Calculus 2502A - Advanced Calculus I
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§14.2: Rules for limits and continuity

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In these notes, we explain how the rules for limits and continuity of functions of a single variable also hold for functions of several variables.

1 Statements

Theorem 1.1 (Rules for limits)**.** *Let f and g be functions of n variables, and let φ be a function of one variable.*

1. *(Sum.) If f and g have a limit at $\vec{a}$, then:*
$$\lim_{\vec{x}\to\vec{a}} \left(f(\vec{x}) + g(\vec{x})\right) = \lim_{\vec{x}\to\vec{a}} f(\vec{x}) + \lim_{\vec{x}\to\vec{a}} g(\vec{x}).$$
2. *(Product.) If f and g have a limit at $\vec{a}$, then:*
$$\lim_{\vec{x}\to\vec{a}} \left(f(\vec{x})g(\vec{x})\right) = \left(\lim_{\vec{x}\to\vec{a}} f(\vec{x})\right)\left(\lim_{\vec{x}\to\vec{a}} g(\vec{x})\right).$$
3. *(Quotient.) If f and g have a limit at $\vec{a}$, and $\lim_{\vec{x}\to\vec{a}} g(\vec{x}) \neq 0$, then:*
$$\lim_{\vec{x}\to\vec{a}} \frac{f(\vec{x})}{g(\vec{x})} = \frac{\lim_{\vec{x}\to\vec{a}} f(\vec{x})}{\lim_{\vec{x}\to\vec{a}} g(\vec{x})}.$$
4. *(Composition.) If $\lim_{\vec{x}\to\vec{a}} f(\vec{x}) = L$ and φ is continuous at L, then*
$$\lim_{\vec{x}\to\vec{a}} \varphi\left(f(\vec{x})\right) = \varphi\left(\lim_{\vec{x}\to\vec{a}} f(\vec{x})\right) = \varphi(L).$$

Proof. 1. Let $\epsilon > 0$. Since f has a limit L_1 at $\vec{a}$, there is a number $\delta_1 > 0$ satisfying the implication
$$0 < |\vec{x} - \vec{a}| < \delta_1 \Rightarrow |f(\vec{x}) - L_1| < \frac{\epsilon}{2}.$$

Likewise, since g has a limit L_2 at $\vec{a}$, there is a number $\delta_2 > 0$ satisfying the implication

$$0 < |\vec{x} - \vec{a}| < \delta_2 \Rightarrow |g(\vec{x}) - L_2| < \frac{\epsilon}{2}.$$

Take $\delta := \min\{\delta_1, \delta_2\}$. For any input $\vec{x}$ satisfying $0 < |\vec{x} - \vec{a}| < \delta$, we have

$$\begin{aligned}
|(f+g)(\vec{x}) - (L_1 + L_2)| &= |f(\vec{x}) - L_1 + g(\vec{x}) - L_2| \\
&\leq |f(\vec{x}) - L_1| + |g(\vec{x}) - L_2| \\
&< \frac{\epsilon}{2} + |g(\vec{x}) - L_2| \quad \text{since } |\vec{x} - \vec{a}| < \delta_1 \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} \quad \text{since } |\vec{x} - \vec{a}| < \delta_2 \\
&= \epsilon.
\end{aligned}$$

This proves $\lim_{\vec{x} \to \vec{a}} (f(\vec{x}) + g(\vec{x})) = L_1 + L_2$.

Parts (2), (3), and (4) are proved similarly. ☐

Corollary 1.2 (Rules for continuity)**.** *Let f and g be functions of n variables, and let φ be a function of one variable.*

1. *(Sum.) If f and g are continuous at $\vec{a}$, then $f + g$ is continuous at $\vec{a}$.*
2. *(Product.) If f and g are continuous at $\vec{a}$, then fg is continuous at $\vec{a}$.*
3. *(Quotient.) If f and g are continuous at $\vec{a}$, and $g(\vec{a}) \neq 0$, then $\frac{f}{g}$ is continuous at $\vec{a}$.*
4. *(Composition.) If f is continuous at $\vec{a}$, and φ is continuous at $f(\vec{a})$, then the composition $\varphi \circ f$ is continuous at $\vec{a}$.*

Proof. 1. We have:

$$\begin{aligned}
\lim_{\vec{x} \to \vec{a}} (f+g)(\vec{x}) &= \lim_{\vec{x} \to \vec{a}} (f(\vec{x}) + g(\vec{x})) \\
&= \lim_{\vec{x} \to \vec{a}} f(\vec{x}) + \lim_{\vec{x} \to \vec{a}} g(\vec{x}) \quad \text{by the sum rule for limits} \\
&= f(\vec{a}) + g(\vec{a}) \quad \text{since } f \text{ and } g \text{ are continuous at } \vec{a} \\
&= (f+g)(\vec{a}).
\end{aligned}$$

Parts (2), (3), and (4) are proved similarly. ☐

2 Examples

To simplify the notation, we will work with functions of two variables.

Exercise 2.1. Using the epsilon-delta definition, show that the projection functions $f(x, y) = x$ and $g(x, y) = y$ are continuous everywhere.

Proposition 2.2. *Polynomials are continuous everywhere.*

Proof. Constant functions are continuous everywhere. We know from 2.1 that the functions $f(x, y) = x$ and $g(x, y) = y$ are continuous everywhere. By the product rule, monomials $cx^i y^j$ for some constant $c \in \mathbb{R}$ and integers $i, j \geq 0$ are also continuous everywhere. A polynomial $p(x, y)$ is a sum of such monomials, and is therefore continuous everywhere, by the sum rule. $\square$

Definition 2.3. A **rational function** is a quotient of two polynomials.

Example 2.4. The function

$$f(x, y) = \frac{5x^2y - y^3 + 1}{2x^9 + xy - 6}$$

is a rational function.

Proposition 2.5. *Rational functions are continuous on their domain.*

Proof. The maximal domain of a rational function $f(x, y) = \frac{p(x,y)}{q(x,y)}$ is the region where the denominator is non-zero:

$$D = \{(x, y) \in \mathbb{R}^2 \mid q(x, y) \neq 0\}.$$

Both functions p and q are polynomials, hence continuous everywhere, by 2.2. By the quotient rule, the function $f = \frac{p}{q}$ is continuous wherever q is non-zero, in particular on the domain of f. $\square$

Example 2.6. For the rational function from Example 2.4, let us compute $\lim_{(x,y)\to(1,2)} f(x, y)$. Note that the denominator $2x^9 + xy - 6$ is non-zero at the point $(1, 2)$:

$$2(1)^9 + (1)(2) - 6 = 2 + 2 - 6 = -2$$

and therefore f is continuous at $(1, 2)$. The limit is obtained by evaluating f at the point:

$$\begin{aligned}
\lim_{(x,y)\to(1,2)} f(x, y) &= f(1, 2) \\
&= \frac{5(1)^2(2) - (2)^3 + 1}{2(1)^9 + (1)(2) - 6} \\
&= \frac{10 - 8 + 1}{-2} \\
&= \boxed{-\frac{3}{2}}.
\end{aligned}$$

Remark 2.7. The same argument would *not* work to compute $\lim_{(x,y)\to(1,4)} f(x,y)$, since the denominator $2x^9 + xy - 6$ is zero at the point $(1,4)$:

$$2(1)^9 + (1)(4) - 6 = 2 + 4 - 6 = 0.$$

We would need to work harder to find this limit or prove that it does not exist.

Example 2.8. Consider the function

$$f(x,y) = \frac{e^{xy}\cos\left(x^3y^2 - 5y^3\right)}{\sqrt{2x+y+1}}$$

and let us find $\lim_{(x,y)\to(1,-1)} f(x,y)$.

Note that xy and $x^3y^2 - 5y^3$ are polynomials, and thus continuous everywhere. By the composition rule, e^{xy} is continuous everywhere and so is $\cos(x^3y^2 - 5y^3)$. By the product rule, the numerator $e^{xy}\cos(x^3y^2 - 5y^3)$ is continuous everywhere.

By the composition rule, the denominator $\sqrt{2x+y+1}$ is continuous wherever the radicand is non-negative: $2x+y+1 \geq 0$. By the quotient rule, f is continuous wherever the denominator is (defined and) non-zero, which is precisely where $2x+y+1 > 0$.

In our case, the radicand $2x+y+1$ is positive at the point $(1,-1)$:

$$2(1) + (-1) + 1 = 2$$

and therefore f is continuous at $(1,-1)$. The limit is obtained by evaluating f at the point:

$$\begin{aligned}
\lim_{(x,y)\to(1,-1)} f(x,y) &= f(1,-1) \\
&= \frac{e^{(1)(-1)}\cos\left((1)^3(-1)^2 - 5(-1)^3\right)}{\sqrt{2(1)+(-1)+1}} \\
&= \frac{e^{-1}\cos(1+5)}{\sqrt{2}} \\
&= \boxed{\frac{e^{-1}\cos(6)}{\sqrt{2}}}.
\end{aligned}$$

3 Bonus Feature

The following example illustrates why using rules for limits and continuity is a good idea.

Example 3.1. Using the epsilon-delta definition, let us show that the function $f(x) = x^3$ is continuous everywhere, i.e., at all $x \in \mathbb{R}$.

We want to show that f is continuous at a, for any $a \in \mathbb{R}$. Let $\epsilon > 0$. Using the factorization $x^3 - a^3 = (x - a)(x^2 + ax + a^2)$, we obtain:

$$\begin{aligned}
|f(x) - f(a)| &= |x^3 - a^3| \\
&= |x - a||x^2 + ax + a^2| \\
&\leq |x - a| \left(|x^2| + |ax| + |a^2|\right) \\
&= |x - a| \left(|x|^2 + |a||x| + |a|^2\right) \\
&\leq |x - a| \left((|a| + 1)^2 + |a|(|a| + 1) + |a|^2\right) \quad \text{if we take } \delta \leq 1 \\
&\leq |x - a| \left((|a| + 1)^2 + (|a| + 1)(|a| + 1) + (|a| + 1)^2\right) \\
&= |x - a|\, 3(|a| + 1)^2 \\
&< \delta\, 3(|a| + 1)^2
\end{aligned}$$

whenever $|x - a| < \delta$ holds, and we *want* that expression to be at most ϵ:

$$\delta\, 3(|a| + 1)^2 \leq \epsilon.$$

By taking $\delta = \min\{1, \frac{\epsilon}{3(|a|+1)^2}\}$, we obtain:

$$\begin{aligned}
|f(x) - f(a)| &< \delta\, 3(|a| + 1)^2 \\
&\leq \frac{\epsilon}{3(|a| + 1)^2}\, 3(|a| + 1)^2 \\
&= \epsilon
\end{aligned}$$

whenever $|x - a| < \delta$ holds. □