Calculus 2302A - Intermediate Calculus I

Fall 2013
Practice Final

Name (please print): ______________________________

Student ID: ________________________

- On the real exam, no calculators, electronic devices, books, or notes may be used.
- Show your work. No credit for answers without justification.
- Good luck!

For the record, most problems are actual exam problems from a Multivariable Calculus course previously taught at another institution.

This practice exam is much longer than the real exam.

List of formulas.

Curvature: $\kappa(t) = |\frac{d\vec{T}}{ds}|$, where $\vec{T}$ = unit tangent vector, and s = arc length.

Newton's Law: $\vec{F} = m\vec{a}$, where $\vec{F}$ = force, m = mass, and $\vec{a}$ = acceleration.

Problem 1. (10 points) Find the acute angle between any two distinct diagonals of a cube. Here, a diagonal is the segment joining two opposite vertices, that is, vertices that do not lie on any common face.

Problem 2. (10 points) Consider the line $-4x + 2y = 1$ in $\mathbb{R}^2$.

(a) Give a parametric equation for the line.

(b) Find a vector in $\mathbb{R}^2$ that is perpendicular to the line.

(c) Find the distance between the line and the point $(4, 3)$ in $\mathbb{R}^2$.

Problem 3. (10 points) Find a vector equation for the line in $\mathbb{R}^3$ through the point $(2, 1, 0)$ that is parallel to the plane $x - y + z = 3$ and perpendicular to the line with parametric equations $x(t) = 2 - t$, $y(t) = 2t$, $z(t) = 1 + t$.

Problem 4. (10 points) Find a parametrization of the portion of the parabola $x = y^2$ going from $(25, 5)$ to $(4, 2)$. Make sure to specify the domain of the parametrization.

Problem 5. (10 points) A particle is moving in space according to the position function

$$\overrightarrow{r}(t) = \left(t^2 + 3, 4t - 5, 3t + 19\right)$$

at time t. Find the tangential component a_T and normal component a_N of the acceleration of the particle.

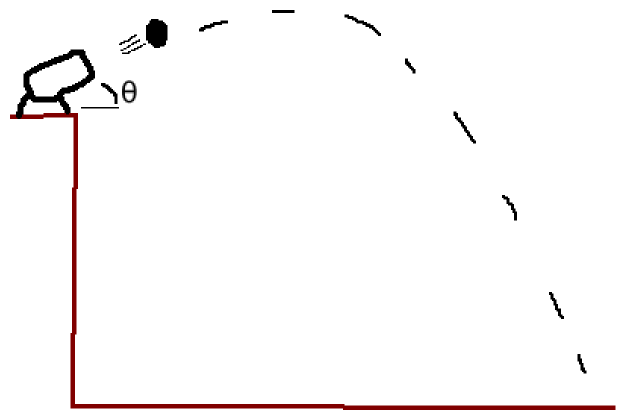


FIGURE 1

Problem 6. (10 points) A cannonball of mass 7 kg is shot from a cliff located 100 meters above ground, with an angle of $\theta = 30°$ above the horizontal, at an initial speed of 10 m/s.

Find the **speed** of the ball when it hits the ground. (Recall that speed is the magnitude of the velocity.)

In this problem, ignore air resistance, and write the gravitational acceleration on Earth as g, which is approximately $g = 9.8\,\mathrm{N/kg} = 9.8\,\mathrm{m/s^2}$.

(More space.)

Problem 7. (10 points) Consider the function $f(x, y) = x^3y^2$.

(a) Compute the partial derivatives f_x and f_y.

(b) Estimate $f(0.9, 2.2)$ using the linear approximation of f at $(1, 2)$.

Problem 8. (10 points) A function $f(x, y)$ has partial derivatives $\frac{\partial f}{\partial x}(4, 3) = 2$ and $\frac{\partial f}{\partial y}(4, 3) = 7$. We want the partial derivatives of f in polar coordinates at that same point.

(a) Recalling that polar coordinates are defined by

$$x(r, \theta) = r \cos \theta$$
$$y(r, \theta) = r \sin \theta,$$

find r, $\cos \theta$, and $\sin \theta$ at the point $(x, y) = (4, 3)$.

(b) Find the partial derivatives $\frac{\partial f}{\partial r}$ and $\frac{\partial f}{\partial \theta}$ at the point $(x, y) = (4, 3)$.

Problem 9. (10 points) Consider the function $f(x, y, z) = xe^y + yz^2$, and the unit vector $\mathbf{u} = (\frac{1}{\sqrt{2}}, 0, \frac{-1}{\sqrt{2}})$. Compute $D_{\mathbf{u}}f(0, 1, 2)$, the directional derivative of f in the direction $\mathbf{u}$ at the point $(0, 1, 2)$.

Problem 10. (10 points) Let C be the curve of intersection of the paraboloid $y = x^2 + z^2$ and the ellipsoid $4x^2 + y^2 + z^2 = 9$. Find a vector equation for the tangent line to C at the point $(-1, 2, 1)$.

Problem 11. (10 points) Consider the function

$$f(x, y) = 2\cos(x) - y^2 + 2y.$$

(a) Find all the critical points of f in the domain $\{(x, y) \in \mathbb{R}^2 \mid -1 < x < 5\}$.

(b) For each point you found in part (a), determine if it is a local maximum, local minimum, or a saddle point. Justify your answer.

Problem 12. (10 points) Find the **minimum value** of the function $f(x, y, z) = x + 2y + 3z$ on the upper sheet of the hyperboloid $x^2 + y^2 - z^2 = -1$. (Here "upper sheet" means the part where $z > 0$.)

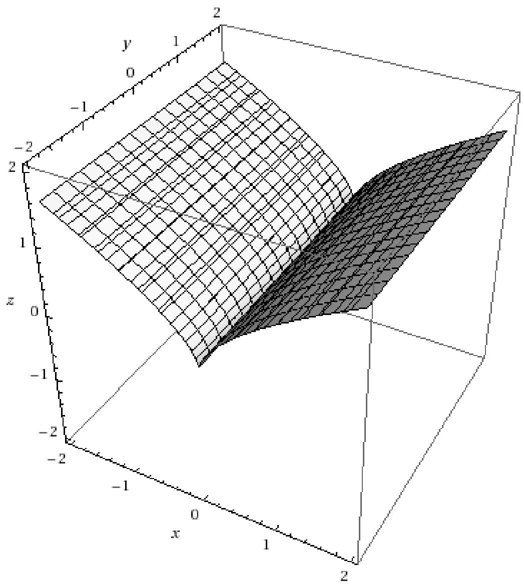
Multiple choice section

Problem 13. (4 points) Three of the following equations are true for **all** vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$ in $\mathbb{R}^3$. Which one is false for some vectors?

(A) $\vec{v} \cdot \vec{w} = \vec{w} \cdot \vec{v}$
(B) $|\vec{v} \times \vec{w}| = |\vec{w} \times \vec{v}|$
(C) $\vec{u} \cdot (\vec{v} \times \vec{w}) = \vec{v} \cdot (\vec{w} \times \vec{u})$
(D) $\vec{u} \times (\vec{v} \times \vec{w}) = (\vec{u} \times \vec{v}) \times \vec{w}$

Problem 14. (4 points) Which of the following pictures represents the surface $x^2 = z^3$ in $\mathbb{R}^3$?

(A)



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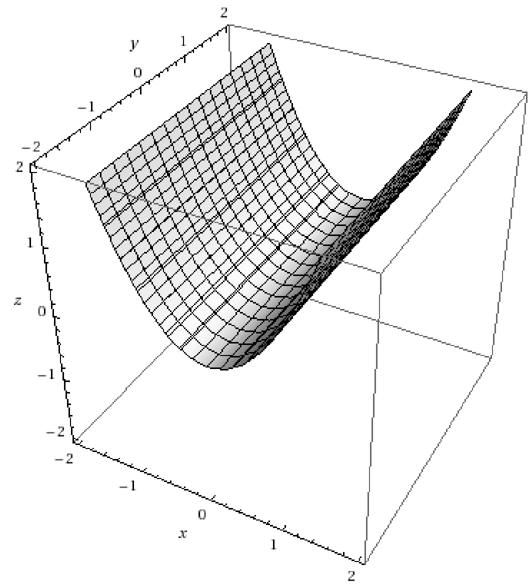

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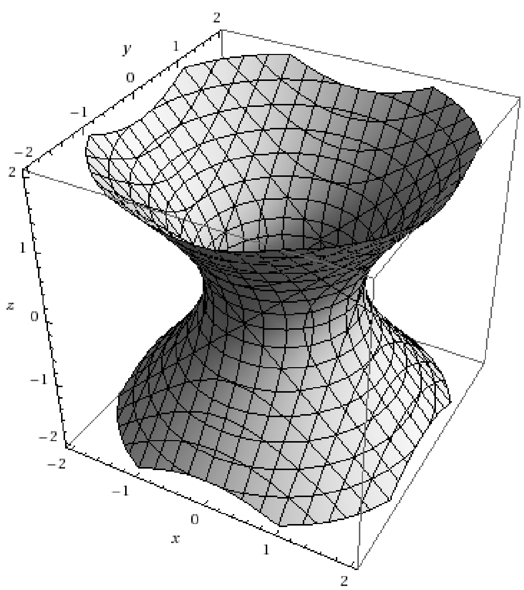

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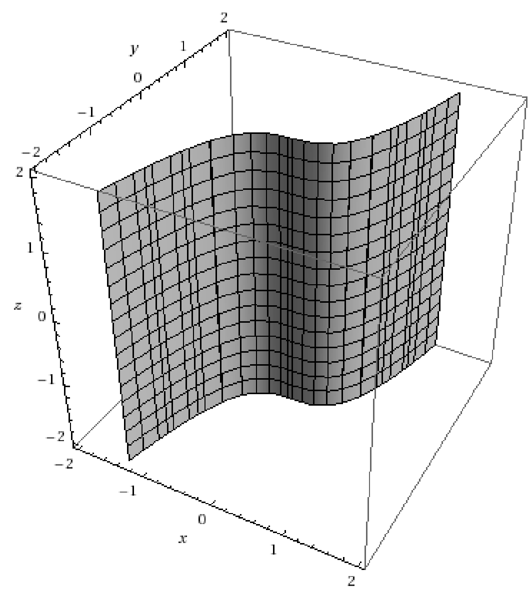
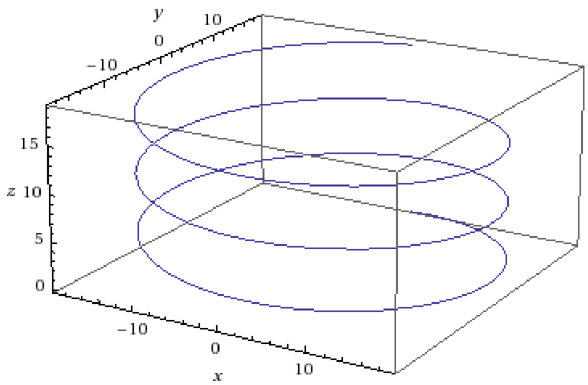


Problem 15. (4 points) Consider the curve parametrized by

$$\vec{r}(t) = (t \sin t, t \cos t, t),\ 0 \leq t \leq 6\pi.$$

Which of the following is a plot of that curve?

(A)



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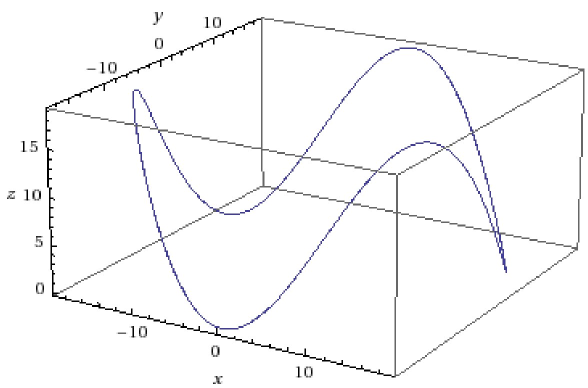

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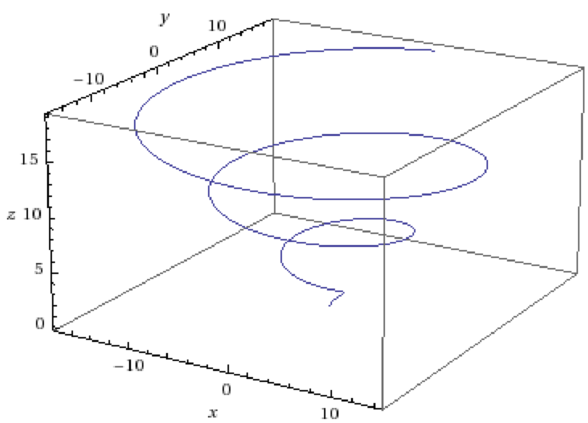

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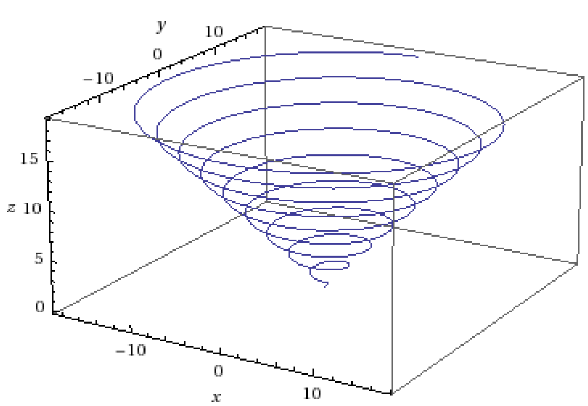
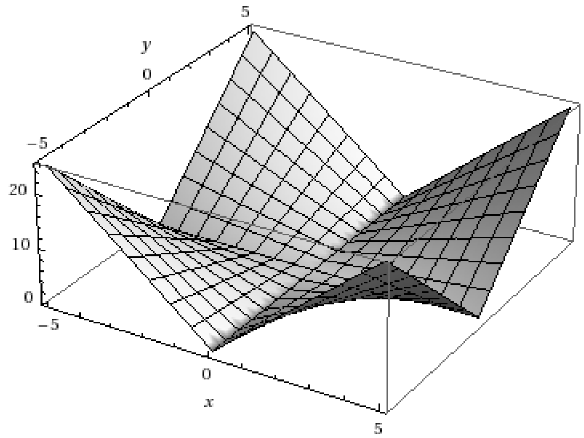


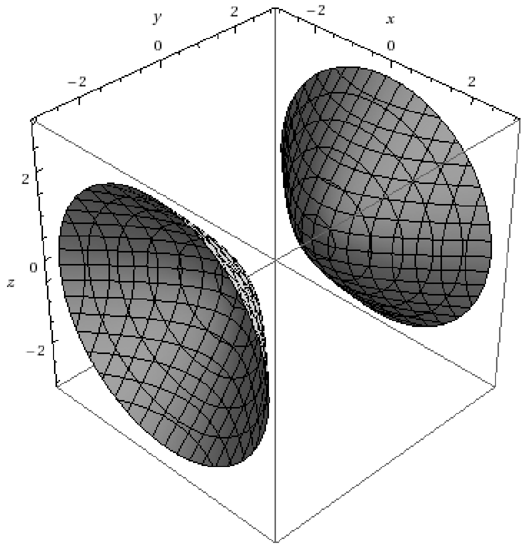


Problem 16. (4 points) Consider the graph depicted above. It is the graph of which function?

(A) $|xy|$
(B) $|x| + |y|$
(C) $(x - y)^2$
(D) $x^2 - y^2$.

Problem 17. (4 points) Which of the following figures represent level sets of the function $f(x,y,z) = x^2 - y^2 + z^2 + 1$?

(A)



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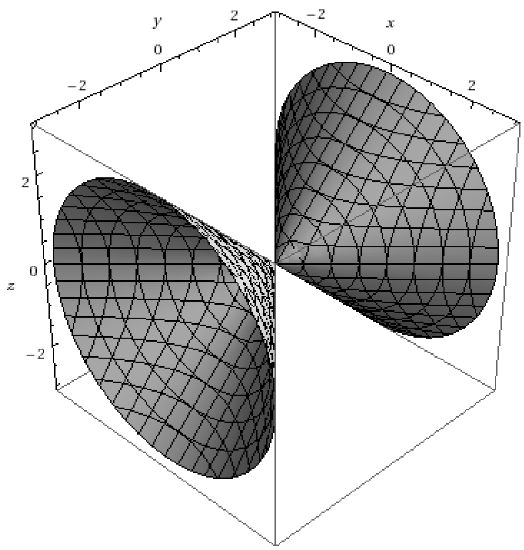
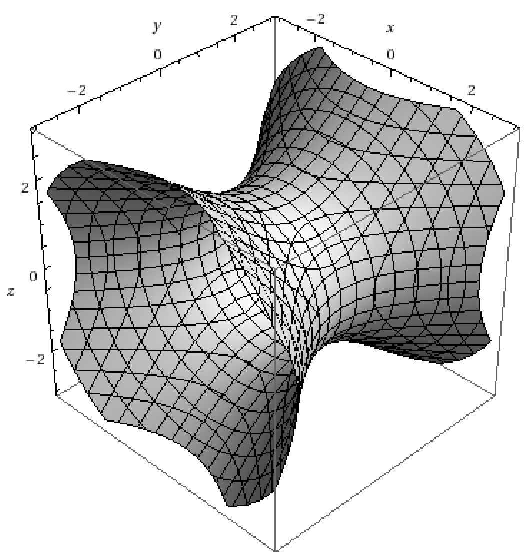
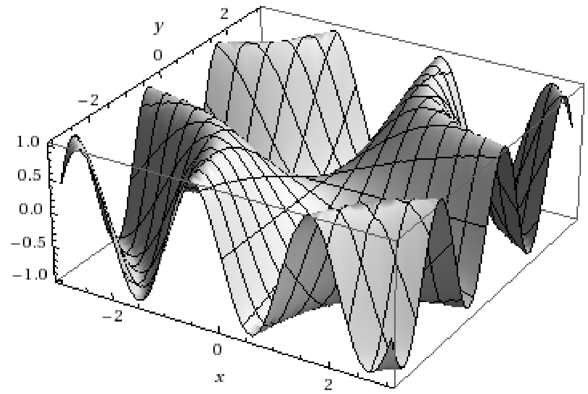


(C)

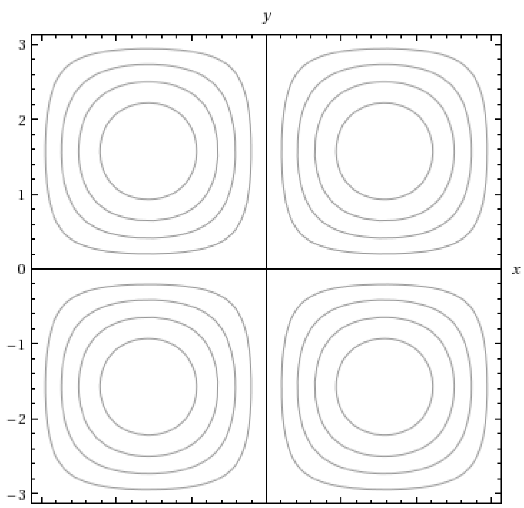


(D) All of the above.



Problem 18. (4 points) Consider the graph depicted above. Which of the following is a contour map of the same function?

(A)



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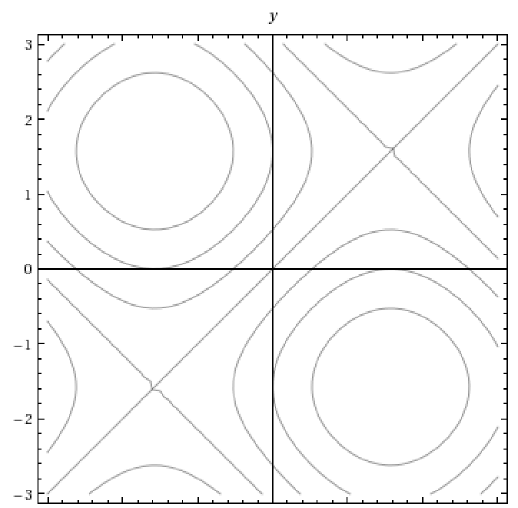
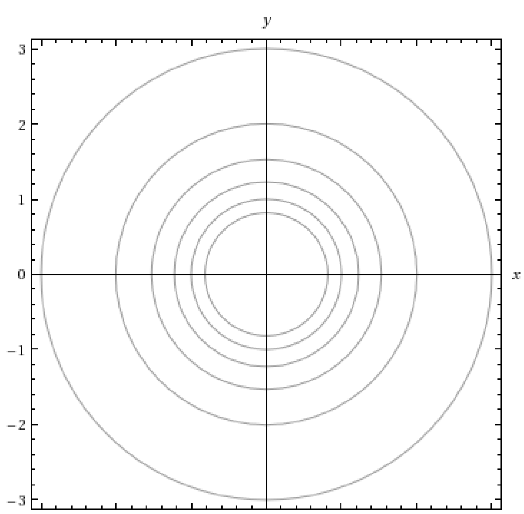
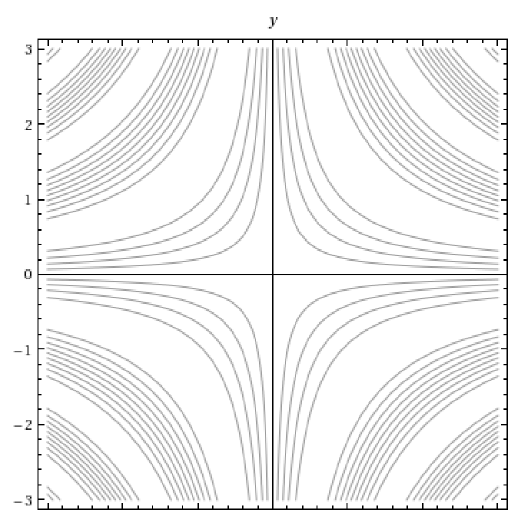


(C)



(D)



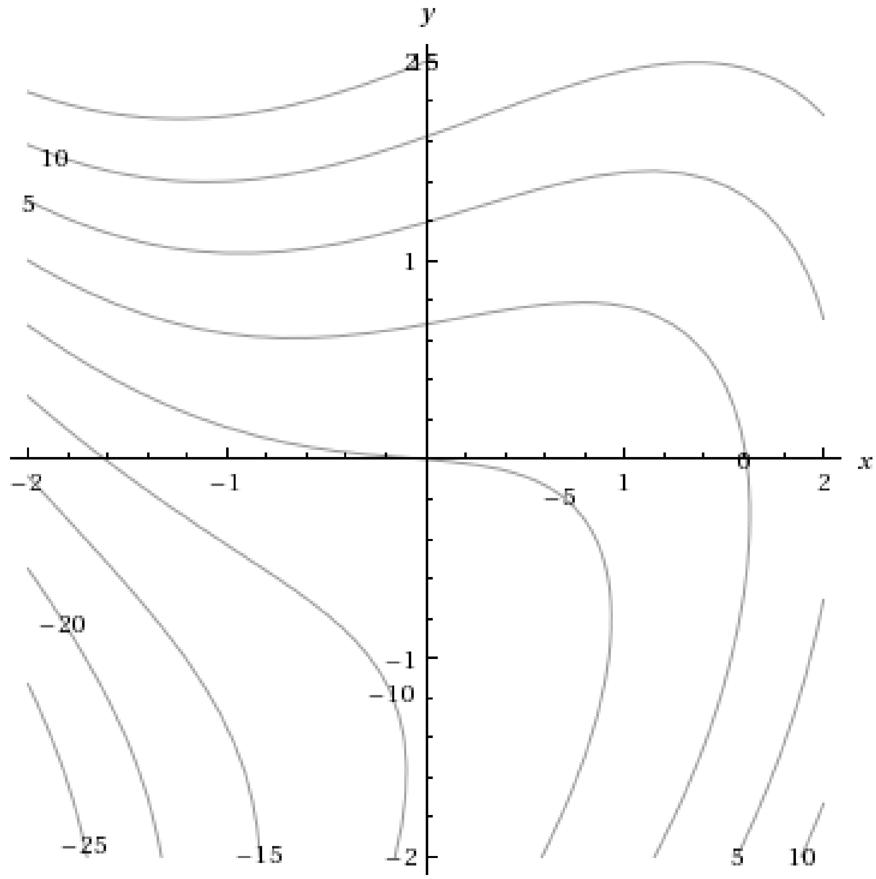
Problem 19. (4 points) Find the following limit, if it exists.

$$\lim_{(x,y)\to(0,0)} \frac{4x^2 + 7x^2y^2 + 4y^2}{x^2 + y^2}$$

(A) 0
(B) 4
(C) 7
(D) It does not exist.

Problem 20. (4 points) Let f be a function of two variables such that $f(x, y) \to 5$ as (x, y) approaches $(1, 2)$ through every straight line through $(1, 2)$. What can you say about the limit $\lim_{(x,y)\to(1,2)} f(x, y)$?

(A) It exists and is equal to 5.
(B) It exists and is equal to $f(1, 2)$.
(C) It may or may not exist, but if it does exist it must be equal to 5.
(D) It does not exist.

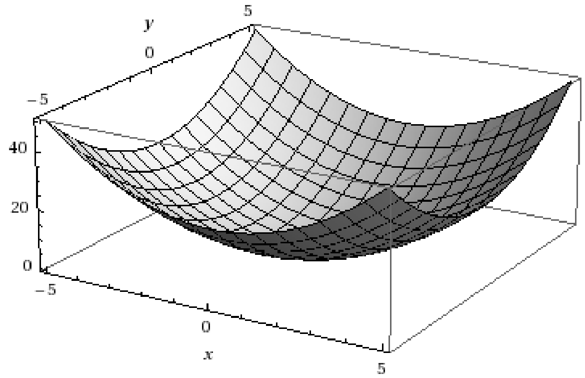


Problem 21. (4 points) The contour map of a function $f(x, y)$ is drawn above. Which of the following quantities is negative at $(0, 0)$? (There is exactly one correct answer.)

(A) $\frac{\partial^2 f}{\partial x \partial y}$
(B) $\frac{\partial^2 f}{\partial x^2}$
(C) $\frac{\partial^2 f}{\partial y^2}$
(D) $\frac{\partial f}{\partial y}$

Problem 22. (4 points) Here are the graphs of some functions $f(x, y)$. Pick the function that satisfies the equation $f_{xx} + f_{yy} = 0$.

(A)



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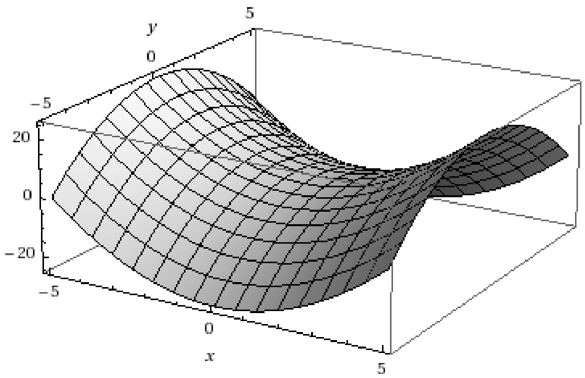

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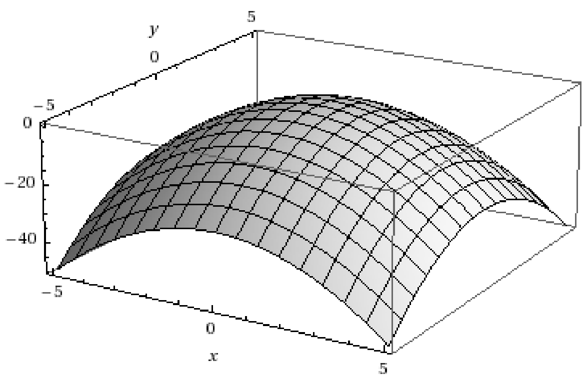
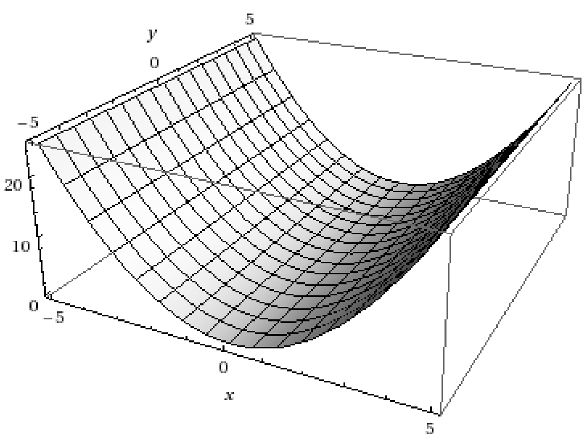
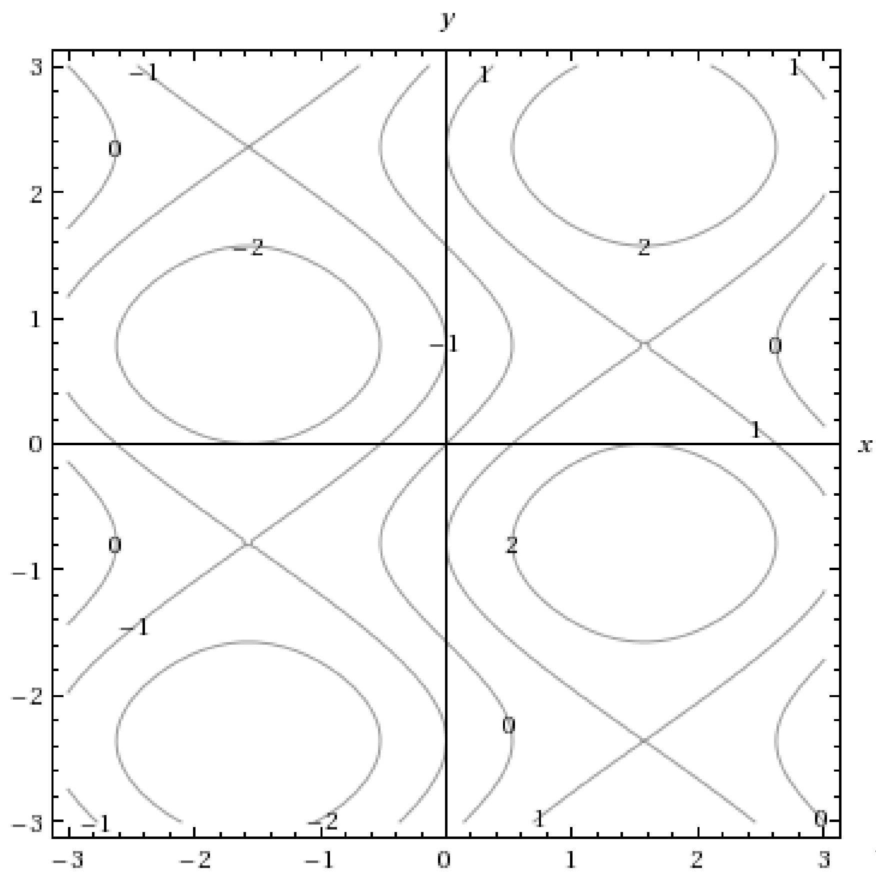


(D)





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Problem 23. (4 points) A contour map of a function f is shown above. What is the gradient of f at $(0, 0)$?

(A) $(2, -2)$
(B) $(0, 0)$
(C) $(-1, 1)$
(D) $(0, 1)$

Problem 24. (4 points) Let f be a function with continuous second partial derivatives. Assume that the following equations hold:

$$f(1,2)=3, \quad \frac{\partial f}{\partial x}(1,2)=0, \quad \frac{\partial f}{\partial y}(1,2)=0, \quad \frac{\partial^2 f}{\partial x^2}(1,2)=1,$$

$$\frac{\partial^2 f}{\partial y^2}(1,2)=2, \quad \text{and} \quad \frac{\partial^2 f}{\partial x \partial y}(1,2)=\frac{\partial^2 f}{\partial y \partial x}(1,2)=3.$$

Pick the statement that is true.

(A) $(1,2)$ is not a critical point of f.
(B) $f(1,2)$ is a local minimum.
(C) $f(1,2)$ is a local maximum.
(D) $f(1,2)$ is a saddle point.

Problem 25. (4 points) The **extreme value theorem** guarantees that every continuous function defined on one of the regions described below must attain a minimum and a maximum. Which region?

(A) $\{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq 1,\ 3 \leq y \leq 7\}$
(B) $\{(x, y) \in \mathbb{R}^2 \mid 1 \leq x \leq 5\}$
(C) $\{(x, y) \in \mathbb{R}^2 \mid 6 < x < 9,\ 3 \leq y \leq 7\}$
(D) $\{(x, y) \in \mathbb{R}^2 \mid x < 1, y < 9\}$