Calculus 2502A - Advanced Calculus I
Fall 2014
§14.8: Example of Lagrange multipliers

Martin Frankland

November 24, 2014

Example 1. Find the minimum and maximum values (if they exist) of the function $f(x, y) = 4x^2 + y^2$ on the hyperbola $xy = 1$, as well as all points where they occur.

Solution. Let C be the hyperbola $xy = 1$. The function f has $\boxed{\text{no maximum}}$ on C, since f is not bounded above on C. Indeed, all points $(x, \frac{1}{x})$ belong to C and we have

$$\lim_{x\to+\infty} f(x, \frac{1}{x}) = \lim_{x\to+\infty} 4x^2 + \frac{1}{x^2} = +\infty.$$

Now let us find the minimum of f on C, which does exist. Write $g(x, y) = xy$. The gradient of g is:

$$\nabla g = (g_x, g_y) = (y, x)$$

which never vanishes on C. Thus, the method of Lagrange multipliers can be used. The gradient of f is:

$$\nabla f = (f_x, f_y) = (8x, 2y).$$

The equation is:

$$\begin{aligned}
&\nabla f = \lambda \nabla g \\
\Leftrightarrow\ &(8x, 2y) = \lambda(y, x) \\
\Leftrightarrow\ &\begin{cases} 8x = \lambda y \\ 2y = \lambda x \end{cases} \\
\Leftrightarrow\ &\begin{cases} \frac{8x}{y} = \lambda \\ \frac{2y}{x} = \lambda \text{ since } x \text{ and } y \text{ are never 0 on } C \end{cases} \\
\Rightarrow\ &\frac{8x}{y} = \frac{2y}{x} \\
\Leftrightarrow\ &8x^2 = 2y^2 \\
\Leftrightarrow\ &y^2 = 4x^2 \\
\Leftrightarrow\ &y = \pm 2x.
\end{aligned}$$

The case $y = -2x$ yields no solutions, since x and y always have the same sign on C. More precisely, substituting $y = -2x$ into the equation of C yields:

$$\begin{aligned}
&xy = 1 \\
\Leftrightarrow\ &x(-2x) = 1 \\
\Leftrightarrow\ &-2x^2 = 1
\end{aligned}$$

which holds for no $x \in \mathbb{R}$. Thus, we conclude $y = 2x$ and substitute into the equation of C:

$$\begin{aligned}
&xy = 1 \\
\Leftrightarrow\ &x(2x) = 1 \\
\Leftrightarrow\ &2x^2 = 1 \\
\Leftrightarrow\ &x^2 = \frac{1}{2} \\
\Leftrightarrow\ &x = \pm\frac{1}{\sqrt{2}}.
\end{aligned}$$

This yields two solutions $\left(\frac{1}{\sqrt{2}}, \frac{2}{\sqrt{2}}\right)$ and $\left(-\frac{1}{\sqrt{2}}, -\frac{2}{\sqrt{2}}\right)$. The value of f at both points happens to be the same:

$$f\left(\frac{1}{\sqrt{2}}, \frac{2}{\sqrt{2}}\right) = f\left(-\frac{1}{\sqrt{2}}, -\frac{2}{\sqrt{2}}\right) = 4\frac{1}{2} + \frac{4}{2} = 4.$$

Therefore, the minimum of f on C is $\boxed{4}$, and it occurs at the points $\boxed{\left(\frac{1}{\sqrt{2}}, \frac{2}{\sqrt{2}}\right)}$ and $\boxed{\left(-\frac{1}{\sqrt{2}}, -\frac{2}{\sqrt{2}}\right)}$.