On good morphisms of exact triangles

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Outline

Adams spectral sequence

Good morphisms

Examples and non-examples

Questions and results

Classical Adams spectral sequence

Given finite spectra X and Y, the classical Adams spectral sequence has the form

$$E_2^{s,t} = \mathrm{Ext}^s_{\mathcal{A}}(H^*Y, \Sigma^t H^*X) \Rightarrow [\Sigma^{t-s}X, Y_p^\wedge]$$

where $\mathcal{A}$ denotes the mod p Steenrod algebra.

For E a nice ring spectrum (e.g. MU or BP), the E-based Adams spectral sequence is:

$$E_2^{s,t} = \mathrm{Ext}^s_{E_*E}(\Sigma^t E_*X, E_*Y) \Rightarrow [\Sigma^{t-s}X, L_E Y].$$

Triangulated version

- Brinkmann (1968): Adams spectral sequence in a triangulated category.
- Franke (1996): Application to $E(1)$-local ($=KU_{(p)}$-local) spectra.
- Further work related to Franke's construction (2007 and on): Roitzheim, Barnes, Patchkoria, and others. Applications to E-local spectra and E-module spectra for various E.
- Christensen (1998): Application to ghost lengths and stable module categories.

Triangulated categories

A **triangulated category** is an additive category $\mathcal{T}$ equipped with a functor $\Sigma : \mathcal{T} \to \mathcal{T}$ that is an equivalence, and with a specified collection of triangles of the form

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X. \tag{1}$$

These must satisfy the following axioms motivated by (co)fiber sequences in topology.

TR0: The triangles are closed under isomorphism. The following is a triangle:

$$X \xrightarrow{1} X \longrightarrow 0 \longrightarrow \Sigma X.$$

TR1: Every map $X \to Y$ is part of a triangle (1).

TR2: (1) is a triangle $\Leftrightarrow$ (2) is a triangle:

$$Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X \xrightarrow{-\Sigma u} \Sigma Y. \tag{2}$$

Triangulated categories, cont'd

$\mathcal{T}$ additive, $\Sigma : \mathcal{T} \to \mathcal{T}$ an equivalence.

TR0: Triangles are closed under isomorphism and contain the trivial triangle.

TR1: Every map appears in a triangle.

TR2: Triangles can be rotated.

TR3: Given a solid diagram

$$\begin{array}{ccccccc} X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & \Sigma X \\ {\scriptstyle f}\downarrow & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \Sigma f} \\ X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & \Sigma X' \end{array}$$

in which the rows are triangles, the dotted fill-in exists making the two squares commute.

TR4: The octahedral axiom holds. (Some details later.)

Examples

Example. The homotopy category of spectra, a.k.a. the stable homotopy category.

Example. The derived category of a ring $D(R)$.

Example. The stable module category of a group algebra $\mathrm{StMod}(kG)$.

Projective and injective classes

Eilenberg and Moore (1965) gave a framework for relative homological algebra in any pointed category. When the category is triangulated, their axioms are equivalent to the following.

Definition. A **projective class** in $\mathcal{T}$ is a pair $(\mathcal{P}, \mathcal{N})$, where $\mathcal{P} \subseteq \operatorname{ob} \mathcal{T}$ and $\mathcal{N} \subseteq \operatorname{mor} \mathcal{T}$, such that:

(i) $\mathcal{P}$ consists of exactly the objects P such that every composite $P \to X \to Y$ is zero for each $X \to Y$ in $\mathcal{N}$.

(ii) $\mathcal{N}$ consists of exactly the maps $X \to Y$ such that every composite $P \to X \to Y$ is zero for each P in $\mathcal{P}$.

(iii) For each X in $\mathcal{T}$, there is a triangle $P \to X \to Y$ with P in $\mathcal{P}$ and $X \to Y$ in $\mathcal{N}$.

An **injective class** in $\mathcal{T}$ is a projective class in $\mathcal{T}^{\mathrm{op}}$.

Examples

Example. In spectra, take:

$$\mathcal{P} = \text{retracts of wedges of spheres } \bigvee_i S^{n_i}$$

$$\mathcal{N} = \text{maps inducing zero on homotopy groups.}$$

$(\mathcal{P}, \mathcal{N})$ is the ghost projective class.

Example. For E any spectrum, take:

$$\mathcal{I} = \text{retracts of products } \prod_i \Sigma^{n_i} E$$

$$\mathcal{N} = \text{maps inducing zero on } E^*(-).$$

Then $(\mathcal{I}, \mathcal{N})$ is an injective class.

For $E = H\mathbb{F}_p$, this injective class leads to the classical (cohomological) Adams spectral sequence.

Examples

Example. For E a homotopy commutative ring spectrum, take:

$$\begin{aligned} \mathcal{I} &= \text{retracts of } E \wedge W \\ \mathcal{N} &= \text{maps } f\colon X \to Y \text{ with } E \wedge f \simeq 0\colon E \wedge X \to E \wedge Y. \end{aligned}$$

The injective class $(\mathcal{I}, \mathcal{N})$ leads to the E-based (homological) Adams spectral sequence.

Remark. We always assume that our projective and injective classes are stable, i.e., closed under suspension and desuspension.

Examples

Example. Let A be a differential graded (dg) algebra. In dg-A-modules, take:

$$\mathcal{P} = \text{retracts of sums } \bigoplus_i A[n_i]$$

$$\mathcal{N} = \text{maps inducing zero on homology.}$$

$(\mathcal{P}, \mathcal{N})$ is the ghost projective class.

The Adams spectral sequence relative to $\mathcal{P}$ is the universal coefficient spectral sequence

$$\mathrm{Ext}^s_{H_*A}(H_*M[t], H_*N) \Rightarrow \mathrm{Ext}^s_A(M[t], N) = D(A)(M[t], N[s])$$

from ordinary Ext to differential Ext.

Remark. A monoidal version also recovers the Eilenberg–Moore spectral sequence

$$\mathrm{Tor}_{H_*A}(H_*M, H_*N) \Rightarrow \mathrm{Tor}_A(M, N).$$

Possible connections with toric topology

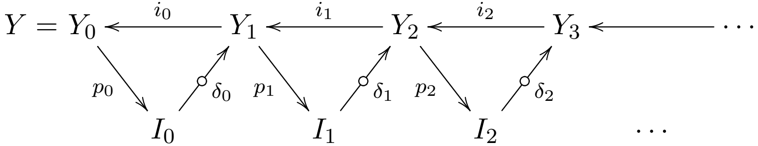
- Tor-algebras $\mathrm{Tor}_A(k,k)$ and Ext-algebras $\mathrm{Ext}_A(k,k)$.
- Computing Massey products in the (co)homology of a dg-algebra. May's convergence theorem (1969).
- Stable splittings:

$$X \overset{\text{stably}}{\sim} \bigvee_i X_i \;?$$

  Detect a map of spectra $\Sigma^\infty X_i \to \Sigma^\infty X$ using an Adams spectral sequence.

Adams resolutions

Definition. An **Adams resolution** of an object Y in $\mathcal{T}$ with respect to an injective class $(\mathcal{I}, \mathcal{N})$ is a diagram



where each I_s is injective, each map i_s is in $\mathcal{N}$, and the triangles are triangles.

Axiom (iii) says that you can form such a resolution.

Applying $\mathcal{T}(X, -)$ leads to an exact couple and therefore a spectral sequence, called the Adams spectral sequence.

$$E_2^{s,t} = \mathrm{Ext}^s_{\mathcal{I}}(\Sigma^t X, Y)$$

Some structural features

Classical	Triangulated version
d_r as an r^{th} order cohomology operation (Maunder 1964)	✓ (Christensen–F. 2017)
Pairing (Moss 1968)	In progress
Convergence theorem (Moss 1970)	Need the pairing

... But why?

Why work with a triangulated category instead of a stable ∞-category or stable model category?

- Fewer hypotheses.
- There's a lot we can do using only the triangulated structure.
- Get derived invariants.

Moss pairing

Theorem (Moss)**.** For spectra X, Y, and Z, there is a natural associative pairing of Adams spectral sequences

$$E_r^{s,t}(Y,Z) \otimes E_r^{s',t'}(X,Y) \to E_r^{s+s',t+t'}(X,Z)$$

satisfying the following properties:

1. It agrees with the Yoneda pairing of Ext classes on E_2 terms.
2. The differentials d_r satisfy the Leibniz rule.
3. The pairing on E_∞ terms is compatible with the composition product
$$[Y,Z] \otimes [X,Y] \xrightarrow{\circ} [X,Z].$$

Cofibers in an Adams tower

Starting point: an $\mathcal{I}$-Adams tower

$$X = X_0 \xleftarrow{x_1} X_1 \xleftarrow{x_2} X_2 \longleftarrow \cdots$$

For intervals $[n, m] \leq [n', m']$, there is a fill-in in the diagram

$$\begin{array}{ccccccc} X_{m'} & \longrightarrow & X_{n'} & \longrightarrow & X_{n'}/X_{m'} & \longrightarrow & \Sigma X_{m'} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ X_m & \longrightarrow & X_n & \longrightarrow & X_n/X_m & \longrightarrow & \Sigma X_m. \end{array}$$

Question. How convenient can those choices be made?

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Mapping cone

Definition (Neeman)**.** The **mapping cone** of a map of triangles

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & \Sigma X \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle \Sigma f} \\ X' & \xrightarrow[u']{} & Y' & \xrightarrow[v']{} & Z' & \xrightarrow[w']{} & \Sigma X' \end{array}$$

is the sequence

$$X' \oplus Y \xrightarrow{\begin{bmatrix} u' & g \\ 0 & -v \end{bmatrix}} Y' \oplus Z \xrightarrow{\begin{bmatrix} v' & h \\ 0 & -w \end{bmatrix}} Z' \oplus \Sigma X \xrightarrow{\begin{bmatrix} w' & \Sigma f \\ 0 & -\Sigma u \end{bmatrix}} \Sigma X' \oplus \Sigma Y.$$

The map of triangles (f, g, h) is **good** if its mapping cone is an exact triangle.

Middling good morphisms

Proposition (Neeman)**.** If the map of triangles (f, g, h) is good, then it extends to a 4×4 diagram

$$
\begin{array}{ccccccc}
X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & \Sigma X \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle \Sigma f} \\
X' & \xrightarrow{u'} & Y' & \xrightarrow{v'} & Z' & \xrightarrow{w'} & \Sigma X' \\
\downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle h'} & & \downarrow{\scriptstyle \Sigma f'} \\
X'' & \xrightarrow{u''} & Y'' & \xrightarrow{v''} & Z'' & \xrightarrow{w''} & \Sigma X'' \\
\downarrow{\scriptstyle f''} & & \downarrow{\scriptstyle g''} & & \downarrow{\scriptstyle h''} & \boxed{-1} & \downarrow{\scriptstyle \Sigma f''} \\
\Sigma X & \xrightarrow[\Sigma u]{} & \Sigma Y & \xrightarrow[\Sigma v]{} & \Sigma Z & \xrightarrow[\Sigma w]{} & \Sigma^2 X
\end{array}
$$

where the first three rows and columns are exact.

Definition (Neeman)**.** A map of triangles is **middling good** if it extends to a 4×4 diagram.

Verdier good morphisms

Definition. A map of triangles

$$\begin{array}{ccccccc}
X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & \Sigma X \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle \Sigma f} \\
X' & \xrightarrow[u']{} & Y' & \xrightarrow[v']{} & Z' & \xrightarrow[w']{} & \Sigma X'
\end{array}$$

is **Verdier good** if h can be constructed as in Verdier's proof of the 4×4 lemma.

Explicitly: ...

Verdier good morphisms, cont'd

... There exists an octahedron for the composite $X \xrightarrow{u} Y \xrightarrow{g} Y'$:

$$\begin{array}{ccccccc}
X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & \Sigma X \\
\| & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle \alpha_1} & & \| \\
X & \xrightarrow{gu} & Y' & \xrightarrow{\widetilde{v}} & A & \xrightarrow{\widetilde{w}} & \Sigma X \\
 & & \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle \beta_1} & & \\
 & & Y'' & = & Y'' & & \\
 & & \downarrow{\scriptstyle g''} & & \downarrow{\scriptstyle \gamma_1} & & \\
 & & \Sigma Y & \xrightarrow[\Sigma v]{} & \Sigma Z, & &
\end{array}$$

Verdier good morphisms, cont'd

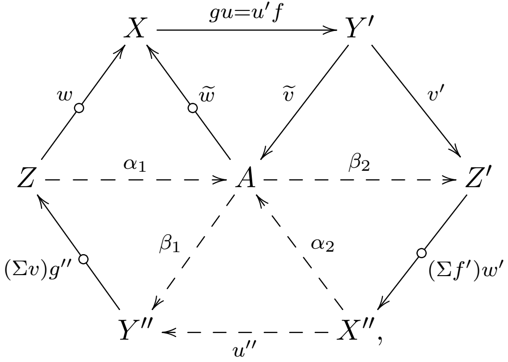
... an octahedron for the composite $X \xrightarrow{f} X' \xrightarrow{u'} Y'$:

$$
\begin{array}{ccccccc}
X & \xrightarrow{f} & X' & \xrightarrow{f'} & X'' & \xrightarrow{f''} & \Sigma X \\
\| & & \downarrow{\scriptstyle u'} & & \downarrow{\scriptstyle \alpha_2} & & \| \\
X & \xrightarrow{u'f=gu} & Y' & \xrightarrow{\widetilde{v}} & A & \xrightarrow{\widetilde{w}} & \Sigma X \\
 & & \downarrow{\scriptstyle v'} & & \downarrow{\scriptstyle \beta_2} & & \\
 & & Z' & = & Z' & & \\
 & & \downarrow{\scriptstyle w'} & & \downarrow{\scriptstyle \gamma_2} & & \\
 & & \Sigma X' & \xrightarrow[\Sigma f']{} & \Sigma X'', & &
\end{array}
$$

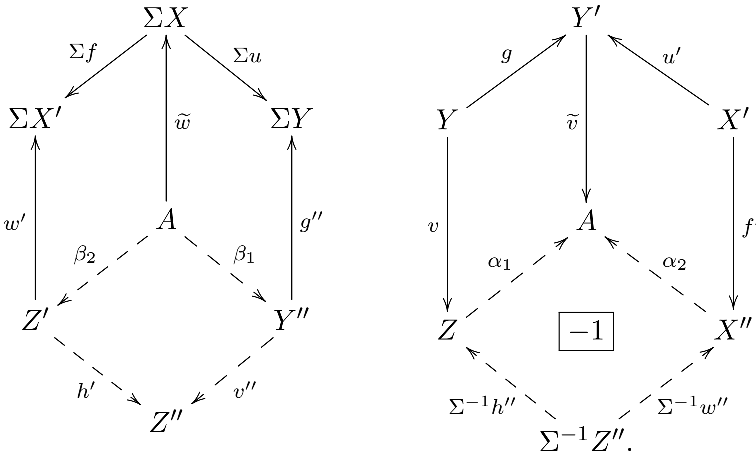
and $h\colon Z \to Z'$ is given by $h = \beta_2 \circ \alpha_1$.

Enhanced 4×4 lemma

Lemma (Miller)**.** A map of triangles (f, g, h) is Verdier good if and only if it extends to a 4×4 diagram and there is an object A (= cofiber of $gu\colon X \to Y'$) together with three diagrams:



Enhanced 4×4 lemma, cont'd



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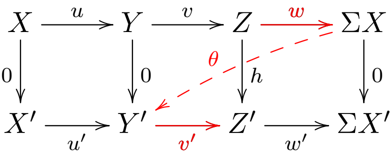
Fill-in of zero

Example (Neeman)**.** The map of triangles



is good $\Leftrightarrow h = v'\theta w$ for some $\theta \colon \Sigma X \to Y'$.

We call such a map h a "**lightning flash**".

Proposition (Christensen–F.)**.** The equivalent conditions above are further equivalent to:

1. The map $(0, 0, h)$ is Verdier good.
2. The map $(0, 0, h)$ is middling good.
3. The Toda bracket $\langle w', h, v \rangle \subseteq \mathcal{T}(\Sigma Y, \Sigma X')$ contains zero.

Similarly for maps $(0, g, 0)$ and $(f, 0, 0)$.

Not middling good

Example. In the derived category $D(\mathbb{Z})$, the map of triangles

$$\begin{array}{ccccccc}
\mathbb{Z}[0] & \xrightarrow{n} & \mathbb{Z}[0] & \xrightarrow{q} & \mathbb{Z}/n[0] & \xrightarrow{\epsilon} & \mathbb{Z}[1] \\
\downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle \epsilon} & & \downarrow{\scriptstyle 0} \\
\mathbb{Z}[0] & \xrightarrow[q]{} & \mathbb{Z}/n[0] & \xrightarrow[\epsilon]{} & \mathbb{Z}[1] & \xrightarrow[-n]{} & \mathbb{Z}[1]
\end{array}$$

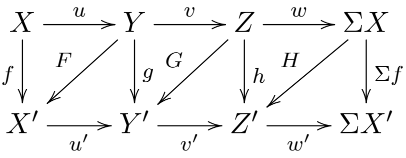
is *not* middling good.

Example. In the stable homotopy category, the map of triangles

$$\begin{array}{ccccccc}
S^0 & \xrightarrow{n} & S^0 & \xrightarrow{q} & M(n) & \xrightarrow{\delta} & S^1 \\
\downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle 0} \\
S^0 & \xrightarrow[q]{} & M(n) & \xrightarrow[\delta]{} & S^1 & \xrightarrow[-n]{} & S^1
\end{array}$$

is *not* middling good.

Definition. A map of triangles (f, g, h) is **nullhomotopic** if there are maps (F, G, H) as in



with

$$\begin{cases} f = Fu + (\Sigma^{-1}w')(\Sigma^{-1}H) \\ g = Gv + u'F \\ h = Hw + v'G. \end{cases}$$

Two maps of triangles (f, g, h) and $(\overline{f}, \overline{g}, \overline{h})$ are **chain homotopic** if their difference is nullhomotopic:

$$(\overline{f} - f, \overline{g} - g, \overline{h} - h) \simeq (0, 0, 0).$$

Chain homotopy invariance

Fact. Chain homotopic maps have isomorphic mapping cones.

$\Rightarrow$ Goodness is invariant under chain homotopy.

What about Verdier goodness?

Proposition (Neeman)**.** If (f, g, h) is Verdier good, then so is $(f, g, h + v'\theta w)$ for any $\theta\colon \Sigma X \to Y'$.

In other words: Adding a lightning flash preserves Verdier goodness.

Contractible triangles

Definition. A triangle $X \to Y \to Z \to \Sigma X$ is **contractible** if its identity map $(1_X, 1_Y, 1_Z)$ is nullhomotopic.

Example. A split triangle $X \to Y \to Z \xrightarrow{0} \Sigma X$ is contractible.

Proposition (Neeman)**.** If the top row or bottom row of

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & \Sigma X \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle \Sigma f} \\ X' & \xrightarrow[u']{} & Y' & \xrightarrow[v']{} & Z' & \xrightarrow[w']{} & \Sigma X' \end{array}$$

is contractible, then the map of triangles is Verdier good.

Middling good but not good

Example (Neeman)**.** The map of triangles

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & \Sigma X \\ \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle w} & & \downarrow{\scriptstyle \Sigma u} \\ Y & \xrightarrow[v]{} & Z & \xrightarrow[w]{} & \Sigma X & \xrightarrow[-\Sigma u]{} & \Sigma Y \end{array}$$

is always middling good.

It is good $\Leftrightarrow$ $w = w\theta w$ for some $\theta\colon \Sigma X \to Z$.

For instance, with $\mathcal{T}(\Sigma X, Z) = 0$ and $w \neq 0$, the map of triangles is *not* good.

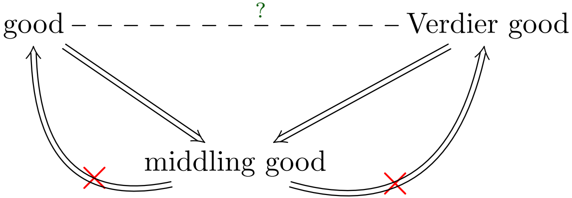
Fact. The map (u, v, w) above is Verdier good $\Leftrightarrow$ $w = w\theta w$ for some $\theta\colon \Sigma X \to Z$.

Middling goodness, cont'd

Corollary. Middling goodness is *not* invariant under chain homotopy.

Indeed, consider $(u, v, w) \simeq (0, 0, w)$.

Summary



Outline

Main questions

1. Is Verdier goodness equivalent to goodness?
   (Does one imply the other?)
2. Is Verdier goodness invariant under chain homotopy?
   (Does nullhomotopic imply Verdier good?)
3. Is Verdier goodness invariant under rotation?

Warning: Composition

Proposition (Neeman+ϵ)**.** Every map of triangles is a composite of two maps that are good and Verdier good.

In particular, a composite of Verdier good maps need not be Verdier good.

Some special cases

Consider a map of triangles (f, g, h).

- If f and g are (split) monomorphisms, then good $\Leftrightarrow$ Verdier good.
- If f and g are (split) epimorphisms, then good $\Leftrightarrow$ Verdier good.
- Case $(1, g, h)$: good $\Rightarrow$ Verdier good (Neeman).
- If one component is zero...

A zero component

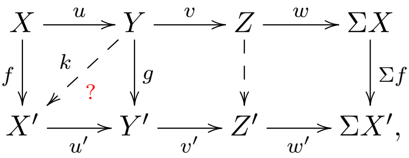
Lemma. A map of triangles $(f, g, 0)$ is nullhomotopic $\Leftrightarrow$ it is nullhomotopic via a nullhomotopy with a single component $(F, 0, 0)$.

Theorem (Christensen–F.)**.**

1. A map of triangles $(f, g, 0)$ is good (if and) only if it is nullhomotopic.
   Likewise for $(f, 0, h)$ and $(0, g, h)$. (Automatic.)
2. For $(f, 0, h)$ and $(0, g, h)$, the condition is further equivalent to Verdier goodness.
3. **If** the triangulated structure admits a 3-triangulated enhancement, then $(f, g, 0)$ being good $\Rightarrow$ Verdier good.

Lifting criterion

Corollary. In the diagram with exact rows



there exists a lift $k\colon Y \to X'$ satisfying $ku = f$ and $u'k = g$
$\Leftrightarrow$ The map $0\colon Z \to Z'$ is a good fill-in.

Remark. This situation appears in the Moss pairing.

Thank you!