On good morphisms of exact triangles

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Outline

Classical Adams spectral sequence

Given finite spectra X and Y, the classical Adams spectral sequence has the form

$$E_2^{s,t} = \mathrm{Ext}^s_{\mathcal{A}}(H^*Y, \Sigma^t H^*X) \Rightarrow [\Sigma^{t-s}X, Y_p^\wedge]$$

where $\mathcal{A}$ denotes the mod p Steenrod algebra.

For E a nice ring spectrum (e.g. MU or BP), the E-based Adams spectral sequence is:

$$E_2^{s,t} = \mathrm{Ext}^s_{E_*E}(\Sigma^t E_*X, E_*Y) \Rightarrow [\Sigma^{t-s}X, L_E Y].$$

Triangulated version

- Brinkmann (1968): Adams spectral sequence in a triangulated category.
- Franke (1996): Application to $E(1)$-local ($=KU_{(p)}$-local) spectra.
- Further work related to Franke's construction (2007 and on): Roitzheim, Barnes, Patchkoria, and others. Applications to E-local spectra and E-module spectra for various E.
- Christensen (1998): Application to ghost lengths and stable module categories.

Projective and injective classes

Definition. A **projective class** in $\mathcal{T}$ is a pair $(\mathcal{P}, \mathcal{N})$, where $\mathcal{P} \subseteq \operatorname{ob} \mathcal{T}$ and $\mathcal{N} \subseteq \operatorname{mor} \mathcal{T}$, such that:

(i) $\mathcal{P}$ consists of exactly the objects P such that every composite $P \to X \to Y$ is zero for each $X \to Y$ in $\mathcal{N}$.

(ii) $\mathcal{N}$ consists of exactly the maps $X \to Y$ such that every composite $P \to X \to Y$ is zero for each P in $\mathcal{P}$.

(iii) For each X in $\mathcal{T}$, there is a triangle $P \to X \to Y$ with P in $\mathcal{P}$ and $X \to Y$ in $\mathcal{N}$.

An **injective class** in $\mathcal{T}$ is a projective class in $\mathcal{T}^{\mathrm{op}}$.

Examples

Example. In spectra, take:

$$\mathcal{P} = \text{retracts of wedges of spheres } \bigvee_i S^{n_i}$$

$$\mathcal{N} = \text{maps inducing zero on homotopy groups.}$$

$(\mathcal{P}, \mathcal{N})$ is the ghost projective class.

Example. For E any spectrum, take:

$$\mathcal{I} = \text{retracts of products } \prod_i \Sigma^{n_i} E$$

$$\mathcal{N} = \text{maps inducing zero on } E^*(-).$$

Then $(\mathcal{I}, \mathcal{N})$ is an injective class.

For $E = H\mathbb{F}_p$, this injective class leads to the classical (cohomological) Adams spectral sequence.

Examples

Example. For E a homotopy commutative ring spectrum, take:

$$\mathcal{I} = \text{retracts of } E \wedge W$$
$$\mathcal{N} = \text{maps } f\colon X \to Y \text{ with } E \wedge f \simeq 0\colon E \wedge X \to E \wedge Y.$$

The injective class $(\mathcal{I}, \mathcal{N})$ leads to the E-based (homological) Adams spectral sequence.

Remark. We always assume that our projective and injective classes are stable, i.e., closed under suspension and desuspension.

Adams resolutions

Definition. An **Adams resolution** of an object Y in $\mathcal{T}$ with respect to an injective class $(\mathcal{I}, \mathcal{N})$ is a diagram

$$\begin{array}{ccccccccc}
Y = Y_0 & \xleftarrow{i_0} & Y_1 & \xleftarrow{i_1} & Y_2 & \xleftarrow{i_2} & Y_3 & \longleftarrow & \cdots \\
 & {\scriptstyle p_0}\searrow \quad \nearrow{\scriptstyle \delta_0} & & {\scriptstyle p_1}\searrow \quad \nearrow{\scriptstyle \delta_1} & & {\scriptstyle p_2}\searrow \quad \nearrow{\scriptstyle \delta_2} & & & \\
 & I_0 & & I_1 & & I_2 & & \cdots &
\end{array}$$

where each I_s is injective, each map i_s is in $\mathcal{N}$, and the triangles are triangles.

Axiom (iii) says that you can form such a resolution.

Applying $\mathcal{T}(X, -)$ leads to an exact couple and therefore a spectral sequence, called the Adams spectral sequence.

$$E_2^{s,t} = \operatorname{Ext}^s_{\mathcal{I}}(\Sigma^t X, Y)$$

Some structural features

Classical	Triangulated version
d_r as an r^{th} order cohomology operation (Maunder 1964)	✓ (Christensen–F. 2017)
Pairing (Moss 1968)	In progress
Convergence theorem (Moss 1970)	Need the pairing

... But why?

Why work with a triangulated category instead of a stable ∞-category or stable model category?

- Fewer hypotheses.
- There's a lot we can do using only the triangulated structure.
- Get derived invariants.

Moss pairing

Theorem (Moss)**.** For spectra X, Y, and Z, there is a natural associative pairing of Adams spectral sequences

$$E_r^{s,t}(Y,Z) \otimes E_r^{s',t'}(X,Y) \to E_r^{s+s',t+t'}(X,Z)$$

satisfying the following properties:

1. It agrees with the Yoneda pairing of Ext classes on E_2 terms.
2. The differentials d_r satisfy the Leibniz rule.
3. The pairing on E_∞ terms is compatible with the composition product
$$[Y,Z] \otimes [X,Y] \xrightarrow{\circ} [X,Z].$$

Cofibers in an Adams tower

Starting point: an $\mathcal{I}$-Adams tower

$$X = X_0 \xleftarrow{x_1} X_1 \xleftarrow{x_2} X_2 \longleftarrow \cdots$$

For intervals $[n, m] \leq [n', m']$, there is a fill-in in the diagram

$$\begin{array}{ccccccc} X_{m'} & \longrightarrow & X_{n'} & \longrightarrow & X_{n'}/X_{m'} & \longrightarrow & \Sigma X_{m'} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ X_m & \longrightarrow & X_n & \longrightarrow & X_n/X_m & \longrightarrow & \Sigma X_m. \end{array}$$

Question. How convenient can those choices be made?

Outline

Mapping cone

Definition (Neeman)**.** The **mapping cone** of a map of triangles

$$\begin{CD} X @>u>> Y @>v>> Z @>w>> \Sigma X \\ @VfVV @VVgV @VVhV @VV\Sigma fV \\ X' @>>u'> Y' @>>v'> Z' @>>w'> \Sigma X' \end{CD}$$

is the sequence

$$X' \oplus Y \xrightarrow{\begin{bmatrix} u' & g \\ 0 & -v \end{bmatrix}} Y' \oplus Z \xrightarrow{\begin{bmatrix} v' & h \\ 0 & -w \end{bmatrix}} Z' \oplus \Sigma X \xrightarrow{\begin{bmatrix} w' & \Sigma f \\ 0 & -\Sigma u \end{bmatrix}} \Sigma X' \oplus \Sigma Y.$$

The map of triangles (f, g, h) is **good** if its mapping cone is an exact triangle.

Middling good morphisms

Proposition (Neeman)**.** If the map of triangles (f, g, h) is good, then it extends to a 4×4 diagram

$$\begin{array}{ccccccc}
X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & \Sigma X \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle \Sigma f} \\
X' & \xrightarrow{u'} & Y' & \xrightarrow{v'} & Z' & \xrightarrow{w'} & \Sigma X' \\
{\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle g'} & & \vdots{\scriptstyle h'} & & \downarrow{\scriptstyle \Sigma f'} \\
X'' & \overset{u''}{\dashrightarrow} & Y'' & \overset{v''}{\dashrightarrow} & Z'' & \overset{w''}{\dashrightarrow} & \Sigma X'' \\
{\scriptstyle f''}\downarrow & & \downarrow{\scriptstyle g''} & & \vdots{\scriptstyle h''} & \boxed{-1} & \downarrow{\scriptstyle \Sigma f''} \\
\Sigma X & \xrightarrow[\Sigma u]{} & \Sigma Y & \xrightarrow[\Sigma v]{} & \Sigma Z & \xrightarrow[\Sigma w]{} & \Sigma^2 X
\end{array}$$

where the first three rows and columns are exact.

Definition (Neeman)**.** A map of triangles is **middling good** if it extends to a 4×4 diagram.

Verdier good morphisms

Definition. A map of triangles

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & \Sigma X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle \Sigma f} \\ X' & \xrightarrow[u']{} & Y' & \xrightarrow[v']{} & Z' & \xrightarrow[w']{} & \Sigma X' \end{array}$$

is **Verdier good** if h can be constructed as in Verdier's proof of the 4×4 lemma.

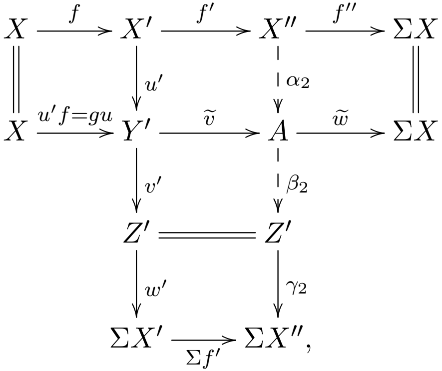
Explicitly: ...

... There exists an octahedron for the composite $X \xrightarrow{u} Y \xrightarrow{g} Y'$:

$$\begin{array}{ccccccc}
X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & \Sigma X \\
\| & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle \alpha_1} & & \| \\
X & \xrightarrow{gu} & Y' & \xrightarrow{\widetilde{v}} & A & \xrightarrow{\widetilde{w}} & \Sigma X \\
 & & \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle \beta_1} & & \\
 & & Y'' & = & Y'' & & \\
 & & \downarrow{\scriptstyle g''} & & \downarrow{\scriptstyle \gamma_1} & & \\
 & & \Sigma Y & \xrightarrow[\Sigma v]{} & \Sigma Z, & &
\end{array}$$

Verdier good morphisms, cont'd

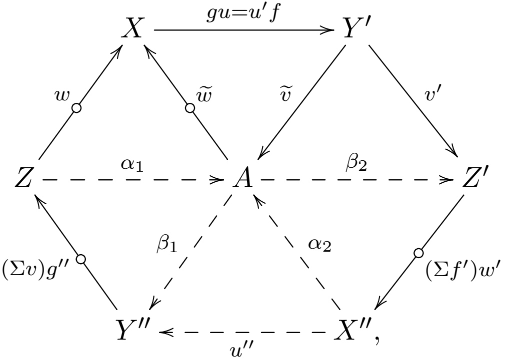
... an octahedron for the composite $X \xrightarrow{f} X' \xrightarrow{u'} Y'$:



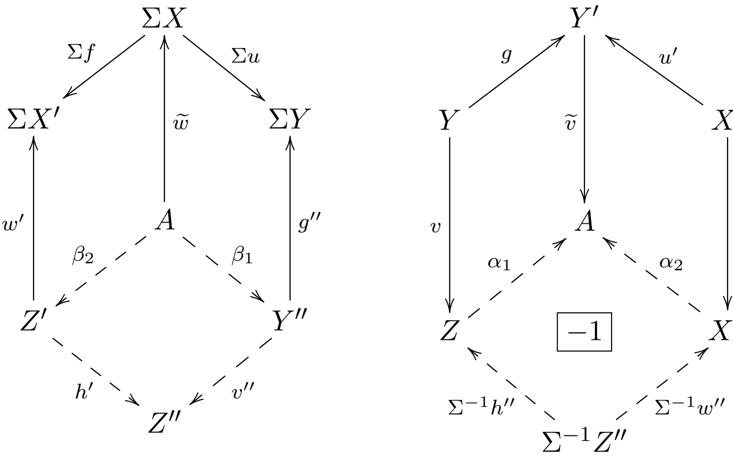
and $h\colon Z \to Z'$ is given by $h = \beta_2 \circ \alpha_1$.

Enhanced 4×4 lemma

Lemma (Miller)**.** A map of triangles (f, g, h) is Verdier good if and only if it extends to a 4×4 diagram and there is an object A (= cofiber of $gu\colon X \to Y'$) together with three diagrams:



Enhanced 4×4 lemma, cont'd



Outline

Adams spectral sequence

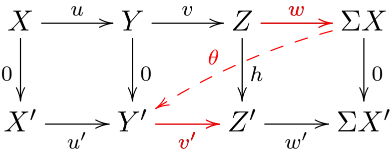
Good morphisms

Examples and non-examples

Questions and results

Fill-in of zero

Example (Neeman)**.** The map of triangles



is good $\Leftrightarrow h = v'\theta w$ for some $\theta\colon \Sigma X \to Y'$.

We call such a map h a "**lightning flash**".

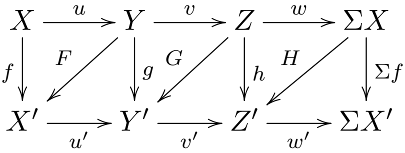
Fact. The map $(0, 0, h)$ above is Verdier good $\Leftrightarrow h = v'\theta w$ for some $\theta\colon \Sigma X \to Y'$.

Similarly for the maps $(0, g, 0)$ and $(f, 0, 0)$.

Remark. Goodness is invariant under rotation. What about Verdier goodness?

Chain homotopy

Definition. A map of triangles (f, g, h) is **nullhomotopic** if there are maps (F, G, H) as in



with

$$\begin{cases} f = Fu + (\Sigma^{-1}w')(\Sigma^{-1}H) \\ g = Gv + u'F \\ h = Hw + v'G. \end{cases}$$

Two maps of triangles (f, g, h) and $(\overline{f}, \overline{g}, \overline{h})$ are **chain homotopic** if their difference is nullhomotopic:

$$(\overline{f} - f, \overline{g} - g, \overline{h} - h) \simeq (0, 0, 0).$$

Chain homotopy invariance

Fact. Chain homotopic maps have isomorphic mapping cones.

$\Rightarrow$ Goodness is invariant under chain homotopy.

What about Verdier goodness?

Proposition (Neeman)**.** If (f, g, h) is Verdier good, then so is $(f, g, h + v'\theta w)$ for any $\theta\colon \Sigma X \to Y'$.

In other words: Adding a lightning flash preserves Verdier goodness.

Contractible triangles

Definition. A triangle $X \to Y \to Z \to \Sigma X$ is **contractible** if its identity map $(1_X, 1_Y, 1_Z)$ is nullhomotopic.

Example. A split triangle $X \to Y \to Z \xrightarrow{0} \Sigma X$ is contractible.

Proposition (Neeman)**.** If the top row or bottom row of

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & \Sigma X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle \Sigma f} \\ X' & \xrightarrow[u']{} & Y' & \xrightarrow[v']{} & Z' & \xrightarrow[w']{} & \Sigma X' \end{array}$$

is contractible, then the map of triangles is Verdier good.

Middling good but not good

Example (Neeman)**.** The map of triangles

$$\begin{array}{ccccccc}
X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & \Sigma X \\
{\scriptstyle u}\downarrow & & \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle w} & & \downarrow{\scriptstyle \Sigma u} \\
Y & \xrightarrow[v]{} & Z & \xrightarrow[w]{} & \Sigma X & \xrightarrow[-\Sigma u]{} & \Sigma Y
\end{array}$$

is always middling good.

It is good $\Leftrightarrow$ $w = w\theta w$ for some $\theta \colon \Sigma X \to Z$.

For instance, with $\mathcal{T}(\Sigma X, Z) = 0$ and $w \neq 0$, the map of triangles is *not* good.

Fact. The map (u, v, w) above is Verdier good $\Leftrightarrow$ $w = w\theta w$ for some $\theta \colon \Sigma X \to Z$.

Not middling good

Example. In the derived category $D(\mathbb{Z})$, the map of triangles

$$\begin{array}{ccccccc}
\mathbb{Z}[0] & \xrightarrow{n} & \mathbb{Z}[0] & \xrightarrow{q} & \mathbb{Z}/n[0] & \xrightarrow{\epsilon} & \mathbb{Z}[1] \\
{\scriptstyle 0}\downarrow & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle \epsilon} & & \downarrow{\scriptstyle 0} \\
\mathbb{Z}[0] & \xrightarrow[q]{} & \mathbb{Z}/n[0] & \xrightarrow[\epsilon]{} & \mathbb{Z}[1] & \xrightarrow[-n]{} & \mathbb{Z}[1]
\end{array}$$

is *not* middling good.

Example. In the stable homotopy category, the map of triangles

$$\begin{array}{ccccccc}
S^0 & \xrightarrow{n} & S^0 & \xrightarrow{q} & M(n) & \xrightarrow{\delta} & S^1 \\
{\scriptstyle 0}\downarrow & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle 0} \\
S^0 & \xrightarrow[q]{} & M(n) & \xrightarrow[\delta]{} & S^1 & \xrightarrow[-n]{} & S^1
\end{array}$$

is *not* middling good.

Middling goodness and Toda brackets

Proposition (Christensen–F.)**.** The map of triangles

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & \Sigma X \\ \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle 0} \\ X' & \xrightarrow[u']{} & Y' & \xrightarrow[v']{} & Z' & \xrightarrow[w']{} & \Sigma X' \end{array}$$

is middling good $\Leftrightarrow$ the Toda bracket $\langle w', h, v \rangle \subseteq \mathcal{T}(\Sigma Y, \Sigma X')$ contains zero.

Middling goodness and Toda brackets, cont'd

Example. The map of triangles

$$\begin{array}{ccccccc}
X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & \Sigma X \\
\downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle w} & & \downarrow{\scriptstyle 0} \\
Y & \xrightarrow[v]{} & Z & \xrightarrow[w]{} & \Sigma X & \xrightarrow[-\Sigma u]{} & \Sigma Y
\end{array}$$

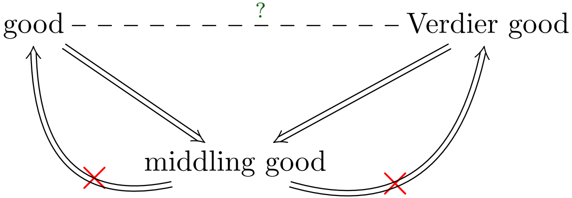
is middling good $\Leftrightarrow 1_{\Sigma Y}$ lies in the indeterminacy subgroup:

$$1_{\Sigma Y} \in (\Sigma u)_* \mathcal{T}(\Sigma Y, \Sigma X) + (\Sigma v)^* \mathcal{T}(\Sigma Z, \Sigma Y).$$

Corollary. Middling goodness is *not* invariant under chain homotopy.

Indeed, consider $(0, 0, w) \simeq (u, v, w)$.

Summary



Outline

Main questions

1. Is Verdier goodness equivalent to goodness?
   (Does one imply the other?)
2. Is Verdier goodness invariant under chain homotopy?
   (Does nullhomotopic imply Verdier good?)
3. Is Verdier goodness invariant under rotation?

Warning: Composition

Proposition (Neeman+ϵ)**.** Every map of triangles is a composite of two maps that are good and Verdier good.

In particular, a composite of Verdier good maps need not be Verdier good.

Some special cases

Consider a map of triangles (f, g, h).

- If f and g are (split) monomorphisms, then good $\Leftrightarrow$ Verdier good.
- If f and g are (split) epimorphisms, then good $\Leftrightarrow$ Verdier good.
- Case $(1, g, h)$: good $\Rightarrow$ Verdier good (Neeman).
- If one component is zero...

A zero component

Lemma. A map of triangles $(f, g, 0)$ is nullhomotopic $\Leftrightarrow$ it is nullhomotopic via a nullhomotopy with a single component $(F, 0, 0)$.

Theorem (Christensen–F.)**.**

1. A map of triangles $(f, g, 0)$ is good (if and) only if it is nullhomotopic.
   Likewise for $(f, 0, h)$ and $(0, g, h)$. (Automatic.)
2. For $(f, 0, h)$ and $(0, g, h)$, the condition is further equivalent to Verdier goodness.

The case $(f, g, 0)$ is still in progress.

Lifting criterion

Corollary. In the diagram with exact rows

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & \Sigma X \\ {\scriptstyle f}\downarrow & \overset{k}{\swarrow} {\scriptstyle ?} & \downarrow{\scriptstyle g} & & \downarrow & & \downarrow{\scriptstyle \Sigma f} \\ X' & \xrightarrow[u']{} & Y' & \xrightarrow[v']{} & Z' & \xrightarrow[w']{} & \Sigma X', \end{array}$$

there exists a lift $k \colon Y \to X'$ satisfying $ku = f$ and $u'k = g$
$\Leftrightarrow$ The map $0 \colon Z \to Z'$ is a good fill-in.

Remark. This situation appears in the Moss pairing.

Thank you!