The DG-category of secondary cohomology operations

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Outline

Secondary Steenrod algebra

Track categories and linearity tracks

Strictification

Computational aspects

Secondary cohomology operations

Secondary cohomology operations arise from relations between primary cohomology operations.

Setup: X is a spectrum, A, B, C, are finite products of Eilenberg–MacLane spectra.

$$\begin{tikzcd}
X \arrow[r, "x"] \arrow[rr, bend left=50, "0"] \arrow[rr, phantom, "\Uparrow"{yshift=3ex}] & A \arrow[r, "a"] \arrow[rr, bend right=50, "0"'] \arrow[rr, phantom, "\Downarrow"{yshift=-3ex}] & B \arrow[r, "b"] & C
\end{tikzcd}$$

Toda bracket $\langle b, a, x \rangle \subseteq [\Sigma X, C]$.

Can reduce the indeterminacy by fixing the nullhomotopy $ba \Rightarrow 0$.

Example

Adem relation $\mathrm{Sq}^3\mathrm{Sq}^1 + \mathrm{Sq}^2\mathrm{Sq}^2 = 0$. Write $H = H\mathbb{F}_2$.

$$X \xrightarrow{x} \Sigma^n H \xrightarrow{\begin{bmatrix}\mathrm{Sq}^1\\ \mathrm{Sq}^2\end{bmatrix}} \Sigma^{n+1} H \times \Sigma^{n+2} H \xrightarrow{[\,\mathrm{Sq}^3\ \ \mathrm{Sq}^2\,]} \Sigma^{n+4} H$$

$$\left\langle [\,\mathrm{Sq}^3\ \ \mathrm{Sq}^2\,], \begin{bmatrix}\mathrm{Sq}^1\\ \mathrm{Sq}^2\end{bmatrix}, x \right\rangle \subseteq [\Sigma X, \Sigma^{n+4} H] \cong H^{n+3}(X; \mathbb{F}_2)$$

$\left\langle [\,\mathrm{Sq}^3\ \ \mathrm{Sq}^2\,], \begin{bmatrix}\mathrm{Sq}^1\\ \mathrm{Sq}^2\end{bmatrix}, - \right\rangle$ is a secondary operation defined on classes x with $\mathrm{Sq}^1 x = 0$ and $\mathrm{Sq}^2 x = 0$, taking values in a quotient of $H^{n+3}(X; \mathbb{F}_2)$.

Maps and homotopies

Upshot

Secondary operations are encoded by maps between (finite products of) Eilenberg–MacLane spectra along with nullhomotopies.

Definition. Let $\mathcal{EM}$ be the topologically enriched category with objects

$$A = \Sigma^{n_1} H\mathbb{F}_p \times \cdots \times \Sigma^{n_k} H\mathbb{F}_p$$

and mapping spaces between them.

Taking the fundamental groupoid of all mapping spaces yields a groupoid-enriched category $\Pi_1 \mathcal{EM}$ which encodes secondary operations.

Algebra or category?

Remark. The homotopy category $\pi_0\mathcal{EM}$ encodes primary operations.

More precisely, $\pi_0\mathcal{EM}$ is the (multi-sorted Lawvere) theory of modules over the Steenrod algebra $\mathcal{A}^* = H\mathbb{F}_p^* H\mathbb{F}_p$.

$$\mathcal{A}^n = [H\mathbb{F}_p, \Sigma^n H\mathbb{F}_p] = \pi_0 \operatorname{Map}(H\mathbb{F}_p, \Sigma^n H\mathbb{F}_p)$$

What should the "secondary Steenrod algebra" be?

Naive attempt: a graded gadget having in degree n the groupoid

$$\Pi_1 \operatorname{Map}(H\mathbb{F}_p, \Sigma^n H\mathbb{F}_p).$$

Secondary Steenrod algebra

Theorem (Baues 2006)**.** Complete structure of the secondary Steenrod algebra as a "secondary Hopf algebra" over $\mathbb{Z}/p^2$.

Theorem (Nassau 2012)**.** A smaller model weakly equivalent to that of Baues.

One important step: **strictification**. Replace the naive structure by one where composition is strictly bilinear.

Theorem (Baues 2006)**.** $\Pi_1 \mathcal{E}\mathcal{M}$ is weakly equivalent to a 1-truncated DG-category over $\mathbb{Z}/p^2$.

This project

- Streamline the strictification result. Really a fact about *coherence in track categories*.
- Get a more general result.
- Different method: no use of Baues–Wirsching cohomology of categories.
- Approach that can be adapted to tertiary operations (longer term goal).

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Failure of bilinearity

Problem

Composition in $\mathcal{EM}$ is bilinear up to homotopy, but not strictly bilinear.

Eilenberg–MacLane spectra are abelian group objects in spectra. Addition is pointwise in the target.

This makes composition left linear (strictly):

$$X \xrightarrow{x} A \xrightarrow{a} B$$

$$(a + a')x = ax + a'x$$
$$a(x + y) \sim ax + ay.$$

The same is true in $\Pi_1\mathcal{EM}$. Let us describe the structure found in $\Pi_1\mathcal{EM}$.

Track categories

Definition. A **track category** $\mathcal{T}$ is a category enriched in groupoids.

The 2-morphisms are called **tracks**, denoted $\alpha\colon f \Rightarrow g$.

Denote the composition of 2-morphisms by $\beta \square \alpha$, depicted as

$$\begin{tikzcd} X \arrow[r, bend left=60, "f"] \arrow[r, "g" description] \arrow[r, bend right=60, "h"'] \arrow[r, phantom, shift left=3, "\Downarrow\alpha"] \arrow[r, phantom, shift right=3, "\Downarrow\beta"] & Y \end{tikzcd}$$

or as a diagram of tracks

$$f \xRightarrow{\alpha} g \xRightarrow{\beta} h.$$

Left linear track categories

Definition. A track category $\mathcal{T}$ is **left linear** if:

1. $\mathcal{T}$ is enriched in pointed groupoids (i.e., strict zero morphisms).
2. Each mapping groupoid $\mathcal{T}(A, B)$ is an abelian group object in groupoids (with basepoint 0).
3. Composition in $\mathcal{T}$ is left linear.

Example. $\Pi_1 \mathcal{EM}$.

Example. $\Pi_1 \mathcal{EM}_{\mathbb{Z}}$, where we replace $H\mathbb{F}_p$ by $H\mathbb{Z}$.

Definition. A **linearity track** is a track of the form

$$\Gamma_a^{x,y} \colon a(x+y) \Rightarrow ax + ay.$$

Linearity tracks

How do linearity tracks in $\mathcal{T} = \Pi_1 \mathcal{E}\mathcal{M}$ arise? From the stability of spectra.

Products are weak coproducts:

$$\mathcal{T}(A \times B, Z) \xrightarrow[\simeq]{(i_A^*, i_B^*)} \mathcal{T}(A, Z) \times \mathcal{T}(B, Z)$$

is an equivalence of groupoids for every object Z.

$$\begin{tikzcd}
X \arrow[r, "{(x,y)}"] \arrow[dr, "x+y"'] & A \times A \arrow[r, "a \times a"] \arrow[d, "+_A"'] \arrow[dr, phantom, "\Gamma_a \Nearrow"] & B \times B \arrow[d, "+_B"] \\
& A \arrow[r, "a"] & B.
\end{tikzcd}$$

where Γ_a is defined by

$$\begin{cases} i_1^* \Gamma_a = \mathrm{id}_a^{\square} \\ i_2^* \Gamma_a = \mathrm{id}_a^{\square}. \end{cases}$$

Linearity track equations

Proposition (Baues 2006)**.** The linearity tracks $\Gamma_a^{x,y}$ constructed above satisfy the following **linearity track equations**.

1. *Precomposition*:

$$\begin{tikzcd}
a(xz+yz) \arrow[r, Rightarrow, "\Gamma_a^{xz,yz}"] \arrow[d, equal] & axz+ayz \arrow[d, equal] \\
a(x+y)z \arrow[r, Rightarrow, "\Gamma_a^{x,y} z"'] & (ax+ay)z.
\end{tikzcd}$$

2. *Postcomposition*:

$$\begin{tikzcd}
ba(x+y) \arrow[r, Rightarrow, "\Gamma_{ba}^{x,y}"] \arrow[d, Rightarrow, "b\Gamma_a^{x,y}"'] & bax+bay \\
b(ax+ay). \arrow[ur, Rightarrow, "\Gamma_b^{ax,ay}"'] &
\end{tikzcd}$$

Linearity track equations, cont'd

3. *Symmetry*: $\Gamma_a^{x,y} = \Gamma_a^{y,x}$.
4. *Left linearity*: $\Gamma_{a+a'}^{x,y} = \Gamma_a^{x,y} + \Gamma_{a'}^{x,y}$.
5. *Associativity*:

$$
\begin{tikzcd}
a(x+y+z) \arrow[r, Rightarrow, "\Gamma_a^{x+y,z}"] \arrow[d, Rightarrow, "\Gamma_a^{x,y+z}"'] & a(x+y)+az \arrow[d, Rightarrow, "\Gamma_a^{x,y}+az"] \\
ax+a(y+z) \arrow[r, Rightarrow, "ax+\Gamma_a^{y,z}"'] & ax+ay+az.
\end{tikzcd}
$$

Linearity track equations, cont'd

6. *Naturality in x and y*: Given tracks $G\colon x \Rightarrow x'$ and $H\colon y \Rightarrow y'$,

$$\begin{tikzcd} a(x+y) \arrow[r, Rightarrow, "\Gamma_a^{x,y}"] \arrow[d, Rightarrow, "a(G+H)"'] & ax+ay \arrow[d, Rightarrow, "aG+aH"] \\ a(x'+y') \arrow[r, Rightarrow, "\Gamma_a^{x',y'}"'] & ax'+ay'. \end{tikzcd}$$

7. *Naturality in a*: Given a track $\alpha\colon a \Rightarrow a'$,

$$\begin{tikzcd} a(x+y) \arrow[r, Rightarrow, "\Gamma_a^{x,y}"] \arrow[d, Rightarrow, "\alpha(x+y)"'] & ax+ay \arrow[d, Rightarrow, "\alpha x+\alpha y"] \\ a'(x+y) \arrow[r, Rightarrow, "\Gamma_{a'}^{x,y}"'] & a'x+a'y. \end{tikzcd}$$

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Truncated Dold-Kan

Lemma. There is an equivalence of categories

$$\mathbf{Gpd}_{\mathrm{ab}} \cong \mathbf{Ch}_{\leq 1}(\mathbb{Z})$$

sending a groupoid $G = (G_1 \rightrightarrows G_0)$ to the 1-truncated chain complex

$$\ker(d_1) \xrightarrow{d_0|} G_0.$$

Lemma. A left linear track category which is also right linear can be identified with a 1-truncated DG-category.

Remark. The equivalence $\mathbf{Gpd}_{\mathrm{ab}} \cong \mathbf{Ch}_{\leq 1}(\mathbb{Z})$ is not monoidal, but close enough.

Strictification theorem

Theorem (Baues, F.)**.** Let $\mathcal{T}$ be a left linear track category admitting linearity tracks that satisfy the linearity track equations. Then $\mathcal{T}$ is weakly equivalent to a 1-truncated DG-category.

If moreover every morphism in $\mathcal{T}$ is p-torsion, then $\mathcal{T}$ is weakly equivalent to a 1-truncated DG-category over $\mathbb{Z}/p^2$.

This recovers Baues' previous result about strictifying $\Pi_1\mathcal{E}\mathcal{M}$.

Corollary. $\Pi_1\mathcal{E}\mathcal{M}_{\mathbb{Z}}$ is weakly equivalent to a 1-truncated DG-category. In other words, the secondary **integral** Steenrod algebra is strictifiable.

Sketch of proof

- Construct a certain pseudo-functor $s\colon \mathcal{B}_0 \to \mathcal{T}$.
- Jazz up $\mathcal{B}_0$ to a 1-truncated DG-category $\mathcal{B}$.
- Get a pseudo-DK-equivalence $s\colon \mathcal{B} \to \mathcal{F}$. By 2-categorical nonsense, induces a weak equivalence. (Adaptation of an argument due to Lack.)

In particular, no cocycle computation in Baues–Wirsching cohomology.

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A different generalization

Theorem (Baues, Pirashvili 2006)**.** Let $\mathcal{T}$ be an additive track theory. If the hom abelian groups in the homotopy category $\pi_0\mathcal{T}$ are 2-torsion or if 2 acts invertibly on them, then $\mathcal{T}$ is strictifiable.

This argument does *not* work for $\Pi_1\mathcal{EM}_{\mathbb{Z}}$.

Their work relies on computations in Hochschild and Shukla cohomology, along with Baues–Wirsching cohomology.

Massey products

Using a strictification of $\mathcal{T}$, we can compute 3-fold Massey products in $\pi_0\mathcal{T}$.

In particular, the (strictified) secondary Steenrod algebra can be used to compute 3-fold Massey products in the Steenrod algebra $\mathcal{A}^*$.

Example. Writing in the Milnor basis:

$$\mathrm{Sq}(0,1,2) \in \langle \mathrm{Sq}(0,2), \mathrm{Sq}(0,2), \mathrm{Sq}(0,2) \rangle \, .$$

Adams differential d_2

Using the secondary Steenrod algebra, one can compute the differential d_2 in the classical Adams spectral sequence [Baues, Jibladze 2011].

Idea: Resolve the secondary cohomology of X as a module over the secondary Steenrod algebra.

The expression for d_2 is a certain representative of a 3-fold Toda bracket involving two primary operations.

Thank you!