

Calculus 2302A - Intermediate Calculus I

Fall 2013

§14.8: Example of Lagrange multipliers

Martin Frankland

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Here is the problem presented in lecture, whose completion was left as an exercise.

Example 1. Find the minimum and maximum values (if they exist) of the function $f(x, y) = 4x^2 + y^2$ on the hyperbola $xy = 1$, as well as all points where they occur.

Solution. Let C be the hyperbola $xy = 1$. The function f has no maximum on C , since f is not bounded above on C . Indeed, all points $(x, \frac{1}{x})$ belong to C and we have

$$\lim_{x \rightarrow +\infty} f(x, \frac{1}{x}) = \lim_{x \rightarrow +\infty} 4x^2 + \frac{1}{x^2} = +\infty.$$

Now let us find the minimum of f on C , which does exist. Write $g(x, y) = xy$. The gradient of g is:

$$\nabla g = (g_x, g_y) = (y, x)$$

which never vanishes on C . Thus, the method of Lagrange multipliers can be used. The gradient of f is:

$$\nabla f = (f_x, f_y) = (8x, 2y).$$

The equation is:

$$\begin{aligned}\nabla f &= \lambda \nabla g \\ \Leftrightarrow (8x, 2y) &= \lambda(y, x) \\ \Leftrightarrow \begin{cases} 8x = \lambda y \\ 2y = \lambda x \end{cases} \\ \Leftrightarrow \begin{cases} \frac{8x}{y} = \lambda \\ \frac{2y}{x} = \lambda \end{cases} &\text{ since } x \text{ and } y \text{ are never } 0 \text{ on } C \\ \Rightarrow \frac{8x}{y} &= \frac{2y}{x} \\ \Leftrightarrow 8x^2 &= 2y^2 \\ \Leftrightarrow y^2 &= 4x^2 \\ \Leftrightarrow y &= \pm 2x.\end{aligned}$$

Note that the case $y = -2x$ yields no solutions, since x and y always have the same sign on C . More precisely, substituting $y = -2x$ into the equation of C yields:

$$\begin{aligned}xy &= 1 \\ \Leftrightarrow x(-2x) &= 1 \\ \Leftrightarrow -2x^2 &= 1\end{aligned}$$

which holds for no $x \in \mathbb{R}$. Thus, we conclude $y = 2x$ and substitute into the equation of C :

$$\begin{aligned}xy &= 1 \\ \Leftrightarrow x(2x) &= 1 \\ \Leftrightarrow 2x^2 &= 1 \\ \Leftrightarrow x^2 &= \frac{1}{2} \\ \Leftrightarrow x &= \pm \frac{1}{\sqrt{2}}.\end{aligned}$$

This yields two solutions $\left(\frac{1}{\sqrt{2}}, \frac{2}{\sqrt{2}}\right)$ and $\left(-\frac{1}{\sqrt{2}}, -\frac{2}{\sqrt{2}}\right)$. The value of f at both points happens to be the same:

$$f\left(\frac{1}{\sqrt{2}}, \frac{2}{\sqrt{2}}\right) = f\left(-\frac{1}{\sqrt{2}}, -\frac{2}{\sqrt{2}}\right) = 4\frac{1}{2} + \frac{4}{2} = 4.$$

Therefore, the minimum of f on C is $\boxed{4}$, and it occurs at the points $\boxed{\left(\frac{1}{\sqrt{2}}, \frac{2}{\sqrt{2}}\right)}$ and

$$\boxed{\left(-\frac{1}{\sqrt{2}}, -\frac{2}{\sqrt{2}}\right)}.$$