Last time:

n	0	1	2	3	4	5	6	7
$\pi_* \mathbb{S}$	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/24$	0	0	$\mathbb{Z}/2$	$\mathbb{Z}/240$
generators	1	η	η^2	ν			ν^2	σ
relations				$\eta^3 = 12\nu$				

$$\mathbb{S} \xrightarrow{2} \mathbb{S} \longrightarrow \mathbb{S}/2 \longrightarrow \mathbb{S}^1$$

$\mathbb{S}/2 \xrightarrow{2 \cdot \mathrm{id}_{\mathbb{S}/2}} \mathbb{S}/2$ is essential

$$\mathbb{S}^1 \xrightarrow{\eta} \mathbb{S} \longrightarrow C\eta \longrightarrow \mathbb{S}^2$$

$\Sigma C\eta \xrightarrow{\eta \cdot \mathrm{id}_{C\eta}} C\eta$ is essential

<u>Theorem</u> Let $x \in \pi_* \mathbb{S}$.

1. If $2x = 0$, then $x\eta \in \langle 2, x, 2 \rangle$.
2. If $\eta x = 0$, then $3x\nu \in \langle \eta, x, \eta \rangle$. (or $3\nu x$?)

(This also works for $x \in \pi_* R$ for any ring spectrum R.)

<u>Examples</u>

$\eta^2 \in \langle 2, \eta, 2 \rangle$ mod 0

$6\nu \in \langle \eta, 2, \eta \rangle$ mod η^3

$\nu^2 \in \langle \eta, \nu, \eta \rangle$ mod 0

Mod 2 Moore spectrum

Compute homotopy groups of $\mathbb{S}/2$ from

$$\mathbb{S} \xrightarrow{2} \mathbb{S} \xrightarrow{j} \mathbb{S}/2 \xrightarrow{\delta} \mathbb{S}^1$$

$$\begin{tikzcd}
\pi_0 \mathbb{S} \arrow[r, "2"] \arrow[d, equal] & \pi_0 \mathbb{S} \arrow[r] \arrow[d, equal] & \pi_0(\mathbb{S}/2) \arrow[d, equal] \\
\mathbb{Z} & \mathbb{Z} & \mathbb{Z}/2 \\
\pi_1 \mathbb{S} \arrow[r, "0"] \arrow[d, equal] & \pi_1 \mathbb{S} \arrow[r, "\cong", "j_*"'] \arrow[d, equal] & \pi_1(\mathbb{S}/2) \arrow[d, equal] \arrow[uull, "0"'] \\
\mathbb{Z}/2 & \mathbb{Z}/2 & \mathbb{Z}/2 \\
\pi_2 \mathbb{S} \arrow[r, "0"] \arrow[d, equal] & \pi_2 \mathbb{S} \arrow[r] \arrow[d, equal] & \pi_2(\mathbb{S}/2) \arrow[d, equal] \arrow[uull] \\
\mathbb{Z}/2 & \mathbb{Z}/2 & \mathbb{Z}/4
\end{tikzcd}$$

generated by $j\eta$

__Lemma 1__ The short exact sequence

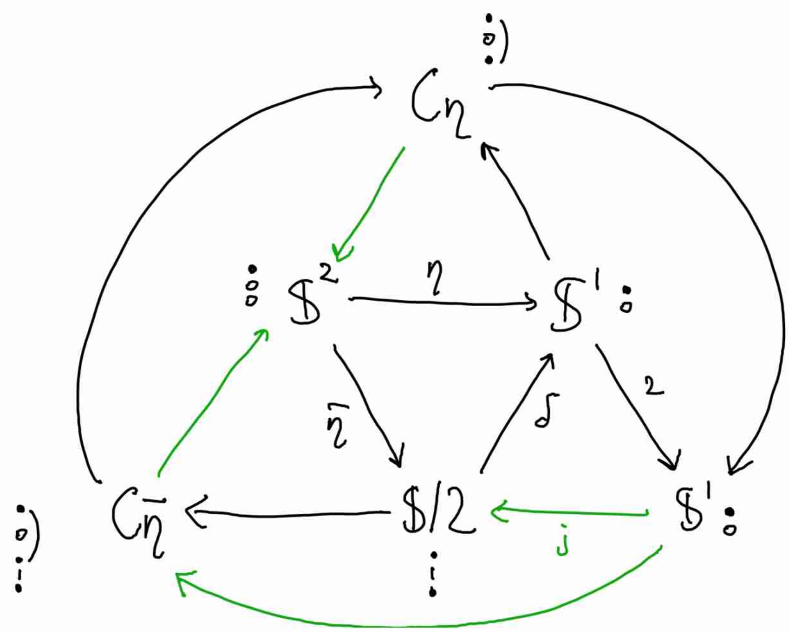
$$0 \longrightarrow \pi_2 \mathbb{S} \longrightarrow \pi_2(\mathbb{S}/2) \xrightarrow{\delta} \pi_1 \mathbb{S} \longrightarrow 0$$

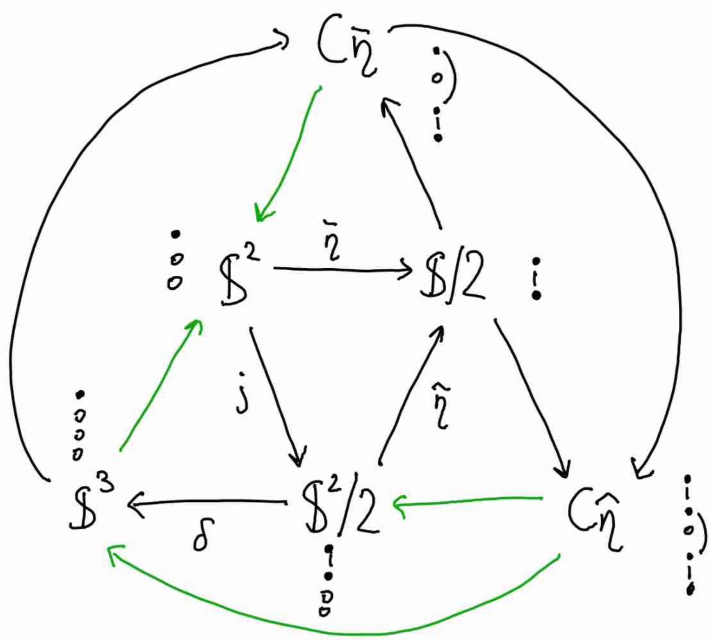
$$\bar{\eta} \longmapsto \eta$$

does not split.

__Proof__ We need to show that $2\bar{\eta} \neq 0$, assume otherwise.

$$\begin{tikzcd}
\mathbb{S}^2 \arrow[r, "2"] & \mathbb{S}^2 \arrow[r, "j"] \arrow[d, "\bar{\eta}"'] & \mathbb{S}^2/2 \arrow[r] \arrow[dl, "\hat{\eta}"] & \mathbb{S}^3 \\
 & \mathbb{S}/2 & &
\end{tikzcd}$$





We have $Sq^1 Sq^2 Sq^1 = Sq^2 Sq^2$ - contradiction. $\square$

$C\hat{\eta} = C(2,\eta,2)$ is a complex that would exist if $0 \in \langle 2,\eta,2 \rangle$.

$$\mathbb{S} \xrightarrow{2} \mathbb{S} \xrightarrow{j} \mathbb{S}/2 \xrightarrow{\delta} \mathbb{S}^1$$

<u>Lemma 2</u> $2 \cdot \mathrm{id}_{\mathbb{S}/2} = j \eta \delta$

<u>Proof</u> Apply $[-, \mathbb{S}/2]$:

$$\begin{tikzcd}
\pi_0(\mathbb{S}/2) \arrow[d, equal] & \pi_0(\mathbb{S}/2) \arrow[l, "0"'] \arrow[d, equal] & {[\mathbb{S}/2, \mathbb{S}/2]} \arrow[l] \arrow[d, equal] \\
\mathbb{Z}/2 & \mathbb{Z}/2 & \mathbb{Z}/4 \\
\pi_1(\mathbb{S}/2) \arrow[urr, bend left, "\delta^*"] \arrow[d, equal] & \pi_1(\mathbb{S}/2) \arrow[l, "0"'] \arrow[d, equal] & \\
\mathbb{Z}/2 & \mathbb{Z}/2 &
\end{tikzcd}$$

since $2 \cdot id_{\mathbb{S}/2} \neq 0$

so $2 \cdot id_{\mathbb{S}/2} = j\eta\delta$

generated by $j\eta$

$\square$

The cone of η

Compute the first few homotopy groups of $C\eta$:

$$\mathbb{S}^1 \xrightarrow{\eta} \mathbb{S} \xrightarrow{j} C\eta \xrightarrow{\delta} \mathbb{S}^2 \quad \text{(different } j \text{ and } \delta\text{)}$$

$$\begin{tikzcd}
\pi_0 \mathbb{S}^1 \arrow[r] \arrow[d, equal] & \pi_0 \mathbb{S} \arrow[r, "\cong"] \arrow[d, equal] & \pi_0 C\eta \arrow[d, equal] \\
0 & \mathbb{Z} & \mathbb{Z} \\
\pi_1 \mathbb{S}^1 \arrow[r] \arrow[d, equal] & \pi_1 \mathbb{S} \arrow[r] \arrow[d, equal] & \pi_1 C\eta \arrow[d, equal] \arrow[uull, bend right] \\
\mathbb{Z} & \mathbb{Z}/2 & 0 \\
\pi_2 \mathbb{S}^1 \arrow[r, "\cong"] \arrow[d, equal] & \pi_2 \mathbb{S} \arrow[r] \arrow[d, equal] & \pi_2 C\eta \arrow[d, equal] \arrow[uull, bend right, "\delta_*"'] \\
\mathbb{Z}/2 & \mathbb{Z}/2 & \mathbb{Z} \\
\pi_3 \mathbb{S}^1 \arrow[r, hook] \arrow[d, equal] & \pi_3 \mathbb{S} \arrow[r, "j_*"] \arrow[d, equal] & \pi_3 C\eta \arrow[d, equal] \arrow[uull, bend right] \\
\mathbb{Z}/2 & \mathbb{Z}/24 & \mathbb{Z}/12
\end{tikzcd}$$

$\ni z$ with $\delta z = 2$

generated by $j\nu$

$$
\begin{tikzcd}
 & & \mathbb{S}^2/2 \arrow[dl] \arrow[ddrr, bend left] & & \\
 & \mathbb{S}^2 \arrow[rr, "2"] \arrow[dr, "z"'] \arrow[dr, phantom, "1", shift left=6] & & \mathbb{S}^2 \arrow[ul] \arrow[dr, "\eta"] \arrow[r, phantom, "2"] & \arrow[r, phantom, "3"] & \\
Cz \arrow[ur] \arrow[uurr, bend left] \arrow[r, phantom, "4"] & & C\eta \arrow[ll] \arrow[ur, "\delta"'] & & \mathbb{S}^1 \arrow[ll] \arrow[llll, bend left]
\end{tikzcd}
$$

Apply π_2 to triangle 1

$$
\begin{tikzcd}
\mathbb{Z} = \pi_2 \mathbb{S}^2 \arrow[rr, hook, "2"] \arrow[dr, "z_*"', "\cong"] & & \pi_2 \mathbb{S}^2 = \mathbb{Z} \\
 & \pi_2 C\eta = \mathbb{Z} \arrow[ur, hook, "\delta_*", "2"'] &
\end{tikzcd}
$$

Apply π_4 to triangle 2:

$$
\begin{tikzcd}
 & \mathbb{Z}/4 = \pi_4(\mathbb{S}^2/2) \arrow[dr, "q_*"] & \\
\mathbb{Z}/2 = \pi_4 \mathbb{S}^2 \arrow[ur, hook] \arrow[rr, hook, "\eta_*"'] & & \pi_4 \mathbb{S}^1 = \mathbb{Z}/24
\end{tikzcd}
$$

It follows that $q_* : \pi_4(\mathbb{S}^2/2) \longrightarrow \pi_4 \mathbb{S}^1$ is a monomorphism.

Compute $\pi_4 C2$ from the outer cofiber sequence ③:

$$\begin{tikzcd}
\pi_4 C2 \arrow[r] \arrow[d, equal] & \pi_4(\mathbb{S}^2/2) \arrow[r, "4_*"] \arrow[d, equal] & \pi_4 \mathbb{S}^1 \arrow[d, equal] \\
0 & \mathbb{Z}/4 & \mathbb{Z}/24 \\
 & & \pi_5 \mathbb{S}^1 \arrow[d, equal] \arrow[uull, bend left] \\
 & & 0
\end{tikzcd}$$

From the cofiber sequence ④:

$$\begin{tikzcd}
\pi_3 \mathbb{S}^2 \arrow[r, "2_*"] \arrow[d, equal] & \pi_3 C\eta \arrow[d, equal] & \\
\mathbb{Z}/12 & \mathbb{Z}/12 & \\
 & & \pi_4 Cx \arrow[d, equal] \arrow[uull, bend left] \\
 & & 0
\end{tikzcd}$$

conclude that $2_* : \pi_3 \mathbb{S}^2 \longrightarrow \pi_3 C\eta$ is a monomorphism.

Lemma 3 $\eta \cdot \mathrm{id}_{C_\eta} = 3 j \nu \delta$

Proof

$$\begin{tikzcd}
\mathbb{Z}/2 = \pi_3 \mathbb{S}^2 \arrow[d, "z_*"'] & \pi_2 \mathbb{S}^2 = \mathbb{Z} \arrow[l] \arrow[d, "\cong"', "z_*"] \\
\pi_3 C_\eta \arrow[d, phantom, "\shortparallel"] \arrow[ddr, bend right, dashed] & \pi_2 C_\eta \arrow[l, "\eta^*"'] \arrow[d, phantom, "\shortparallel"] \\
\mathbb{Z}/12 \text{ gen. by } j\nu & \mathbb{Z} \\
& \pi_1 C_\eta \arrow[d, phantom, "\shortparallel"] & {[\Sigma C_\eta, C_\eta]} \arrow[l] \arrow[d, phantom, "\shortparallel"] \\
& 0 & \mathbb{Z}/6 \text{ generated by } j\nu\delta
\end{tikzcd}$$

im η^* is the subgroup of order 2

$2\eta = 0$ so $\eta \cdot \mathrm{id}_{C_\eta}$ has order 2 so $\eta \cdot \mathrm{id}_{C_\eta} = 3 j \nu \delta$. $\square$

Proof of Thm (for $2x = 0 \Rightarrow \eta x \in \langle 2, x, 2 \rangle$)

$$\mathbb{S} \xrightarrow{2} \mathbb{S} \xrightarrow{j} \mathbb{S}/2 \xrightarrow{\delta} \mathbb{S}^1$$

$$\begin{tikzcd}
\mathbb{S}^k \arrow[r, "2"] & \mathbb{S}^k \arrow[r, "j"] \arrow[dr, "x"'] & \mathbb{S}^k/2 \arrow[r, "\delta"] \arrow[d, "\bar{x}"] & \mathbb{S}^{k+1} \arrow[d, "x\eta"] \\
& & \mathbb{S} \arrow[r, "2"'] & \mathbb{S}
\end{tikzcd}$$

$2\,\mathrm{id}_{\mathbb{S}^k/2} = j \eta \delta \Rightarrow 2\bar{x} = \bar{x} j \eta \delta = x \eta \delta \Rightarrow x\eta \in \langle 2, x, 2 \rangle$ $\square$

Obstructions for constructions of finite spectra

The following is very sketchy!

Let $a \in \pi_p \mathbb{S}$ and $b \in \pi_q \mathbb{S}$. Then $C(a,b)$ is a complex with the cell structure

$$\begin{tikzcd}[row sep=small] p+q+2 & \circ \arrow[d, dash, "b"] \\ p+1 & \circ \arrow[d, dash, "a"] \\ 0 & \circ \end{tikzcd}$$

More precisely $C(a,b)$ is any complex constructed as follows:

$$\begin{tikzcd} \mathbb{S}^p \arrow[r, "a"] & \mathbb{S} \arrow[r, "j_a"] & Ca \arrow[r] & \mathbb{S}^{p+1} \arrow[r, "a", "(-\Sigma a)"'] & \mathbb{S}^1 \\ & & \mathbb{S}^{p+q+r+1} \arrow[ul, "\langle a,b,c \rangle \ni x"] \arrow[r, "c"'] & \mathbb{S}^{p+q+1} \arrow[ul, "\tilde{b}"'] \arrow[u, "b"'] & \end{tikzcd}$$

if $ab=0$

$$\mathbb{S}^{p+q+1} \xrightarrow{\tilde{b}} Ca \longrightarrow C(a,b) \longrightarrow \mathbb{S}^{p+q+2}$$

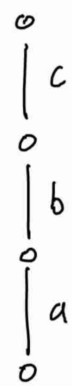
$C(a,b)$ exists if and only if $ab=0$.

<u>Example</u> $\eta^2 \neq 0$ since $C(\eta,\eta)$ doesn't exist

Continue with $c \in \pi_r \mathbb{S}$:

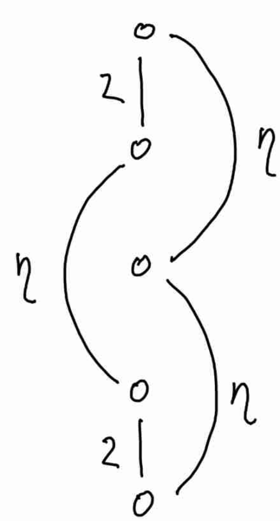
$$\begin{tikzcd}
\mathbb{S}^{p+q+1} \arrow[r, "\tilde{b}"] & Ca \arrow[r] & C(a,b) \arrow[r] & \mathbb{S}^{p+q+2} \arrow[r, "\tilde{b}"] & \Sigma Ca \\
& & & \mathbb{S}^{p+q+r+2} \arrow[u, "c"'] \arrow[ul, "\tilde{c}"] \arrow[r, "x"'] & \mathbb{S}' \arrow[u, "j_a"']
\end{tikzcd}$$

$$\mathbb{S}^{p+q+r+2} \xrightarrow{\tilde{c}} C(a,b) \longrightarrow C(a,b,c) \longrightarrow \mathbb{S}^{p+q+r+3}$$



$C(a,b,c)$ exists if and only if $ab=0$, $bc=0$ and $0 \in \langle a,b,c\rangle$

<u>Example</u> $0 \notin \langle 2,\eta,2\rangle$ since $C(2,\eta,2)$ doesn't exist, but there is complex



since $\eta^2 \in \langle 2,\eta,2\rangle$.