Calculus 2502A - Advanced Calculus I

Fall 2014
Practice Final

Name (please print): ______________________________

Student ID: ________________________

- On the real exam, no calculators, electronic devices, books, or notes may be used.
- Show your work. No credit for answers without justification.
- Good luck!

This practice exam is longer than the real exam.

List of formulas.

Curvature: $\kappa(t) = |\frac{d\vec{T}}{ds}|$, where $\vec{T}$ = unit tangent vector, and s = arc length.

Newton's Law: $\vec{F} = m\vec{a}$, where $\vec{F}$ = force, m = mass, and $\vec{a}$ = acceleration.

Problem 1. (10 points) Find the acute angle between any two distinct diagonals of a cube. Here, a diagonal is the segment joining two opposite vertices, that is, vertices that do not lie on any common face.

Problem 2. (10 points) Consider the line $-4x + 2y = 1$ in $\mathbb{R}^2$.

(a) Give a parametric equation for the line.

(b) Find a vector in $\mathbb{R}^2$ that is perpendicular to the line.

(c) Find the distance between the line and the point $(4, 3)$ in $\mathbb{R}^2$.

Problem 3. (10 points) Find a vector equation for the line in $\mathbb{R}^3$ through the point $(2, 1, 0)$ that is parallel to the plane $x - y + z = 3$ and perpendicular to the line with parametric equations $x(t) = 2 - t$, $y(t) = 2t$, $z(t) = 1 + t$.

Problem 4. (10 points) Let C be a curve with a smooth parametrization $\vec{r}: I \to \mathbb{R}^3$, that is, $\vec{r}\,'(t) \neq \vec{0}$ holds for all $t \in I$. Assume I is an interval. Show that the curvature of C is identically zero if and only if C lies entirely on a line.

(More space.)

Problem 5. (10 points) Let C be a curve with parametrization $\vec{r}: I \to \mathbb{R}^3$ whose component functions are polynomials of degree at most 2. Show that C is a *planar* curve, i.e., C lies entirely within some plane in $\mathbb{R}^3$.

Problem 6. (10 points) Consider the function $f(x, y) = x^3y^2$. Find the linear approximation of f at $(1, 2)$ and use it to estimate $f(0.9, 2.2)$

Problem 7. (10 points) A function $f(x, y)$ has partial derivatives $\frac{\partial f}{\partial x}(4, 3) = 2$ and $\frac{\partial f}{\partial y}(4, 3) = 7$.

Find the partial derivatives $\frac{\partial f}{\partial r}$ and $\frac{\partial f}{\partial \theta}$ at the point $(x, y) = (4, 3)$, where (r, θ) are the usual polar coordinates.

Problem 8. (10 points) Consider the function $f(x,y,z) = xe^y + yz^2$, and the unit vector $\vec{u} = (\frac{1}{\sqrt{2}}, 0, \frac{-1}{\sqrt{2}})$. Compute $D_{\vec{u}}f(0,1,2)$, the directional derivative of f in the direction $\vec{u}$ at the point $(0,1,2)$.

Problem 9. (10 points) Let C be the curve of intersection of the paraboloid $y = x^2 + z^2$ and the ellipsoid $4x^2 + y^2 + z^2 = 9$. Find a vector equation for the tangent line to C at the point $(-1, 2, 1)$.

Problem 10. (10 points) Consider the function

$$f(x, y) = 2\cos(x) - y^2 + 2y.$$

(a) Find all the critical points of f in the domain $\{(x, y) \in \mathbb{R}^2 \mid -1 < x < 5\}$.

(b) For each point you found in part (a), determine if it is a local maximum, local minimum, or a saddle point. Justify your answer.

Problem 11. (10 points) Find the **minimum value** of the function $f(x, y, z) = x + 2y + 3z$ on the upper sheet of the hyperboloid $x^2 + y^2 - z^2 = -1$. (Here "upper sheet" means the part where $z > 0$.)

Problem 12. (10 points) Let $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the function defined by

$$f(x,y) = \big(3x + y^2 + 1, xy + 5\big)$$

and consider the function $h\colon \mathbb{R}^3 \to \mathbb{R}^2$ defined by $h(x,y,z) = f\left(ye^z, x^3 + z - 2\right)$.

Find the linear approximation of h at the point $(1,2,0)$ and use it to approximate $h(1.01, 1.95, 0.03)$.

(More space.)

Multiple choice section

Please circle one answer for each problem.

Problem 13. (4 points) Four of the following equations are true for *all* vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$ in $\mathbb{R}^3$. Which one is *false* for some vectors?

(A) $\vec{v} \cdot \vec{w} = \vec{w} \cdot \vec{v}$

(B) $|\vec{v} \times \vec{w}| = |\vec{w} \times \vec{v}|$

(C) $\vec{u} \cdot (\vec{v} \times \vec{w}) = \vec{v} \cdot (\vec{w} \times \vec{u})$

(D) $\vec{u} \times (\vec{v} \times \vec{w}) = (\vec{u} \times \vec{v}) \times \vec{w}$

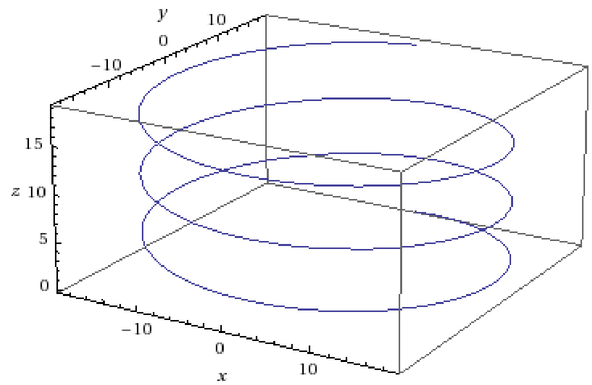
(E) $\vec{u} \times (\vec{v} + \vec{w}) = (\vec{u} \times \vec{w}) + (\vec{u} \times \vec{v})$

Problem 14. (4 points) Consider the curve parametrized by

$$\vec{r}(t) = (t\sin t, t\cos t, t)\,,\ 0 \le t \le 6\pi.$$

Which of the following is a plot of that curve?

(A)



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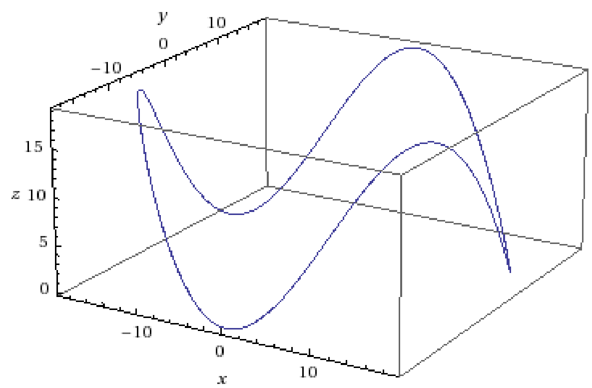

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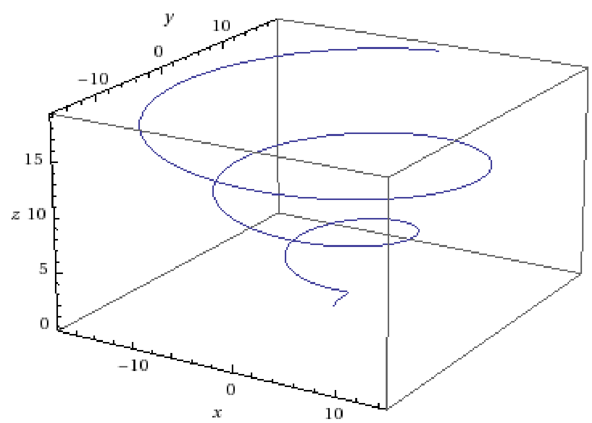

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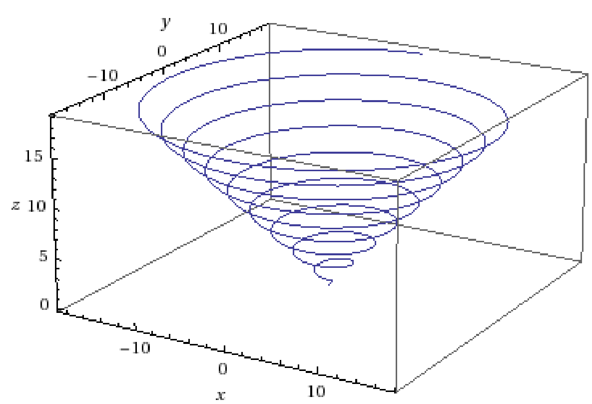
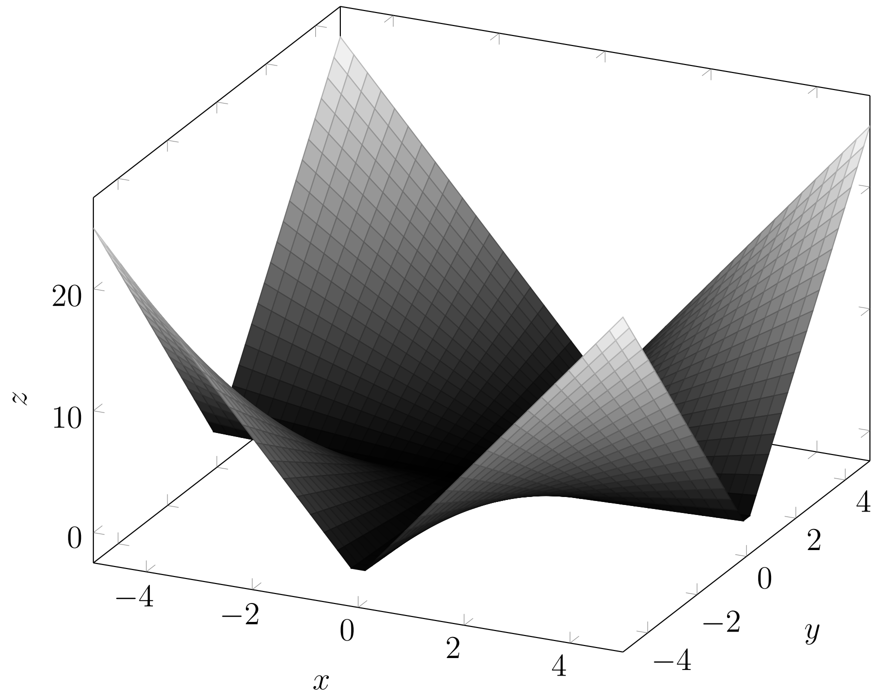




Problem 15. (4 points) Consider the graph depicted above. It is the graph of which function?

(A) $|xy|$

(B) $|x| + |y|$

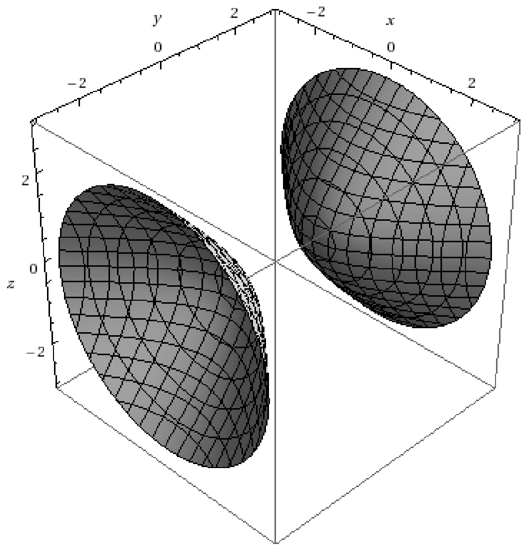
(C) $|x|^2 + |y|^2$

(D) $x^2 - y^2$

(E) $(x - y)^2$

Problem 16. (4 points) Which of the following figures represent level sets of the function $f(x, y, z) = x^2 - y^2 + z^2 + 1$?

(A)



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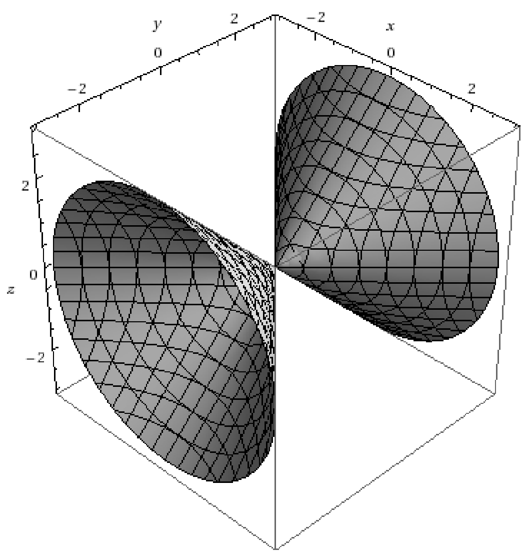

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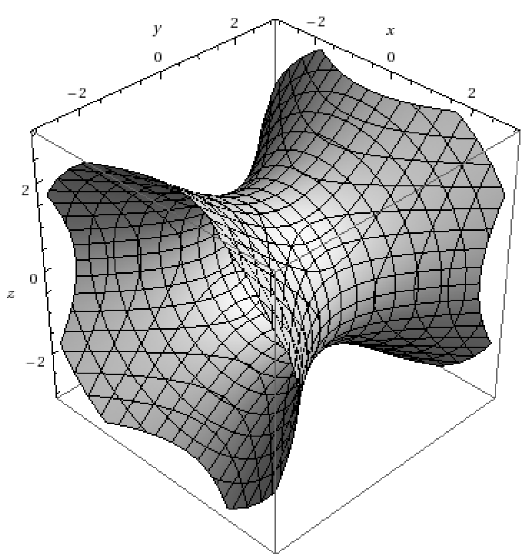
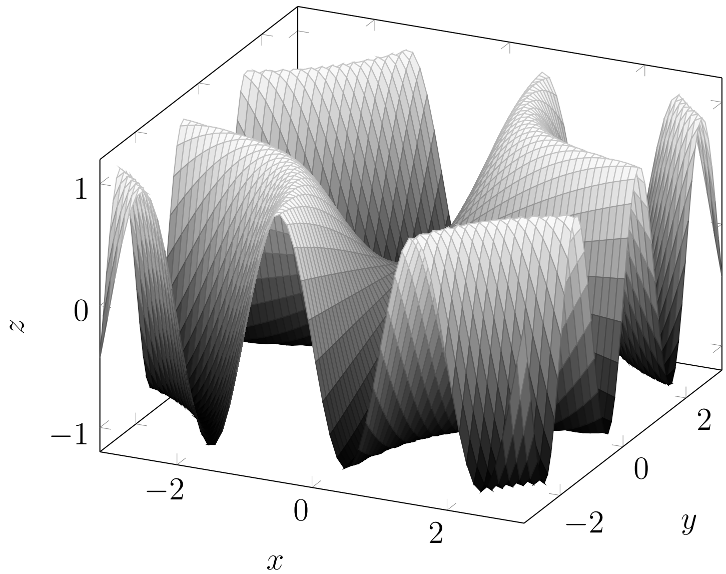
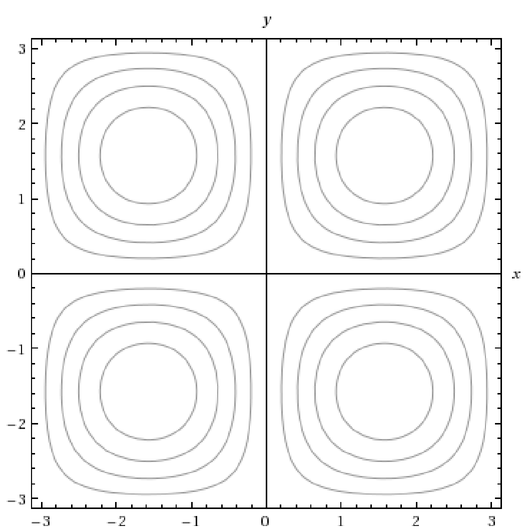


(D) All of the above.



Problem 17. (4 points) Consider the graph depicted above. Which of the following is a contour plot of the same function?

(A)



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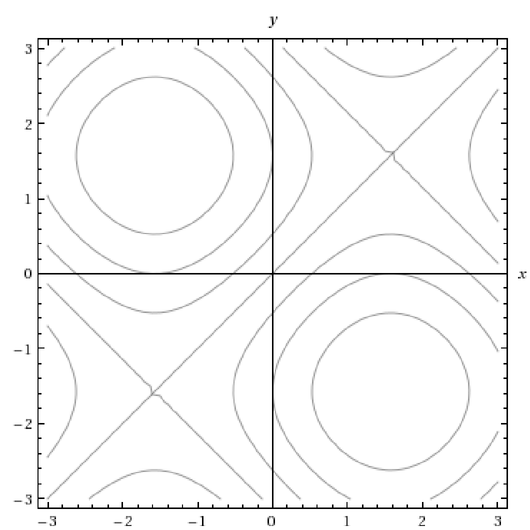
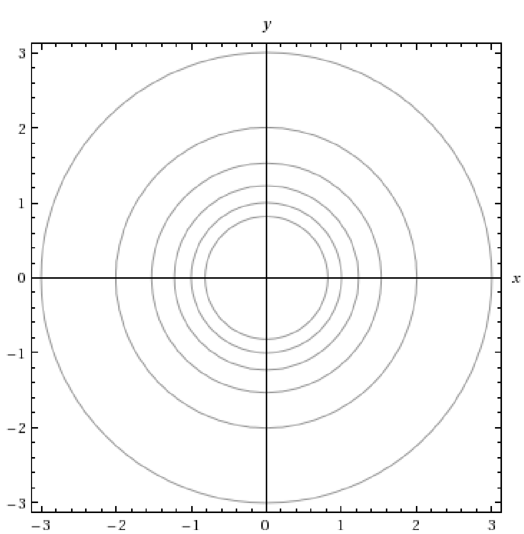
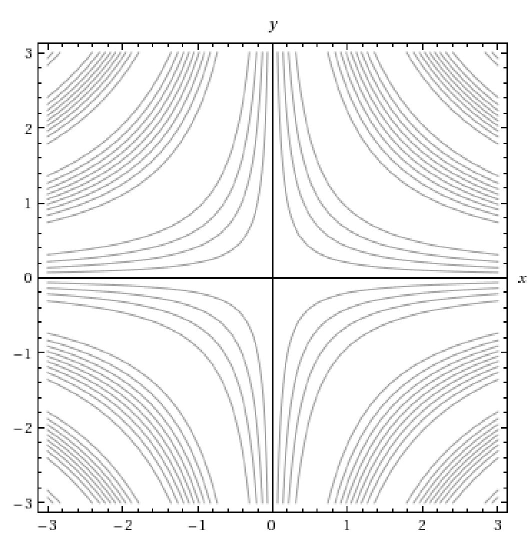


(C)



(D)



Problem 18. (4 points) Find the following limit, if it exists.

$$\lim_{(x,y)\to(0,0)} \frac{4x^2 + 7x^2y^2 + 4y^2}{x^2 + y^2}$$

(A) 0

(B) 4

(C) 7

(D) 14

(E) It does not exist.

Problem 19. (4 points) Let f be a function of two variables such that $f(x, y) \to 5$ as (x, y) approaches $(1, 2)$ along every straight line through $(1, 2)$. What can you say about the limit $\lim_{(x,y)\to(1,2)} f(x, y)$?

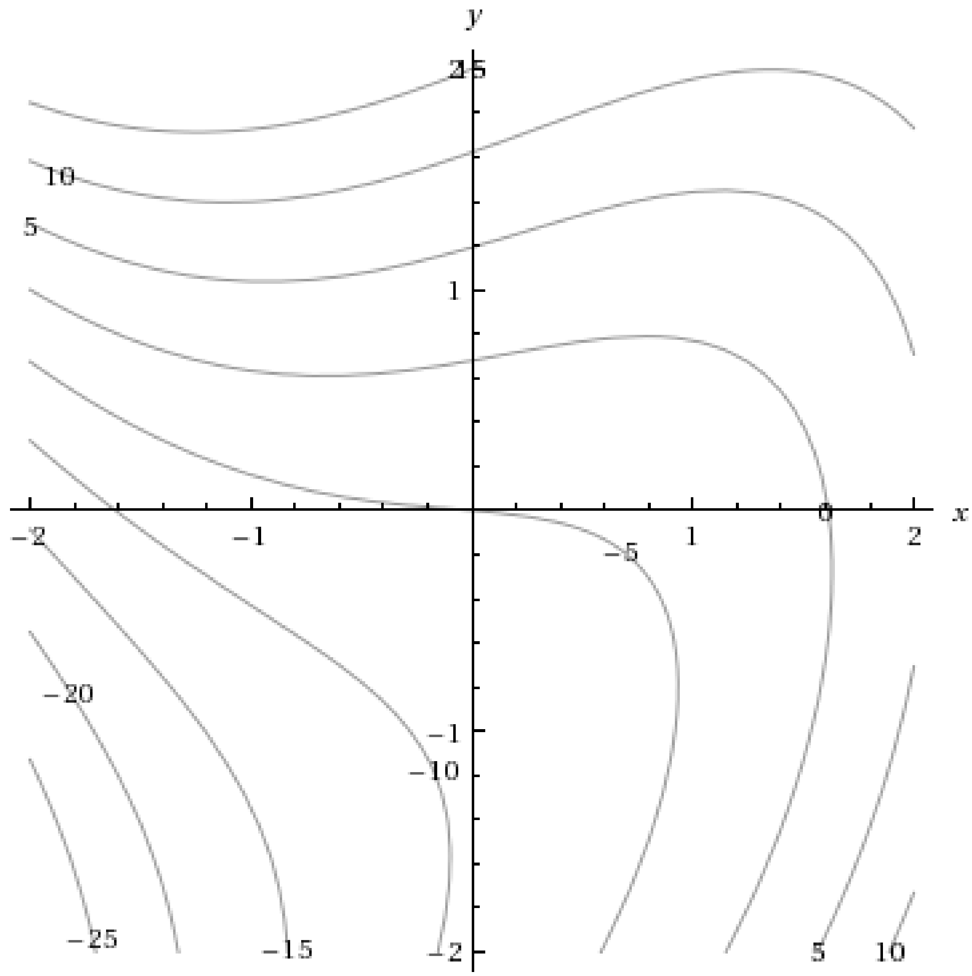
(A) It exists and is equal to 5.

(B) It exists and is equal to $f(1, 2)$.

(C) It may or may not exist, but if it does exist it must be equal to 5.

(D) It may or may not exist, and it could be any number.

(E) It does not exist.



Problem 20. (4 points) The contour plot of a function $f(x, y)$ is drawn above. Which of the following quantities is negative at $(0, 0)$? (There is exactly one correct answer.)

(A) $\frac{\partial^2 f}{\partial x^2}$

(B) $\frac{\partial^2 f}{\partial x \partial y}$

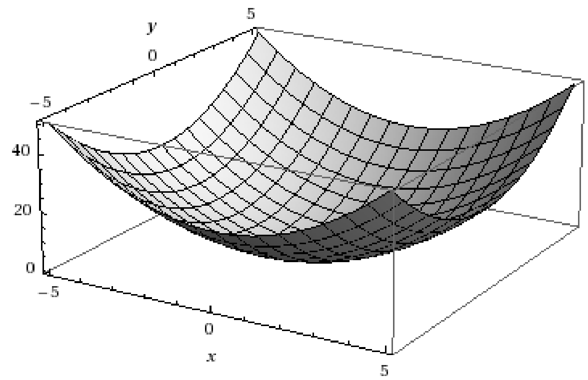
(C) $\frac{\partial^2 f}{\partial y^2}$

(D) $\frac{\partial f}{\partial x}$

(E) $\frac{\partial f}{\partial y}$

Problem 21. (4 points) Here are the graphs of some functions $f(x, y)$. Pick the function that satisfies the equation $f_{xx} + f_{yy} = 0$.

(A)



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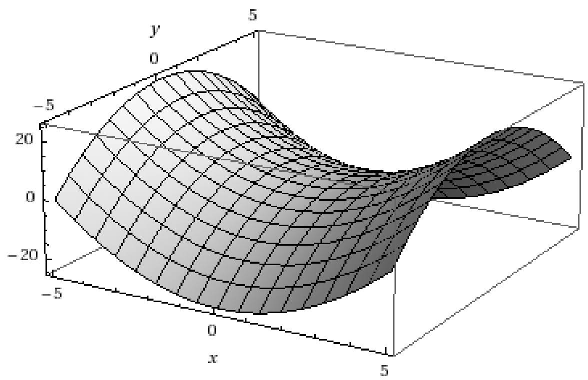
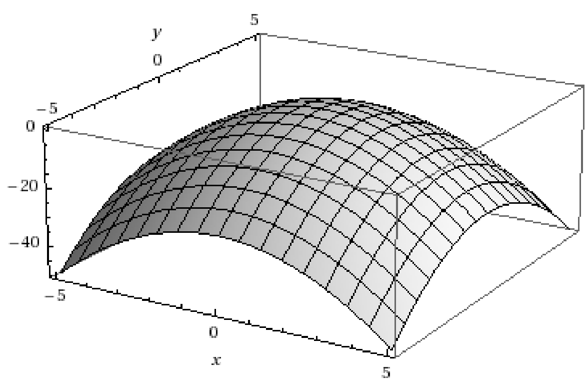


(C)



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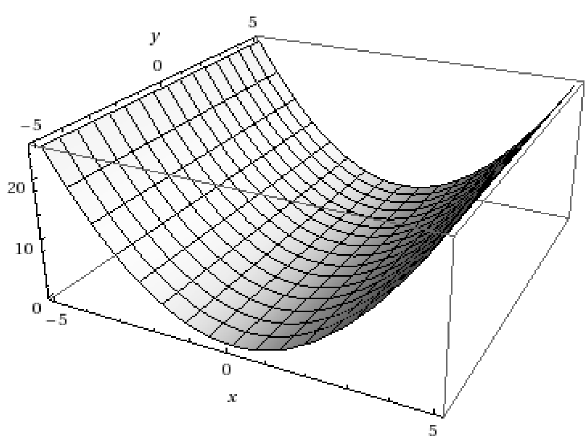
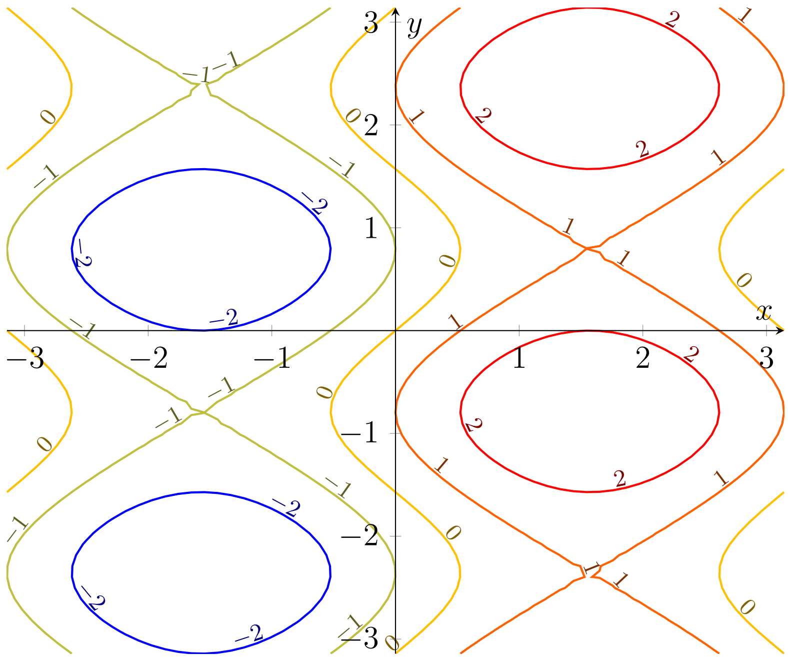




Problem 22. (4 points) A contour plot of a function f is shown above. What is the gradient of f at $(0, 0)$?

(A) $(2, -2)$

(B) $(0, 0)$

(C) $(-1, 1)$

(D) $(0, 1)$

(E) $(-2, 0)$

Problem 23. (4 points) Let f be a function with continuous second partial derivatives. Assume that the following equations hold:

$$f(1,2) = 3$$

$$\frac{\partial f}{\partial x}(1,2) = 0$$

$$\frac{\partial f}{\partial y}(1,2) = 0$$

$$\frac{\partial^2 f}{\partial x^2}(1,2) = 1$$

$$\frac{\partial^2 f}{\partial y^2}(1,2) = 2$$

$$\frac{\partial^2 f}{\partial x \partial y}(1,2) = \frac{\partial^2 f}{\partial y \partial x}(1,2) = 3.$$

Pick the statement that is true.

(A) $(1,2)$ is not a critical point of f.

(B) $f(1,2)$ is a local minimum.

(C) $f(1,2)$ is a local maximum.

(D) $f(1,2)$ is a saddle point.

(E) $(1,2)$ is a critical point of f whose type cannot be determined from the provided information.

Problem 24. (4 points) The *extreme value theorem* guarantees that every continuous function defined on one of the regions described below must attain a minimum and a maximum. Which region?

(A) $\{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq 1,\ 3 \leq y \leq 7\}$

(B) $\{(x, y) \in \mathbb{R}^2 \mid 1 \leq x \leq 5\}$

(C) $\{(x, y) \in \mathbb{R}^2 \mid 6 < x < 9,\ 3 \leq y \leq 7\}$

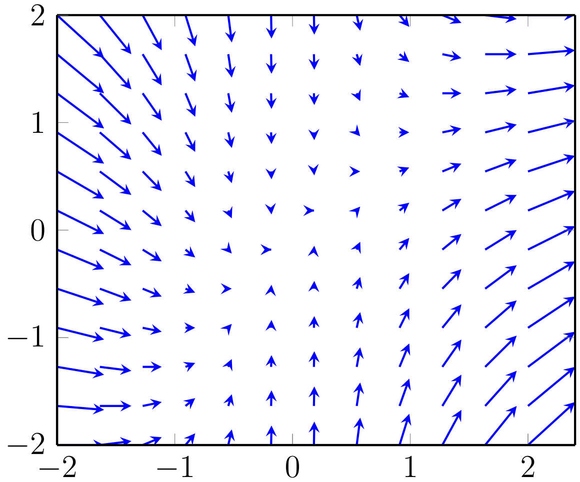
(D) $\{(x, y) \in \mathbb{R}^2 \mid x \leq 1, y \leq 9\}$

(E) $\{(x, y) \in \mathbb{R}^2 \mid 4x^2 + y^2 < 16\}$

Problem 25. (4 points) Choose the figure which represents the vector field

$$\vec{F}(x,y) = \left(y^2, x+y\right).$$

(A)



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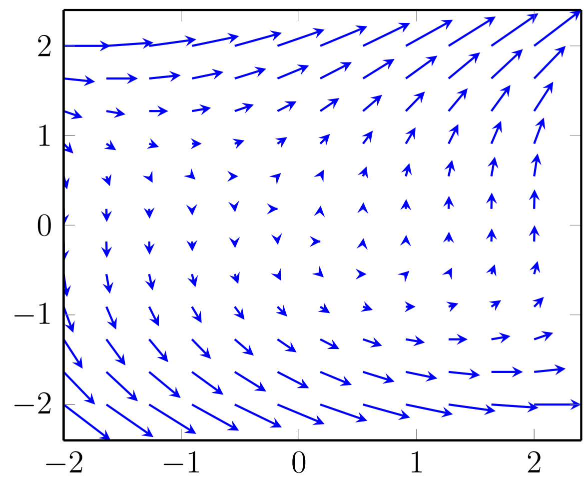

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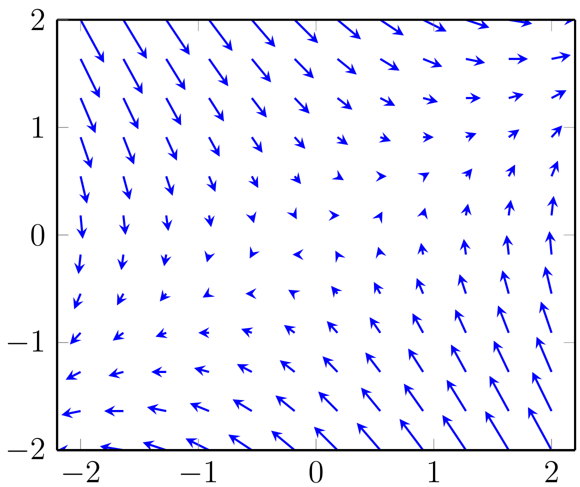

(D)