Hopf classes

The mod 2 Steenrod algebra:

$$\mathcal{A}_2 \cong \mathbb{F}_2\langle Sq^1, Sq^2, Sq^3, \dots\rangle / \text{Adem relations}$$

$$|Sq^i| = i$$

$$S^3 \xrightarrow{\eta} S^2 \longrightarrow \mathbb{C}P^2 \longrightarrow \Sigma S^3$$

$$0 \leftarrow H^*S^2 \leftarrow H^*\mathbb{C}P^2 \leftarrow H^*\Sigma S^3 \leftarrow 0$$

$$H^*\mathbb{C}P^2 \cong \mathbb{F}_2[x]/x^3 \quad |x| = 2$$

$Sq^2 x = x^2$ so η is stably essential,

we say that Sq^2 detects η.

$S^1 \xrightarrow{2} S^1 \longrightarrow \mathbb{R}P^2$ $\quad H^*\mathbb{R}P^2 = \mathbb{F}_2[w]/w^3 \quad |w| = 1 \quad$ Sq^1 detects 2

$S^7 \xrightarrow{\nu} S^4 \longrightarrow \mathbb{H}P^2$ $\quad H^*\mathbb{H}P^2 = \mathbb{F}_2[y]/y^3 \quad |y| = 4 \quad$ Sq^4 detects ν

$S^{15} \xrightarrow{\sigma} S^8 \longrightarrow \mathbb{O}P^2$ $\quad H^*\mathbb{O}P^2 = \mathbb{F}_2[z]/z^3 \quad |z| = 8 \quad$ Sq^8 detects σ

$$S^1 \xrightarrow{2} S^1 \longrightarrow M(2) \longrightarrow S^2$$

the mod 2 Moore space

apply $- \wedge M(2)$:

$$\Sigma M(2) \xrightarrow{2} \Sigma M(2) \longrightarrow M(2) \wedge M(2) \longrightarrow \Sigma^2 M(2)$$

Künneth Theorem: $\tilde{H}^*(M(2) \wedge M(2)) \cong \tilde{H}^* M(2) \otimes \tilde{H}^* M(2)$

Cartan Formula: $Sq^m(xy) = \sum_{i+j=m} Sq^i x \cdot Sq^j y$

4	$w^2 \otimes w^2$	
3	$w \otimes w^2$	$w^2 \otimes w$
2	$w \otimes w$	

$$Sq^2(w \otimes w) = Sq^2 w \otimes Sq^0 w + Sq^1 w \otimes Sq^1 w + Sq^0 w \otimes Sq^2 w$$
$$= w^2 \otimes w^2 \neq 0$$

Sq^2 detects $\Sigma M(2) \xrightarrow{2} \Sigma M(2)$

Similarly: Sq^4 detects $S^3 \wedge C\eta \xrightarrow{\eta \wedge id_{C\eta}} S^2 \wedge C\eta$.

<u>Triangulated structure of SHC</u>

$$\begin{tikzcd}
 & & B \arrow[dl] \arrow[d] \arrow[dr, "0"] & & & & & & \\
\Sigma^{-1} Y \arrow[r] & \Sigma^{-1} Cf \arrow[r] & X \arrow[r, "f"'] \arrow[dr, "0"'] & Y \arrow[r] \arrow[d] & Cf \arrow[r] \arrow[dl] & \Sigma X \arrow[r, "-\Sigma f"'] & \Sigma Y \arrow[r] & \Sigma Cf & \cdots \\
 & & & A & & & & &
\end{tikzcd}$$

$$\begin{tikzcd}
X \arrow[rr, "f"] & & Y \arrow[dl] \\
& Cf \arrow[ul, green] &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
A \arrow[r, green] & B
\end{tikzcd}$$

means

$$A \longrightarrow \Sigma B$$

Verdier's octahedron:

$$\begin{tikzcd}
& & C(gf) \arrow[dl, green] \arrow[ddrr, bend left] & & \\
& X \arrow[rr] \arrow[dr, "f"] & & Z \arrow[ul] \arrow[dr] & \\
Cf \arrow[ur, green] \arrow[uurr, bend left] & & Y \arrow[ll] \arrow[ur, "g"'] & & Cg \arrow[ll, green] \arrow[llll, green, bend left]
\end{tikzcd}$$

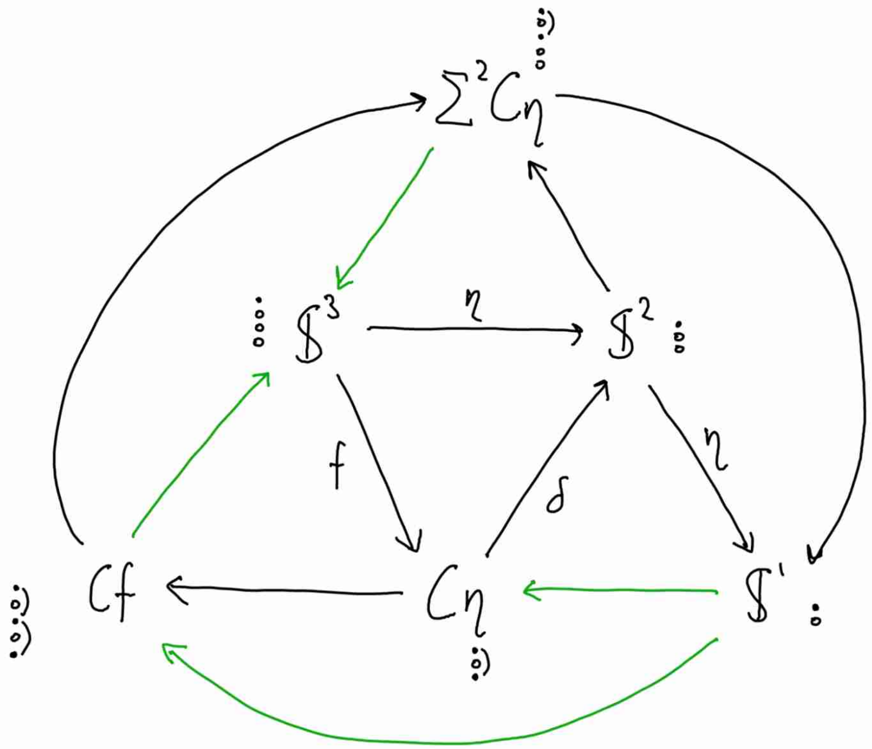
$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d] & Y \arrow[r, "g"] \arrow[d] & Z \arrow[d] \\
* \arrow[r] \arrow[ur, phantom, "\ulcorner", very near end] & Cf \arrow[r] \arrow[d] \arrow[ur, phantom, "\ulcorner", very near end] & C(gf) \arrow[d] \\
& * \arrow[r] \arrow[ur, phantom, "\ulcorner", very near end] & C(g)
\end{tikzcd}$$

Composites of Hopf maps

Lemma 1 η^2 is essential.

Proof Assume that $\eta^2 = 0$

$$
\begin{tikzcd}
 & & & \mathbb{S}^3 \arrow[dl, "f"'] \arrow[d, "\eta"] & \\
\mathbb{S}^1 \arrow[r, "\eta"'] & \mathbb{S} \arrow[r] & C\eta \arrow[r, "\delta"'] & \mathbb{S}^2 \arrow[r, "\eta"'] & \mathbb{S}^1
\end{tikzcd}
$$



$Sq^2 Sq^2 = Sq^3 Sq^1$ – contradiction $\square$

Similarly:

$Sq^4 Sq^4 = Sq^7 Sq^1 + Sq^6 Sq^2 \Rightarrow \nu^2 \neq 0$

$Sq^8 Sq^8 = Sq^{15} Sq^1 + Sq^{14} Sq^2 + Sq^{12} Sq^4 \Rightarrow \sigma^2 \neq 0$

$Sq^1 Sq^4 = Sq^4 Sq^1 + Sq^2 Sq^3 \Rightarrow 2\nu \neq 0$

$Sq^1 Sq^8 = Sq^8 Sq^1 + Sq^2 Sq^7 \Rightarrow 2\sigma \neq 0$

$Sq^2 Sq^8 = Sq^9 Sq^1 + Sq^8 Sq^2 + Sq^4 Sq^6 \Rightarrow \eta\sigma \neq 0$

$2\eta = \eta\nu = \nu\sigma = 0$

Let $k > 0$ and $\mathbb{S}^k \xrightarrow{f} \mathbb{S}$

and Φ a (higher) cohomology operation (of degree $k+1$).

We say that Φ detects f if it acts non-trivially on the generator of $H^0 Cf$.

<u>Lemma 2</u> If $X \xrightarrow{f} Y$ is null in cohomology, θ is a primary cohomology operation of degree q and $y \in H^n Y$ such that $\theta y = 0$. TFAE:

- θ acts non-trivially on all lifts of y to Cf
- $\theta_f(y) = \langle \theta, y, f \rangle$ is non-trivial

<u>Proof</u>

$$\begin{tikzcd}
X \arrow[r, "f"] & Y \arrow[r] \arrow[dr, "y"'] & Cf \arrow[r] \arrow[d, "\tilde{y}"] & \Sigma X \arrow[d, "\hat{y} \in \langle \theta, y, f \rangle"] \\
& & \Sigma^n H \arrow[r, "\theta"'] & \Sigma^{n+q} H
\end{tikzcd}$$

$f^* = 0 \Rightarrow Cf \longrightarrow \Sigma X$ is injective on cohomology

$\Rightarrow \theta\tilde{y} = 0$ iff $\hat{y} = 0$ $\square$

E.g.

$$\begin{tikzcd}
\mathbb{S}^1 \arrow[r, "\eta"] & \mathbb{S} \arrow[r] \arrow[dr, "\iota"'] & C\eta \arrow[r] \arrow[d, "x"] & \mathbb{S}^2 \arrow[d, "\langle Sq^2, \iota, \eta \rangle \neq 0"] \\
& & H \arrow[r, "Sq^2"'] & \Sigma^2 H
\end{tikzcd}$$

Lemma 3 The secondary operation associated to the relation $Sq^2Sq^2 + Sq^3Sq^1 = 0$ detects η^2.

Proof

$$\begin{tikzcd}
 & & C_{\eta^2} \arrow[dl] \arrow[ddrr, bend left] & & \\
 & \mathbb{S}^2 \arrow[rr, "\eta^2"] \arrow[dr, "\eta"] & & \mathbb{S} \arrow[ul] \arrow[dr] & \\
\Sigma C_\eta \arrow[ur] \arrow[uurr, bend left, "f"] & & \mathbb{S}^1 \arrow[ur, "\eta"] \arrow[ll] & & C_\eta \arrow[ll] \arrow[llll, bend left]
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbb{S}^1 \arrow[r, "\eta"] \arrow[d] & \mathbb{S} \arrow[d] \arrow[dr, "\iota"] & & & \\
\Sigma C_\eta \arrow[r, "f"'] & C_{\eta^2} \arrow[r, "u"'] & H \arrow[r, "\begin{bmatrix} Sq^1 \\ Sq^2 \end{bmatrix}"] & \Sigma H \vee \Sigma^2 H \arrow[r, "{[Sq^3 \ Sq^2]}"] & \Sigma^4 H
\end{tikzcd}$$

$\langle Sq^2, \iota, \eta \rangle \neq 0$ in $H^2 \mathbb{S}^2$ (Lemma 2)

$\Rightarrow \langle Sq^2, u, f \rangle \neq 0$ in $H^1 \Sigma C_\eta$ (naturality)

$\Rightarrow Sq^2 \langle Sq^2, u, f \rangle \neq 0$ in $H^3 \Sigma C_\eta$ (Sq^2 detects η)

$Sq^3 \langle Sq^1, u, f \rangle = 0$ in $H^3 \Sigma C_\eta$ (degrees)

$$[Sq^3\ Sq^2]\left\langle \begin{bmatrix} Sq^1 \\ Sq^2 \end{bmatrix}, \upsilon, f \right\rangle \neq 0 \text{ in } H^3\Sigma C\eta$$

$$\Rightarrow f^*\left\langle [Sq^3\ Sq^2], \begin{bmatrix} Sq^1 \\ Sq^2 \end{bmatrix}, \upsilon \right\rangle \neq 0 \text{ in } H^3\Sigma C\eta \quad \text{(PS Formula)}$$

$$\Rightarrow \left\langle [Sq^3\ Sq^2], \begin{bmatrix} Sq^1 \\ Sq^2 \end{bmatrix}, \upsilon \right\rangle \neq 0 \text{ in } H^3 C\eta^2 \quad (f^* \text{ injective in degree } 3) \ \square$$

The Adams Spectral Sequence

Convention: all spectra are 2-complete

<u>Theorem</u> For spectra X and Y of finite type there is a spectral sequence

$$E_2^{s,t} = Ext_{\mathcal{A}}^{s,t}(H^*Y, H^*X) \Rightarrow [X, Y]_{t-s}$$

t - cohomological degree

s - Ext degree, filtration $\qquad d_r : E_r^{s,t} \longrightarrow E_r^{s+r,t+r-1}$

$t-s$ - total degree

Adams resolution of $Y = Y_0$

$$\begin{tikzcd}
\cdots Y_3 \arrow[r, "i_2"] & Y_2 \arrow[r, "i_1"] \arrow[d] & Y_1 \arrow[r, "i_0"] \arrow[d, "p_1"] & Y_0 \arrow[d, "p_0"] \\
\cdots C_{i_2} \arrow[u] & C_{i_1} \arrow[l] \arrow[ul] & C_{i_0} \arrow[l] \arrow[ul] &
\end{tikzcd}$$

p_0^* surjective

C_{i_0} — GEM of finite type

$$
\begin{tikzcd}
 & \cdots & H^*Y_2 \arrow[dr, bend right] & & H^*Y_1 \arrow[ll, "0"'] \arrow[dr, bend right] & & H^*Y_0 \arrow[ll, "0"'] \\
\cdots & H^*Ci_2 \arrow[rr] & & H^*Ci_1 \arrow[ur] \arrow[rr] & & H^*Ci_0 \arrow[ur] &
\end{tikzcd}
$$

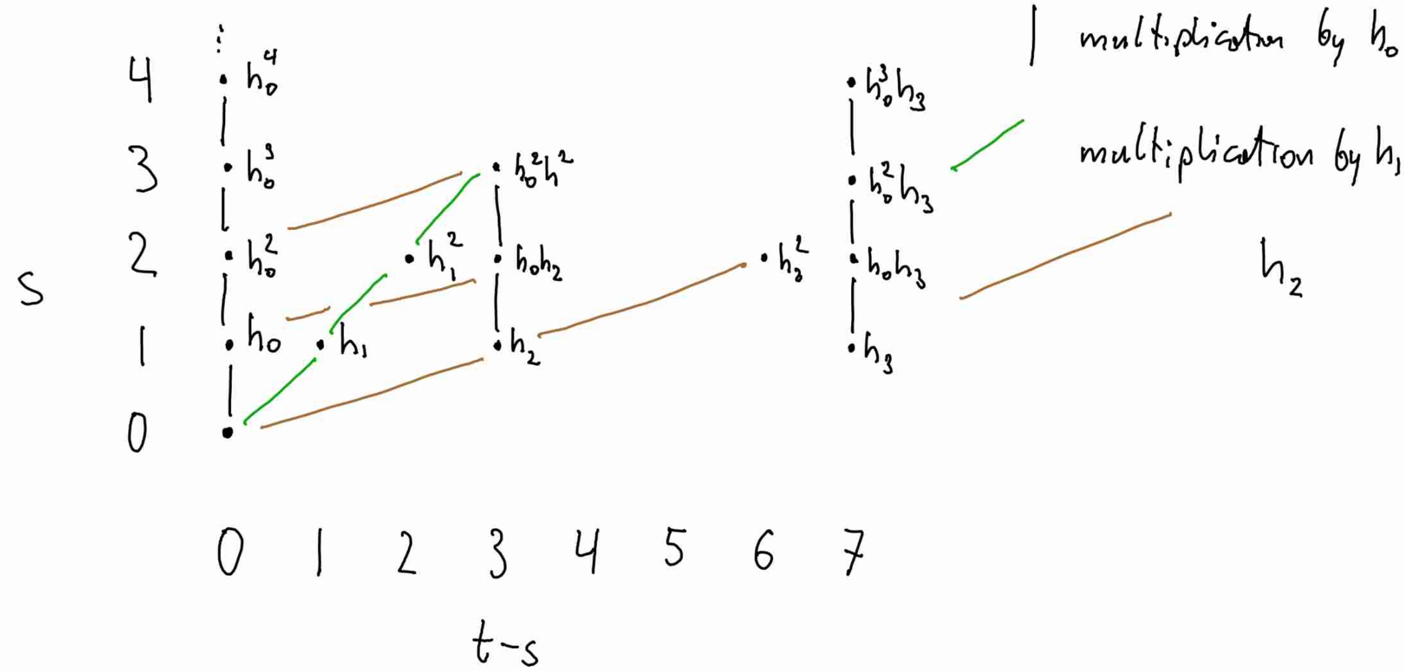
H^*Ci_0 – $\mathcal{A}$-free resolution of H^*Y_0, it computes $\mathrm{Ext}_{\mathcal{A}}^{*,*}(H^*Y, -)$

$$
\begin{tikzcd}
 & & & & & \mathbb{S}^2 \arrow[d, "\eta"] \arrow[ddllll, bend right] \\
 & & & & & \mathbb{S}^1 \arrow[d, "\eta"] \arrow[ddll, bend right] \\
\cdots & Y_2 \arrow[rr, "i_1"] & & Y_1 \arrow[rr, "i_0"] \arrow[dl] & & \mathbb{S} \arrow[dl] \\
 & \cdots & Ci_1 \arrow[ul] & & H \arrow[ll] \arrow[ul] &
\end{tikzcd}
$$

module of relations between Steenrod squares

$$
\begin{tikzcd}
 & & & & & H^*\mathbb{S}^2 \arrow[d, "\eta^*=0"] \\
 & & & & & H^*\mathbb{S}^1 \arrow[d, "\eta^*=0"] \\
\cdots & H^*Y_2 \arrow[uurrrr, bend left] \arrow[dr, bend right] & & H^*Y_1 \arrow[ll, "0"'] \arrow[urr, bend left] \arrow[dr, bend right] & & H^*\mathbb{S} \arrow[ll, "0"'] \\
 & \cdots & H^*Ci_1 \arrow[ur] \arrow[rr] & & H^*H \arrow[ur] &
\end{tikzcd}
$$

$\mathrm{Ext}_{\mathcal{A}}^{s,t}(\mathbb{F}_2, \mathbb{F}_2)$



<u>Theorem (Adams)</u>

- $\mathrm{Ext}_{\mathcal{A}}^{1,1+*}(\mathbb{F}_2, \mathbb{F}_2)$ has a basis $\{h_i \mid i \geq 0\}$ $h_i \in \mathrm{Ext}_{\mathcal{A}}^{1,2^i-2}$.
- $\mathrm{Ext}_{\mathcal{A}}^{2,2+*}(\mathbb{F}_2, \mathbb{F}_2)$ has a basis $\{h_ih_j \mid i,j \geq 0 \ \ |i-j| \neq 1\}$

(in particular $2\eta = \eta\nu = \nu\sigma = 0$).

n	0	1	2	3	4	5	6	7
$\pi_n \mathbb{S}_2^\wedge$	$\mathbb{Z}_2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/8$	0	0	$\mathbb{Z}/2$	$\mathbb{Z}/16$
generators	1	η	η^2	ν			ν^2	σ
relations				$\nu = 4\eta^3$				
$\pi_n \mathbb{S}$	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/24$	0	0	$\mathbb{Z}/2$	$\mathbb{Z}/240$
generators	1	η	η^2	ν			ν^2	σ
relations				$\nu = 12\eta^3$				