Calculus 2502A - Advanced Calculus I
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§14.2: Examples of limits

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The following example illustrates the strategy of switching to polar coordinates.

Example 1. Find the limit

$$\lim_{(x,y)\to(0,0)} \frac{5x^3 - 2x^2y + 4xy^2 + y^6}{x^2 + y^2}$$

if it exists, or show that it does not exist.

Solution. Using polar coordinates

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

the limit can be expressed as

$$\begin{aligned}
&\lim_{r\to 0} \frac{5(r\cos\theta)^3 - 2(r\cos\theta)^2(r\sin\theta) + 4(r\cos\theta)(r\sin\theta)^2 + (r\sin\theta)^6}{r^2} \\
&= \lim_{r\to 0} \frac{5r^3\cos^3\theta - 2r^3\cos^2\theta\sin\theta + 4r^3\cos\theta\sin^2\theta + r^6\sin^6\theta}{r^2} \\
&= \lim_{r\to 0} \left(5r\cos^3\theta - 2r\cos^2\theta\sin\theta + 4r\cos\theta\sin^2\theta + r^4\sin^6\theta\right) \\
&= 0 + 0 + 0 + 0 \\
&= \boxed{0}.
\end{aligned}$$

Indeed, for any integers $m, n \geq 0$ and any $\theta \in \mathbb{R}$, we have the upper bound:

$$\begin{aligned}
\left|\cos^m\theta\sin^n\theta\right| &= \left|\cos\theta\right|^m \left|\sin\theta\right|^n \\
&\leq (1)^m(1)^n \\
&= 1
\end{aligned}$$

and therefore the limit: $\lim_{r\to 0} r\cos^m\theta\sin^n\theta = 0$. $\square$

Although there is no analogue of l'Hôpital's rule in higher dimension, some problems can be reduced to a one-dimensional problem, as in the following example.

Example 2. Find the limit

$$\lim_{(x,y)\to(0,0)} \frac{5x^2+4y^2}{\sqrt{5x^2+4y^2+1}-1}$$

if it exists, or show that it does not exist.

Solution. Given $\lim_{(x,y)\to(0,0)} 5x^2+4y^2 = 0$ *and* the fact $5x^2+4y^2 \neq 0$ for $(x,y)\neq(0,0)$ in a small enough neighborhood of $(0,0)$, the composition rule yields:

$$\begin{aligned}\lim_{(x,y)\to(0,0)} \frac{5x^2+4y^2}{\sqrt{5x^2+4y^2+1}-1} &= \lim_{u\to 0}\frac{u}{\sqrt{u+1}-1}\\ &= \lim_{u\to 0}\frac{1}{1/2\sqrt{u+1}} \quad \text{by l'Hôpital's rule}\\ &= \frac{1}{1/2} \quad \text{by the rules for continuity}\\ &= \boxed{2}. \quad \square\end{aligned}$$

Remark 3. For your numerical pleasure, if we define $f(u) = \frac{u}{\sqrt{u+1}-1}$, then we have the values

$$f(0.1) \approx 2.04881$$
$$f(0.01) \approx 2.00499$$
$$f(0.001) \approx 2.00050.$$

The next two examples illustrate the strategy of finding upper bounds and the Squeeze theorem.

Example 4. Find the limit

$$\lim_{(x,y)\to(0,0)} \frac{7x^3e^{xy}}{5x^2+4y^2+x^2y^2+y^6}$$

if it exists, or show that it does not exist.

Solution. Consider the upper bound:

$$\begin{aligned}\left|\frac{7x^3e^{xy}}{5x^2+4y^2+x^2y^2+y^6}\right| &\leq \left|\frac{7x^3e^{xy}}{4x^2+4y^2}\right|\\ &= \frac{7}{4}\left|\frac{x^3}{x^2+y^2}\right| e^{xy}.\end{aligned}$$

By the product rule, we have

$$\begin{aligned}\lim_{(x,y)\to(0,0)} \frac{7}{4}\,\frac{x^3 e^{xy}}{x^2+y^2} &= \frac{7}{4}\left(\lim_{(x,y)\to(0,0)} \frac{x^3}{x^2+y^2}\right)\left(\lim_{(x,y)\to(0,0)} e^{xy}\right)\\ &= \frac{7}{4}(0)(1)\\ &= 0\end{aligned}$$

which guarantees:

$$\lim_{(x,y)\to(0,0)} \frac{7x^3 e^{xy}}{5x^2+4y^2+x^2y^2+y^6} = \boxed{0}. \quad \square$$

Example 5. Find the limit

$$\lim_{(x,y)\to(0,0)} \frac{x^2\tan(xy)}{x^2+y^2}$$

if it exists, or show that it does not exist.

Solution. For all $(x,y) \neq (0,0)$, the inequality

$$0 \le \frac{x^2}{x^2+y^2} \le 1$$

holds. This implies the inequality

$$\left|\frac{x^2\tan(xy)}{x^2+y^2}\right| \le \tan(xy) \xrightarrow{(x,y)\to(0,0)} \tan(0) = 0.$$

Therefore, we have:

$$\lim_{(x,y)\to(0,0)} \frac{x^2\tan(xy)}{x^2+y^2} = \boxed{0}. \quad \square$$