(\mathscr{O}_X) \qquad L(\mathscr{O}_Y) \longleftarrow L(\mathscr{O}_X) : \mathbb{R}f_*,$$

which are adjoints (according to the model we chose $\mathbb{L}f^*$ is a bit tricky to define explicitly). The morphism

$$\mathbb{L}f^* : L(\mathscr{O}_Y) \longrightarrow L(\mathscr{O}_X)$$

always preserves perfect complexes are induces a morphism

$$\mathbb{L}f^* : L_{pe}(Y) \longrightarrow L_{pe}(X).$$

This construction provides a functor

$$L_{pe} : k-Sch^{op} \longrightarrow Ho(dg-cat^{tr}),$$

from the (opposite of) category of k-schemes to the category of triangulated dg-categories. The existence of this functor is the starting point of an extremelly rich source of questions about its behaviour. Following the philosophy of non-commutative geometry according to M. Kontsevich, $Ho(dg-cat^{tr})$ can be considered as the category of *non-commutative schemes*, and the functor L_{pe} above is simply passing from the commutative to the non-commutative setting. Contrary to what this might suggest, at leas as first naive thoughts, the functor L_{pe} is far from being an embedding and its general study leads to very interesting questions.

(a) A first observation is that two schemes X and Y can be such that $L_{pe}(X) \simeq L_{pe}(Y)$ without being isomorphic, as shown by many well known examples of *derived equivalences* (see [23] for more about this). The *fibers* of the functor L_{pe}, that is the set of schemes, up to isomorphisms, having the same dg-categories of perfect complexes, is expected to be finite, at least when we restrict to smooth and projective schemes. It is shown in [1] that these fibers are *discrete* and countable.
(b) The functor L_{pe} sends direct product of k-schemes into tensor product in $Ho(dg-cat^{tr})$, at least under reasonable conditions (see [28], [3]). In other words, L_{pe} is a symmetric monoidal functor, when $k-Sch$ is considered as a symmetric monoidal category for the direct monoidal structure.
(c) The image of the smooth and proper k-schemes inside $Ho(dg-cat^{tr})$ has an explicit description: its objects are smooth and projective k-schemes, and morphisms between two such schemes X and Y are given by quasi-isomorphism classes of perfect complexes on $X\times_k Y$ (see 5.3.2 below). This category is very close to the category of Chow motives, for which morphisms are rather correspondences up to rationnal equivalences. By analogy, $Ho(dg-cat^{tr})$ can be used in order to define a notion of non-commutative motives (see [18]). Constructing *realisations* for these non-commutative motives has lead to the notion of non-commutative Hodge structures (see [16]), and to the construction of the non-commutative Gauss–Manin connexion (see [34]).

We now come back to the descent property. The following proposition will not be proved in these notes. We refer to [12] for more details about the descent for perfect complexes.

Proposition 5.3.1 *Let $X=U\cup V$, where U and V are two Zariski open subschemes. Then the following square*

$$\begin{array}{ccc} L_{pe}(X) & \longrightarrow & L_{pe}(U) \\ \downarrow & & \downarrow \\ L_{pe}(V) & \longrightarrow & L_{pe}(U\cap V) \end{array}$$

is homotopy cartesian in the model category $dg-cat$.

Let us also mention the following related statement.

Proposition 5.3.2 *Let X and Y be two smooth and proper schemes over $Speck$. Then, there exists a natural isomorphism in $Ho(dg-cat)$*

$$\mathbb{R}\underline{Hom}(L_{pe}(X),L_{pe}(Y)) \simeq L_{pe}(X\times_k Y).$$

For a proof we refer the reader to [28]. It should be emphasised here that the corresponding statement is false on the level of derived categories. More precisely, let $E \in D_{parf}(X \times_k Y)$ and let

$$\begin{array}{cccc} \phi_E : D_{parf}(X) & \longrightarrow & D_{parf}(Y) \\ F & \mapsto & \mathbb{R}(p_Y)_*(E \otimes^{\mathbb{L}} p_X^*(F)) \end{array}$$

be the corresponding functor. The construction $E \mapsto \phi_E$ defines a functor

$$\phi_- : D_{parf}(X \times_k Y) \longrightarrow Fun^{tr}(D_{parf}(X), D_{parf}(Y)),$$

where the right hand side is the category of triangulated functors from $D_{parf}(X)$ to $D_{parf}(Y)$. When X and Y are projective over $Spec\, k$ (and that k is field) then it is known that this functor is essentially surjective (see [23]). In general it is not known if ϕ_- is essentially surjective or not. In any case, even for very simple X and Y the functor ϕ_- is not faithful, and thus is not an equivalence of categories in general. Suppose for instance that $X = Y = E$ and elliptic curve over $k = \mathbb{C}$, and let $\Delta \in D_{parf}(X \times_k X)$ be the structure sheaf of the diagonal. The image by ϕ_- of the objects Δ and $\Delta[2]$ are respectively the identity functor and the shift by 2 functor. Because X is of cohomological dimension 1 we have $Hom(id, id[2]) = 0$, where this hom is computed in $Fun^{tr}(D_{parf}(X), D_{parf}(X))$. However, $Hom(\Delta, \Delta[2]) \simeq HH^2(X) \simeq H^1(E, \mathscr{O}_E) \simeq k$.

5.4 Saturated dg-Categories and Secondary K-Theory

We arrive at the last section of these lectures. We have seen that dg-categories can be used in order to replace derived categories, and that they can be used in order to define K-theory. In this section we will see that dg-categories can also be considered as *coefficients* that can themselves be used in order to define a secondary version of K-theory. For this I will present an analogy between the categories $Ho(dg-cat^{tr})$ and $k-Mod$. Throught this analogy projective k-modules of finite rank correspond to the notion of *saturated dg-categories*. I will then show how to define secondary K-theory spectrum $K^{(2)}(k)$ using saturated dg-categories, and give some ideas of how to define analogs of the rank and chern character maps in order to see that this secondary K-theory $K^{(2)}(k)$ is non-trivial. I will also mention a relation between $K^{(2)}(k)$ and the Brauer group, analog to the well known relation between K-theory and the Picard group.

We start by the analogies between the categories $k-Mod$ of k-modules and $Ho(dg-cat^{tr})$. The true analogy is really between $k-Mod$ and the homotopy theory of triangulated dg-categories, e.g. the simplicial category $Ldg-cat^{tr}$ obtained by simplicial localization (see [29]). The homotopy category $Ho(dg-cat^{tr})$ is sometimes too coarse to see the analogy. We will however restrict ourselves with $Ho(dg-cat^{tr})$, even though some of the facts below about $Ho(dg-cat^{tr})$ are not completely intrinsic and requires to lift things to the model category of dg-categories.

(a) The category $k-Mod$ is a closed symmetric monoidal category for the usual tensor product. In the same way, $Ho(dg-cat^{tr})$ has a closed symmetric monoidal structure induced from the one of $Ho(dg-cat)$ (see $Sect.$ 3.2). Explicitly, if T and T' are two triangulated dg-category we form $T\otimes^{\mathbb{L}}T' \in Ho(dg-cat)$. This is not a triangulated dg-category anymore and we set

$$T\widehat{\otimes}^{\mathbb{L}}T' := (\widehat{T\otimes^{\mathbb{L}}T'})_{pe} \in Ho(dg-cat^{tr}).$$

The unit of this monoidal structure is the triangulated hull of $\mathbf{1}$, i.e. the dg-category of cofibrant and perfect complexes of k-modules. The corresponding internal Homs is the one of $Ho(dg-cat)$, as we already saw that $\mathbb{R}\underline{Hom}(T,T')$ is triangulated if T and T' are.

(b) The category $k-Mod$ has a zero object and finite sums are also finite products. This is again true in $Ho(dg-cat^{tr})$. The zero dg-category (with one object and 0 as endomorphism ring of this object) is a zero object in $Ho(dg-cat^{tr})$. Also, for two triangulated dg-categories T and T' their sum $T\coprod T'$ as dg-categories is not triangulated anymore. Their direct sum in $Ho(dg-cat^{tr})$ is the triangulated hull of $T\coprod T'$, that is

$$\widehat{T\coprod T'}_{pe} \simeq \widehat{T}_{pe}\times \widehat{T'}_{pe} \simeq T\times T'.$$

We note that this second remarkable property of $Ho(dg-cat^{tr})$ is not satisfied by $Ho(dg-cat)$ itself. We can say that $Ho(dg-cat^{tr})$ is *semi-additive*, which is justified by the fact that the Homs in $Ho(dg-cat^{tr})$ are abelian monoids (or abelian semi-groups).

(c) The category $k-Mod$ has arbitrary limits and colimits. The corresponding statement is not true for $Ho(dg-cat^{tr})$. However, we have homotopy limits and homotopy colimits in $Ho(dg-cat^{tr})$, whose existence are insured by the model category structure on $dg-cat$.

(d) There is a natural notion of short exact sequences in $k-Mod$. In the same way, there is a natural notion of short exact sequences in $Ho(dg-cat^{tr})$. These are the sequences of the form

$$T_0 \xrightarrow{j} T \xrightarrow{p} (\widehat{T/T_0})_{pe},$$

where i is a quasi-fully faithful functor between triangulated dg-categories, and $(\widehat{T/T_0})_{pe}$ is the quotient defined as the triangulated hull of the homotopy push-out of dg-categories

$$\begin{array}{ccc} T_0 & \longrightarrow & T \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & T/T_0. \end{array}$$

These sequences are natural in terms of the homotopy theory of triangulated dg-categories as it can be shown that quasi-fully faithful dg-functors are precisely the *homotopy monomorphisms* in $dg-cat$, i.e. the morphisms $T \longrightarrow T'$ such that the diagonal map

$$T \longrightarrow T \times^h_{T'} T$$

is a quasi-equivalence (the right hand side is a homotopy pull-back). This defines a dual notion of homotopy epimorphisms of triangulated dg-categories as being the morphism $T \longrightarrow T'$ such that for any triangulated dg-categories T'' the induced morphism

$$\mathbb{R}\underline{Hom}(T', T'') \longrightarrow \mathbb{R}\underline{Hom}(T, T'')$$

is a homotopy monomorphisms (i.e. is quasi-fully faithful). In the exact sequences above j is a homotopy monomorphism, p is a homotopy epimorphism, p is the cokernel of j and j is the kernel of p. The situation is therefore really close to the situation in $k-Mod$.

If $k-Mod$ and $Ho(dg-cat^{tr})$ are so analoguous then we should be able to say what is the analog property of being projective of finite rank, and to define a K-group or even a K-theory spectrum of such objects. It turns that this can be done and that the theory can actually be pushed rather far. Also, we will see that this new K-theory might have some geometric and arithmetic significance.

It is well know that the projective modules of finite rank over k are precisely the dualizable (also called rigid) objects in the closed monoidal category $k-Mod$. Recall that any k-module M has a dual $M^\vee := \underline{Hom}(M,k)$, and that there always exists an evaluation map

$$M^\vee \otimes M \longrightarrow \underline{Hom}(M,M).$$

The k-module M is dualizable if this evaluation map is an isomorphism, and this is known to be equivalent to the fact that M is projective of finite rank.

We will take this as a definition of *projective triangulated dg-categories of finite rank*. The striking fact is that these dg-categories have already been studied for other reasons under the name of *saturated dg-categories*, or *smooth and proper dg-categories*.

Definition 5.4.1 *A triangulated dg-category T is* saturated *if it is dualizale in $Ho(dg-cat^{tr})$, i.e. if the evaluation morphism*

$$\mathbb{R}\underline{Hom}(T, \widehat{\mathbf{1}}_{pe}) \widehat{\otimes}^{\mathbb{L}} T \longrightarrow \mathbb{R}\underline{Hom}(T,T)$$

is an isomorphism in $Ho(dg-cat^{tr})$.

The saturated triangulated dg-categories can be characterized nicely using the notion of smooth and proper dg-algebras (see [19, 30, 32]). Recall that a dg-algebra B is smooth if B is a compact object in $D(B \otimes^{\mathbb{L}} B^{op})$. Recall also that such a dg-algebra is proper if its underlying complex if perfect (i.e. if B is compact in $D(k)$). The following proposition can be deduced from the results of [32]. We omit the proof in these notes (see however [32] for some statements about saturated dg-categories).

Proposition 5.4.2 *A triangulated dg-category is saturated if and only if it is Morita equivalent to a smooth and proper dg-algebra.*

This proposition is interesting as it allows us to show that there are many examples of saturated dg-categories. The two main examples are the following.

(a) Let X be a smooth and proper k-scheme. Then $L_{pe}(X)$ is a saturated dg-category (see [32]).
(b) For any k-algebra, which is projective of finite rank as a k-module and which is of finite global cohomlogical dimension, the dg-category $\widehat{A}_{pe}$ of perfect complexes of A-modules is saturated.

The symmetric monoidal category $Ho(dg-cat^{sat})$ of saturated dg-categories is rigid. Note that any object T has a dual $T^{\vee} := \mathbb{R}\underline{Hom}(T, \widehat{\mathbf{1}}_{pe})$. Moroever, it can be shown that $T^{\vee} \simeq T^{op}$ is simply the opposite dg-category (this is only true when T is saturated). In particular, for T and T' two saturated dg-categories we have the following important formula

$$T^{op} \widehat{\otimes}^{\mathbb{L}} T' \simeq \mathbb{R}\underline{Hom}(T, T').$$

We can now define the secondary K-group. We start by $\mathbb{Z}[sat]$, the free abelian group on isomorphism classes (in $Ho(dg-cat^{tr})$) of satuared dg-categories. We define $K_0^{(2)}(k)$ to be the quotient of $\mathbb{Z}[sat]$ by the relation

$$[T] = [T_0] + [\widehat{(T/T_0)}_{pe}]$$

for any full saturated sub-dg-category $T_0 \subset T$ with quotient $\widehat{(T/T_0)}_{pe}$.

More generally, we can consider a certain Waldhausen category *Sat*, whose objects are cofibrant dg-categories T such that $\widehat{T}_{pe}$ is saturated, whose morphisms are morphisms of dg-categories, whose equivalences are Morita equivalences, and whose cofibrations are cofibrations of dg-categories which are also fully faithful. From this we can construct a spectrum, denoted by $K^{(2)}(k)$ by applying Waldhausen's construction, called the *secondary K-theory spectrum of k*. We have

$$\pi_0(K^{(2)}(k)) \simeq K_0^{(2)}(k).$$

We no finish with some arguments that $K^{(2)}(k)$ to show that is non trivial and interesting.

First of all, we have the following two basic properties.

(a) $k \mapsto K^{(2)}(k)$ defines a functor from the category of commutative rings to the homotopy category of spectra. To a map of rings $k \to k'$ we associate the base change $- \otimes_k^{\mathbb{L}} k'$ from saturated dg-categories over k to saturated dg-categories over k', which induces a functor of Waldhausen categories and thus a morphism on the corresponding K-theory spectra.

(b) If $k = colim_i k_i$ is a filtered colimit of commutative rings then we have

$$K^{(2)}(k) \simeq colim_i K^{(2)}(k_i).$$

This follows from the non trivial statement that the homotopy theory of saturated dg-categories over k is the filtered colimit of the homotopy theories of saturated dg-categories over the k_i (see [31]).

(c) The monoidal structure on $Ho(dg - cat^{tr})$ induces a commutative ring structure on $K_0^{(2)}(k)$. I guess that this monoidal structure also induces a E_∞-multiplication on $K^{(2)}(k)$.

Our next task is to prove that $K^{(2)}(k)$ is non zero. For this we construct a rank map

$$rk_0^{(2)} : K_0^{(2)}(k) \longrightarrow K_0(k)$$

which an analog of the usual rank map (also called the trace map)

$$rk_0 : K_0(k) \longrightarrow HH_0(k) = k.$$

Let T be a saturated dg-category. As T is dualizable in $Ho(dg - cat^{tr})$ there exists a trace map

$$\mathbb{R}\underline{Hom}(T,T) \simeq T^{op}\widehat{\otimes}^{\mathbb{L}} T \longrightarrow \widehat{\mathbf{1}}_{pe},$$

which is the dual of the identity map

$$id : \widehat{\mathbf{1}}_{pe} \longrightarrow T^{op}\widehat{\otimes}^{\mathbb{L}} T.$$

The image of the identity provides a perfect complex of k-modules, and thus an element

$$rk_0^{(2)}(T) \in K_0(k).$$

This defines the map

$$rk_0^{(2)} : K_0^{(2)}(k) \longrightarrow K_0(k).$$

It can be shown that $rk_0^{(2)}(T)$ is in fact $HH.(T)$, the Hochschild homology complex of T.

Lemma 5.4.3 *For any saturated dg-category T we have*

$$rk_0^{(2)}(T) = [HH.(T)] \in K_0(k),$$

where $HH.(T)$ is the (perfect) complex of Hochschild homology of T.

In particular we see that for X a smooth and proper k-scheme we have

$$rk_0^{(2)}(L_{pe}(X)) = [HH.(X)] \in K_0(k).$$

When $k = \mathbb{C}$ then $HH_*(X)$ can be identified with Hodge cohomology $H^*(X, \Omega_X^*)$, and thus $rk_0^{(2)}(L_{pe}(X))$ is then the euler characteristic of X. In other words, we can say that the rank of $L_{pe}(X)$ is $\chi(X)$. The map $rk_0^{(2)}$ shows that $K_0^{(2)}(k)$ is non zero.

The usual rank $rk_0 : K_0(k) \longrightarrow HH_0(k) = k$ is only the zero part of a rank map

$$rk_* : K_*(k) \longrightarrow HH_*(k).$$

In the same way, it is possible to define a secondary rank map

$$rk_*^{(2)} : K_*^{(2)}(k) \longrightarrow K_*(S^1 \otimes^{\mathbb{L}} k),$$

where $S^1 \otimes^{\mathbb{L}} k$ is a simplicial ring that can be defined as

$$S^1 \otimes^{\mathbb{L}} k = k \otimes^{\mathbb{L}}_{k \otimes^{\mathbb{L}}_{\mathbb{Z}} k} k.$$

Note that by definition of Hochschild homology we have

$$HH.(k) \simeq S^1 \otimes^{\mathbb{L}} k,$$

so we can also write

$$rk_*^{(2)} : K_*^{(2)}(k) \longrightarrow K_*(HH.(k)).$$

Using this map I guess it could be possible to check that the higher K-groups $K_i^{(2)}(k)$ are also non zero in general. Actually, I think it is possible to construct an analog of the Chern character

$$Ch : K_*(k) \longrightarrow HC_*(k)$$

as a map

$$Ch^{(2)} : K_*^{(2)}(k) \longrightarrow HC_*^{(2)}(k) := K_*^{S^1}(S^1 \otimes^{\mathbb{L}} k),$$

where the right hand side is the S^1-equivariant K-theory of $S^1 \otimes^{\mathbb{L}} k$ (note that S^1 acts on $S^1 \otimes^{\mathbb{L}} k$), which we take as a definition of secondary cyclic homology (see [33,34] for more details about this construction).

To finish we show that $K_0^{(2)}(k)$ has a relation with the Brauer group, analog to the relation between $K_0(k)$ and the Picard group. For this, we define $Br_{dg}(k)$ to be the group of isomorphism classes of invertible objects (for the monoidal structure) in $Ho(dg-cat^{tr})$. As being invertible is stronger than being dualizable we have a natural map

$$Br_{dg}(k) \longrightarrow K_0^{(2)}(k)$$

analog to the natural map

$$Pic(k) \longrightarrow K_0(k).$$

Now, by definition $Br_{dg}(k)$ can also be described as the Morita equivalence classes of *Azumaya's dg-algebras*, that is of dg-algebras B satisfying the following two properties

(a)
$$B^{op} \otimes^{\mathbb{L}} B \longrightarrow \mathbb{R}\underline{End}_{C(k)}(B)$$
is a quasi-isomorphism.

(b) The underlying complex of B is a compact generator of $D(k)$.

In particular, a non-dg Azumaya's algebra over k defines an element in $Br_{dg}(k)$, and we thus get a map $Br(k) \longrightarrow Br_{dg}(k)$, from the usual Brauer group of k (see [21]) to the dg-Brauer group of k. Composing with the map $Br_{dg}(k) \longrightarrow K_0^{(2)}(k)$ we get a map

$$Br(k) \longrightarrow K_0^{(2)}(k),$$

from the usual Brauer group to the secondary K-group of k. I do not know if this map is injective in general, but I guess it should be possible to prove that it is non zero in some examples by using the Chern character mentioned above.

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References

1. Anel, M., Toën, B.: Dénombrabilité des classes d'équivalences dérivées de variétés algébriques. J. Algebr. Geom. **18**(2), 257–277 (2009)
2. Artin, M., Grothendieck, A., Verdier, J.L.: Théorie des topos et cohomologie étale des schémas- Tome 1. In: Lecture Notes in Mathematics, vol. 269, Springer, Berlin (1972)
3. Francis, J., Ben-Zvi, D., Nadler, D.: Integral transforms and drinfeld centers in derived algebraic geometry. preprint arXiv arXiv:0805.0157
4. Bondal, A., Kapranov, M.: Enhanced triangulated categories, Math. USSR Sbornik **70**, 93–107 (1991)
5. Bourbaki, N.: Eléments de mathématique. Algbre. Chapitre 10. Algèbre homologique. Springer, Berlin (2007)
6. Deligne, P.: Equations différentielles à points singuliers réguliers. Lecture Notes in Mathematics, vol.163. Springer, Berlin (1970)
7. Dugger, D., Shipley, B.: *K*-theory and derived equivalences. Duke Math J. **124**(3), 587–617 (2004)
8. Elmendorf, A.D., Kriz, I., Mandell, M.A., May, J.P.: Rings, modules, and algebras in stable homotopy theory. In: Mathematical Surveys and Monographs, vol. 47. American Mathematical Society, Providence, RI (1997)
9. Gabriel, P., Zisman, M.: Calculus of fractions and homotopy theory. In: Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 35. Springer, New York (1967)
10. Gabriel, P., Popesco, N.: Caractérisation des catégories abéliennes avec générateurs et limites inductives exactes. C. R. Acad. Sci. Paris **258**, 4188–4190 (1964)
11. Griffiths, P., Harris, J.: Principles of algebraic geometry, Wiley Classics Library. Wiley, New York (1994)
12. Hirschowitz, A., Simpson, C.: Descente pour les n-champs. Preprint math.AG/9807049.
13. Hovey, M.: Model categories. In: Mathematical surveys and monographs, vol. **63**, American Mathematical Society, Providence (1998)
14. Hovey, M.: Model category structures on chain complexes of sheaves. Trans. Am. Math. Soc. **353**(6), 2441–2457 (2001)
15. Joyal, A., Tierney, M.: Strong stacks and classifying spaces. Category theory (Como, 1990). Lecture Notes in Mathematics vol. 1488, pp. 213–236. Springer, Berlin (1991)
16. Katzarkov, L., Kontsevich, M., Pantev, T., Hodge theoretic aspects of mirror symmetry. prépublication arXiv:0806.0107

17. Kelly, G.: Basic concepts of enriched category theory. In: London Mathematical Society Lecture Note Series, vol. **64**. Cambridge Universtity Press, Cambridge (1982)
18. Kontsevich, M.: Note of motives in finite characteristic. preprint arXiv math/0702206.
19. Kontsevich, M., Soibelman, Y.: Notes on A-infinity algebras, A-infinity categories and non-commutative geometry. preprint math.RA/060624
20. May, J.P.: Simplicial objects in algebraic topology. Van Nostrand Mathematical Studies, vol. 11. D. Van Nostrand, Princeton (1967)
21. Milne, J.: Etale cohomology. In: Princeton mathematical series 33. Princeton University Press, Princeton (1980)
22. Neeman, A.: Triangulated categories. In: Annals of Mathematics Studies, vol. 148. Princeton University Press, Princeton, NJ (2001)
23. Rouquier, R.: Catégories dérivées et géométrie birationnelle (d'après Bondal, Orlov, Bridgeland, Kawamata et al.). Séminaire Bourbaki, vol. 2004/2005. Astrisque No. **307**, Exp. No. 946, viii, pp. 283–307 (2006)
24. Schlichting, M.: A note on K-theory and triangulated categories. Invent. Math. **150**, 111–116 (2002)
25. Schlichting, M.: Higher Algebraic K-theory, this volume
26. Simpson, C.: Higgs bundles and local systems. Inst. Hautes tudes Sci. Publ. Math. **75**, 5–95 (1992)
27. Tabuada, G.: Une structure de catégorie de modèles de Quillen sur la catégorie des dg-catégories. In: Comptes Rendus de l'Acadadémie de Sciences de Paris, vol. **340**, pp. 15–19 (2005)
28. Toën, B.: The homotopy of dg-categories and derived Morita theory. Invent. Math. **167**(3), 615–667 (2007)
29. Toën, B.: Higher and derived stacks: A global overview. Algebraic geometry–Seattle 2005. Part 1, 435–487, Proceedings of Symposia in Pure Mathematics, vol. 80, Part 1. American Mathematical Society, Providence, RI (2009)
30. Toën, B.: Finitude homotopique des dg-algèbres propres et lisses. Proc. Lond. Math. Soc. (3) **98**(1), 217–240 (2009)
31. Toën, B.: Anneaux de définitions des dg-algèbres propres et lisses. Bull. Lond. Math. Soc. **40**(4), 642–650 (2008)
32. Toën, B.: Vaquié, M.: Moduli of objects in dg-categories. Ann. Sci. de l'ENS **40**(3), 387–444 (2007)
33. Toën, B.: Vezzosi, G.: Chern character, loop spaces and derived algebraic geometry. In: Algebraic Topology. (Proc. The Abel Symposium 2007). Abel Symposia, vol. 4, pp. 331–354. Springer, Berlin (2009)
34. Toën, B.: Vezzosi, M.: ∞-Catégories monoidales rigides, traces et caractères de Chern preprint arXiv 0903.3292