Towards the dual motivic Steenrod algebra in positive characteristic

Martin Frankland
University of Regina

Joint with Markus Spitzweck

Session on Topology
2018 CMS Winter Meeting
Vancouver
December 9, 2018

Outline

1. Setup
2. Dual motivic Steenrod algebra
3. Reduction step
4. Hopkins–Morel–Hoyois isomorphism
5. Slices of MGL

Setup

Setup: motivic homotopy theory.

Work over a nice scheme S, called the *base scheme*. (Noetherian, separated, of finite Krull dimension.)

Sm_S = category of smooth schemes of finite type over S.

Motivic spaces = localization of simplicial presheaves on Sm_S with respect to $\mathbb{A}^1$-equivalences and Nisnevich hypercovers.

Motivic spectra = stabilization of pointed motivic spaces with respect to $S^1 \wedge -$ and $\mathbb{G}_m \wedge -$.

Notation: Let $\mathrm{SH}(S)$ denote the motivic stable homotopy category over S. It is a compactly generated tensor triangulated category.

Bigraded spheres

Motivic spheres:

$$S^{p,q} = (S^1)^{\wedge(p-q)} \wedge (\mathbb{G}_m)^{\wedge q}$$

and corresponding suspension functors:

$$\Sigma^{p,q} X = S^{p,q} \wedge X = X(q)[p].$$

Unit for the smash product $\wedge$:

$$\mathbb{1}_S = S^{0,0} = \Sigma^\infty S_+.$$

Example.

- $\mathbb{G}_m = \mathbb{A}^1 - \{0\} = S^{1,1}$.
- $\mathbb{A}^n - \{0\} \simeq S^{2n-1,n}$.
- $\mathbb{P}^1 \simeq \mathbb{A}^1/(\mathbb{A}^1 - \{0\}) \simeq S^1 \wedge \mathbb{G}_m = S^{2,1}$.

Motivic Eilenberg–MacLane spectra

$H\mathbb{Z}$ is a motivic spectrum representing motivic cohomology in $\mathrm{SH}(S)$. $H\mathbb{Z}$ is an E_∞ ring spectrum, in an essentially unique way. Likewise for $H\mathbb{F}_\ell$.

This yields well-behaved categories of modules over $H\mathbb{Z}$ or $H\mathbb{F}_\ell$.

Notation: In case of ambiguity, write the dependency on the base scheme as $H\mathbb{F}_\ell^S$.

Remark. $H\mathbb{F}_\ell$ has complicated homotopy:

$$\begin{aligned}\pi_{p,q}H\mathbb{F}_\ell &= [S^{p,q}, H\mathbb{F}_\ell] \\ &= [S^{0,0}, \Sigma^{-p,-q}H\mathbb{F}_\ell] \\ &= [\Sigma^\infty S_+, \Sigma^{-p,-q}H\mathbb{F}_\ell] \\ &= H^{-p,-q}(S; \mathbb{F}_\ell),\end{aligned}$$

motivic cohomology of the base scheme S.

Classical dual Steenrod algebra

Theorem (Milnor 1958)**.** Let p be an odd prime. The mod p dual Steenrod algebra is

$$\pi_*(H\mathbb{F}_p \wedge H\mathbb{F}_p) = \Lambda_{\mathbb{F}_p}[\tau_0, \tau_1, \tau_2, \ldots] \otimes_{\mathbb{F}_p} \mathbb{F}_p[\xi_1, \xi_2, \ldots],$$

with $|\tau_i| = 2p^i - 1$ and $|\xi_i| = 2p^i - 2$.

In particular, additive basis given by monomials:

$$\tau_0^{\epsilon_0} \xi_1^{r_1} \tau_1^{\epsilon_1} \xi_2^{r_2} \tau_2^{\epsilon_2} \cdots ,$$

with $\epsilon_i \in \{0, 1\}$, $r_i \geq 0$, and finitely many factors — the (dual) *Milnor basis.*

Dual motivic Steenrod algebra

Motivic analogue [Voevodsky]: Classes $\tau_i, \xi_i \in \pi_{*,*}(H\mathbb{F}_\ell \wedge H\mathbb{F}_\ell)$ with bidegree

$$|\tau_i| = (2\ell^i - 1, \ell^i - 1),\ i \geq 0$$
$$|\xi_i| = (2\ell^i - 2, \ell^i - 1),\ i \geq 1.$$

Consider monomials of the form

$$\tau_0^{\epsilon_0}\xi_1^{r_1}\tau_1^{\epsilon_1}\xi_2^{r_2}\tau_2^{\epsilon_2}\cdots \in \pi_{*,*}(H\mathbb{F}_\ell \wedge H\mathbb{F}_\ell)$$

as before.

Consider the induced map of $H\mathbb{F}_\ell$-modules

$$\bigoplus_{\text{monomials}} \Sigma^{p_\alpha,q_\alpha} H\mathbb{F}_\ell \xrightarrow{\Psi^S} H\mathbb{F}_\ell \wedge H\mathbb{F}_\ell.$$

Dual motivic Steenrod algebra, II

Theorem (Voevodsky 2003)**.** Assume that the base scheme S is a field of characteristic zero. Then the map of $H\mathbb{F}_\ell$-modules Ψ^S is an equivalence.

Theorem (Hoyois–Kelly–Østvær 2017)**.** Likewise over a field of characteristic $p \neq \ell$.

Theorem (F.–Spitzweck)**.** Let $S = \mathrm{Spec}(\Bbbk)$ with $\Bbbk$ a field of characteristic p, and $p = \ell$. Then the map $\Psi^{\Bbbk}$ is a split monomorphism, i.e., admits a retraction.

Remark. Also for S essentially smooth over such a field, by base change.

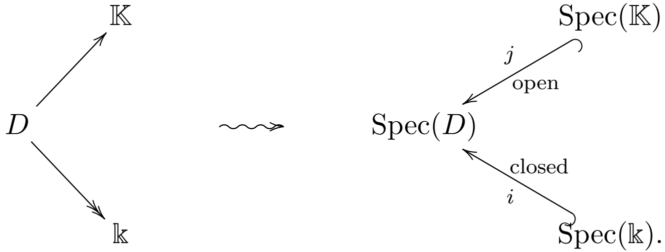
Strategy

Let $\Bbbk$ be a field of characteristic p.

Let D be a discrete valuation ring having $\Bbbk$ as residue field, and a fraction field $\mathbb{K} = \mathrm{Frac}(D)$ of characteristic zero.

Example. $\Bbbk = \mathbb{F}_p$, $D = \mathbb{Z}_p$, the p-adic integers, and $\mathbb{K} = \mathbb{Q}_p$, the p-adic rationals.

Consider the ring maps and induced maps of affine schemes:



Main ingredient

What happens over the closed point $\mathrm{Spec}(\Bbbk)$?

Theorem (Spitzweck 2013)**.** There is an equivalence in $\mathrm{SH}(\Bbbk)$

$$i^! H\mathbb{F}_p^D \simeq \Sigma^{-2,-1} H\mathbb{F}_p^{\Bbbk}.$$

Proposition (F.–Spitzweck)**.** There is an equivalence of $H\mathbb{F}_p^{\Bbbk}$-module spectra

$$i^* j_* H\mathbb{F}_p^{\mathbb{K}} \simeq H\mathbb{F}_p^{\Bbbk} \oplus \Sigma^{-1,-1} H\mathbb{F}_p^{\Bbbk}.$$

One application: Reduction step

Definition. An object of $\mathrm{SH}(S)$ is **lisse** if it lies in the full localizing triangulated subcategory of $\mathrm{SH}(S)$ generated by the strongly dualizable objects.

Remark. In the classical stable homotopy category, *every* object is lisse in this sense.

In fact, this is one of the axioms of a stable homotopy theory in the sense of Hovey–Palmieri–Strickland.

If $\mathbb{K}$ is a field of characteristic zero, then every object of $\mathrm{SH}(\mathbb{K})$ is lisse (Röndigs–Østvær 2008).

Reduction step, II

Proposition (F.–Spitzweck)**.** If $H\mathbb{F}_p^{\mathbb{Z}_p}$ is lisse in $\mathrm{SH}(\mathbb{Z}_p)$, then the map

$$\bigoplus_{\text{monomials}} \Sigma^{p_\alpha, q_\alpha} H\mathbb{F}_p^{\Bbbk} \xrightarrow{\Psi^{\Bbbk}} H\mathbb{F}_p^{\Bbbk} \wedge H\mathbb{F}_p^{\Bbbk}$$

is an equivalence.

In other words, the structure theorem for the dual motivic Steenrod algebra would hold over $\Bbbk$.

Hopkins–Morel–Hoyois isomorphism

In classical homotopy theory: complex cobordism MU, with

$$\pi_* MU \cong \mathbb{Z}[a_1, a_2, \ldots],\ |a_i| = 2i.$$

The orientation map $MU \to H\mathbb{Z}$ induces a map

$$MU/(a_1, a_2, \ldots) \xrightarrow{\simeq} H\mathbb{Z}$$

which is an equivalence.

Hopkins–Morel–Hoyois, II

In motivic homotopy theory: algebraic cobordism spectrum MGL, with $a_i \in \pi_{2i,i}\mathrm{MGL}$.

The orientation map $\mathrm{MGL} \to H\mathbb{Z}$ induces a map

$$\mathrm{MGL}/(a_1, a_2, \ldots) \xrightarrow{\Phi} H\mathbb{Z}$$

Theorem (Hopkins–Morel 2004; Hoyois 2015)**.** Let $S = \mathrm{Spec}(\Bbbk)$ for a field $\Bbbk$.

1. In the case $\mathrm{char}(\Bbbk) = 0$, then Φ is an equivalence.
2. In the case $\mathrm{char}(\Bbbk) = p$, then Φ becomes an equivalence after inverting p.

Hopkins–Morel–Hoyois, III

Key step: For any prime number $\ell \neq p$,

$$H\mathbb{F}_\ell \wedge \mathrm{MGL}/(a_1, a_2, \ldots) \xrightarrow{H\mathbb{F}_\ell \wedge \Phi} H\mathbb{F}_\ell \wedge H\mathbb{Z}$$

is an equivalence.

Key ingredient: dual motivic Steenrod algebra, to compute the right-hand side.

Slice filtration

Definition. Let $\mathrm{SH}(S)^{\mathrm{eff}} \subseteq \mathrm{SH}(S)$ be the localizing triangulated subcategory generated by objects $\Sigma^\infty X_+$ for $X \in \mathrm{Sm}_S$. Those are called **effective spectra**.

n**-effective** spectra: $\Sigma^{*,n}\mathrm{SH}(S)^{\mathrm{eff}}$.

n-effective covers $f_n E \to E$ form a natural tower

$$\cdots \longrightarrow f_2 E \longrightarrow f_1 E \longrightarrow f_0 E \longrightarrow f_{-1} E \longrightarrow \cdots .$$

$$\begin{array}{ccccccccc} \cdots \longrightarrow & f_2 E & \longrightarrow & f_1 E & \longrightarrow & f_0 E & \longrightarrow & f_{-1} E & \longrightarrow \cdots . \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\ & s_2 E & & s_1 E & & s_0 E & & s_{-1} E & \end{array}$$

The cofiber $s_n E$ is called the n**-slice** of E.

Slices of MGL

Theorem (Voevodsky 2004; Levine 2008)**.** Over a base field, $s_0 \mathbb{1} \cong H\mathbb{Z}$.

In particular, all slices $s_n E$ are canonically $H\mathbb{Z}$-modules.

Canonical map $L_* \to \pi_{2*,*}\mathrm{MGL}$, where L_* is the Lazard ring $L_n \cong \pi_{2n} MU$.

Induces a map of $H\mathbb{Z}$-modules

$$\Sigma^{2n,n} H\mathbb{Z} \otimes_{\mathbb{Z}} L_n \xrightarrow{\psi_S} s_n\mathrm{MGL}.$$

Slices of MGL, II

Theorem (Voevodsky)**.** Over a base field $\mathbb{K}$ of characteristic zero, $\psi_{\mathbb{K}}$ is an equivalence.

Theorem (Spitzweck 2010)**.** Hopkins–Morel over S implies the conjecture on $s_n(\mathrm{MGL})$, i.e., that ψ_S is an equivalence.

Theorem (Hoyois 2015)**.** Over a base field $\Bbbk$ of characteristic p, $\psi_{\Bbbk}$ becomes an equivalence after inverting p.

Theorem (F.–Spitzweck)**.** Over a field $\Bbbk$ of characteristic p, the map $\psi_{\Bbbk}$ is a split monomorphism, i.e., admits a retraction.

Thank you!