Calculus 2302A - Intermediate Calculus I
Fall 2013
§14.4: Differentiability

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1 Definition and basic properties

Definition 1.1. A function f of two variables is **differentiable** at a point (x_0, y_0) if there exist numbers $A, B \in \mathbb{R}$ such that

$$f(x_0 + h, y_0 + k) = f(x_0, y_0) + Ah + Bk + E(h, k) \tag{1}$$

where the error term $E(h, k)$ satisfies

$$\lim_{(h,k)\to(0,0)} \frac{E(h, k)}{\sqrt{h^2 + k^2}} = 0.$$

In other words, the error term $E(h, k)$ has "order higher than 1".

Proposition 1.2. *If f is differentiable at (x_0, y_0), then the partial derivatives of f exist at (x_0, y_0), and in fact the numbers A and B in Definition 1.1 must be*

$$A = f_x(x_0, y_0)$$
$$B = f_y(x_0, y_0).$$

Proof. Varying only the x component about (x_0, y_0) corresponds to setting $k = 0$ in (1). One has

$$f(x_0 + h, y_0) = f(x_0, y_0) + Ah + E(h, 0)$$

and therefore

$$\begin{aligned} f_x(x_0, y_0) &= \lim_{h\to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h} \\ &= \lim_{h\to 0} \frac{Ah + E(h, 0)}{h} \\ &= A + \lim_{h\to 0} \frac{E(h, 0)}{h} \\ &= A + 0 \\ &= A. \end{aligned}$$

Likewise, varying only the y component about (x_0, y_0) corresponds to setting $h = 0$ in (1). One has

$$f(x_0, y_0 + k) = f(x_0, y_0) + Bk + E(0, k)$$

and therefore

$$\begin{aligned} f_y(x_0, y_0) &= \lim_{k \to 0} \frac{f(x_0, y_0 + k) - f(x_0, y_0)}{k} \\ &= \lim_{k \to 0} \frac{Bk + E(0, k)}{k} \\ &= B + \lim_{k \to 0} \frac{E(0, k)}{k} \\ &= B + 0 \\ &= B. \end{aligned}$$

□

Consequently, Definition 1.1 could have been equivalently stated as follows.

Definition 1.3. A function f of two variables is **differentiable** at a point (x_0, y_0) if the partial derivatives $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$ exist, and

$$f(x_0 + h, y_0 + k) = f(x_0, y_0) + f_x(x_0, y_0)\, h + f_y(x_0, y_0)\, k + E(h, k)$$

where the error term $E(h, k)$ satisfies

$$\lim_{(h,k) \to (0,0)} \frac{E(h, k)}{\sqrt{h^2 + k^2}} = 0.$$

Roughly speaking: f is differentiable at (x_0, y_0) if f is well approximated by its linear approximation close to (x_0, y_0). Geometrically, it means that the graph of f has a tangent plane at the point $(x_0, y_0, f(x_0, y_0))$.

Remark 1.4. The error term $E(h, k)$ measures the discrepancy between f and its linearization $L(x, y)$ at (x_0, y_0):

$$\begin{aligned} E(h, k) &= f(x_0 + h, y_0 + k) - [f(x_0, y_0) + f_x(x_0, y_0)\, h + f_y(x_0, y_0)\, k] \\ &= f(x_0 + h, y_0 + k) - L(x_0 + h, y_0 + k). \end{aligned}$$

As for functions of a single variable, differentiability is stronger than continuity.

Proposition 1.5. *If f is differentiable at (x_0, y_0), then f is continuous at (x_0, y_0).*

Proof. Exercise # 45. □

2 Examples

Example 2.1. A polynomial of degree one $f(x, y) = ax + by + c$ is differentiable everywhere, i.e., at any point $(x, y) \in \mathbb{R}^2$. Indeed, f *equals* its linearization at any point, so that the error term E in Definition 1.3 is identically zero.

Example 2.2. The function $f(x, y) = \sqrt{x^2 + y^2}$ is *not* differentiable at $(0, 0)$. Indeed, the partial derivative of $f_x(0, 0)$ does not exist, since the limit:

$$\begin{aligned} \lim_{h \to 0} \frac{f(0 + h, 0) - f(0, 0)}{h} &= \lim_{h \to 0} \frac{\sqrt{h^2 + 0} - 0}{h} \\ &= \lim_{h \to 0} \frac{|h|}{h} \\ &= \lim_{h \to 0} \operatorname{sign}(h) \end{aligned}$$

does not exist.

Note, however, that f is continuous at $(0, 0)$, and in fact everywhere.

Example 2.3. As in the previous example, the function $f(x, y) = \sqrt{(x - x_0)^2 + (y - y_0)^2}$ is *not* differentiable at (x_0, y_0).

Example 2.4 (Exercise # 46a)**.** The function

$$f(x, y) = \begin{cases} \frac{xy}{x^2+y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

is *not* differentiable at $(0, 0)$. Indeed, f is not even continuous at $(0, 0)$. As we have seen in §14.2, the limit

$$\lim_{(x,y) \to (0,0)} \frac{xy}{x^2 + y^2}$$

does not exist.

Note, however, that f *does* have both partial derivatives at $(0, 0)$. Indeed, f restricted to the axes is identically zero, and so we have

$$\begin{aligned} f_x(0, 0) &= \lim_{h \to 0} \frac{f(0 + h, 0) - f(0, 0)}{h} \\ &= \lim_{h \to 0} \frac{0 - 0}{h} \\ &= 0 \end{aligned}$$

$$\begin{aligned} f_y(0, 0) &= \lim_{h \to 0} \frac{f(0, 0 + h) - f(0, 0)}{h} \\ &= \lim_{h \to 0} \frac{0 - 0}{h} \\ &= 0. \end{aligned}$$

It turns out that the "linearization" of f at $(0,0)$

$$\begin{aligned}L(x,y) &= f(0,0) + f_x(0,0)(x-0) + f_y(0,0)(y-0)\\ &= 0 + 0(x-0) + 0(y-0)\\ &= 0\end{aligned}$$

is a very bad approximation of f.

The following example is worse.

Example 2.5. The function

$$f(x,y) = \begin{cases} \frac{y^3}{x^2+y^2} & \text{if } (x,y) \neq (0,0)\\ 0 & \text{if } (x,y) = (0,0)\end{cases}$$

is *not* differentiable at $(0,0)$. It *is* continuous at $(0,0)$, as we have

$$\lim_{(x,y)\to(0,0)} \frac{y^3}{x^2+y^2} = 0 = f(0,0).$$

Moreover, it has both partial derivatives at $(0,0)$:

$$\begin{aligned}f_x(0,0) &= \lim_{h\to 0} \frac{f(0+h,0) - f(0,0)}{h}\\ &= \lim_{h\to 0} \frac{0}{h}\\ &= 0\\ f_y(0,0) &= \lim_{h\to 0} \frac{f(0,0+h) - f(0,0)}{h}\\ &= \lim_{h\to 0} \frac{h^3/h^2}{h}\\ &= \lim_{h\to 0} \frac{h}{h}\\ &= 1.\end{aligned}$$

However, the error term

$$\begin{aligned}E(h,k) &= f(x_0+h, y_0+k) - f(x_0,y_0) - f_x(x_0,y_0)h - f_y(x_0,y_0)k\\ &= f(h,k) - f(0,0) - f_x(0,0)h - f_y(0,0)k\\ &= \frac{k^3}{h^2+k^2} - k\\ &= \frac{-h^2k}{h^2+k^2}\end{aligned}$$

does *not* satisfy

$$\lim_{(h,k)\to(0,0)} \frac{|E(h,k)|}{\sqrt{h^2+k^2}} = 0$$

since this limit does not exist.

Upshot

Propositions 1.2 and 1.5 say that *if* f is differentiable at (x_0, y_0) *then* f is continuous at (x_0, y_0) and the partial derivatives $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$ exist. Example 2.5 shows that those two conditions, even together, are not enough to guarantee that f is differentiable at (x_0, y_0).

3 Rules for differentiability

The following theorem will guarantee that many nice, familiar functions are differentiable.

Theorem 3.1 (Rules for differentiability)**.** *Let f and g be functions of two variables, and let φ be a function of one variable.*

1. *(Sum.) If f and g are differentiable at (x_0, y_0), then $f + g$ is differentiable at (x_0, y_0).*
2. *(Product.) If f and g are differentiable at (x_0, y_0), then fg is differentiable at (x_0, y_0).*
3. *(Quotient.) If f and g are differentiable at (x_0, y_0), and $g(x_0, y_0) \neq 0$, then $\frac{f}{g}$ is differentiable at (x_0, y_0).*
4. *(Composition.) If f is differentiable at (x_0, y_0), and φ is differentiable at $f(x_0, y_0)$, then the composition $\varphi \circ f$ is differentiable at (x_0, y_0).*

In all these cases, the partial derivatives are computed with the usual rules for differentiation:

1. $(f + g)_x = f_x + g_x$
2. $(fg)_x = f_x g + f g_x$
3. $\left(\frac{f}{g}\right)_x = \frac{f_x g - f g_x}{g^2}$
4. $(\varphi \circ f)_x(x, y) = \varphi'\left(f(x, y)\right) f_x(x, y)$

and likewise for the partial derivatives with respect to y.

Example 3.2. Polynomials are differentiable everywhere. This follows from Example 2.1, the product rule, and the sum rule.

Example 3.3. Rational functions $f = \frac{p}{q}$ are differentiable on their (maximal) domain, namely wherever q is non-zero. This follows from Example 3.2 and the quotient rule.

For example, the function

$$f(x, y) = \frac{8x^5y^3 + 3xy^2 - 10}{7x^2y - 6x + 1}$$

is differentiable wherever the condition $7x^2y - 6x + 1 \neq 0$ holds.

Example 3.4. The function

$$f(x,y) = \frac{5x^3y + e^{xy}\sqrt{x+3y+9}}{1 - x^2 - 4y^2}$$

is differentiable on the region

$$\{(x,y) \in \mathbb{R}^2 \mid x + 3y + 9 > 0 \text{ and } 1 - x^2 - 4y^2 \neq 0\}.$$

Note that the (maximal) domain of f is the region

$$D = \{(x,y) \in \mathbb{R}^2 \mid x + 3y + 9 \geq 0 \text{ and } 1 - x^2 - 4y^2 \neq 0\}$$

and in fact we know that f is continuous on D. Differentiability of f is not guaranteed on the line $x + 3y + 9 = 0$, because the function $\varphi(t) = \sqrt{t}$ is not differentiable at $t = 0$.

Example 3.5. The function

$$f(x,y) = \sqrt{(x-x_0)^2 + (y-y_0)^2}$$

from Example 2.3 is differentiable everywhere *except* at (x_0, y_0).

This follows from the composition rule, and the fact that the function $\varphi(t) = \sqrt{t}$ is differentiable at all $t > 0$.

Just for fun, let us compute the partial derivatives of f:

$$\begin{aligned}
f_x(x,y) &= \frac{1}{2\sqrt{(x-x_0)^2 + (y-y_0)^2}} 2(x-x_0)(1) \\
&= \frac{x-x_0}{\sqrt{(x-x_0)^2 + (y-y_0)^2}} \\
f_y(x,y) &= \frac{1}{2\sqrt{(x-x_0)^2 + (y-y_0)^2}} 2(y-y_0)(1) \\
&= \frac{y-y_0}{\sqrt{(x-x_0)^2 + (y-y_0)^2}}.
\end{aligned}$$

4 Sufficient conditions

We have seen that the existence of partial derivatives does not guarantee differentiability. The following theorem provides stronger conditions that do guarantee differentiability.

Theorem 4.1. *If both partial derivatives f_x and f_y exist in a neighborhood of (x_0, y_0), and f_x and f_y are* continuous *at (x_0, y_0), then f is differentiable at (x_0, y_0).*

Example 4.2. Theorem 4.1 provides an alternate proof of the following facts.

- Polynomials are differentiable everywhere.
- Rational functions are differentiable on their (maximal) domain.
- The function
$$f(x, y) = x^3y + y^5e^x + \cos(xy)$$
is differentiable everywhere, i.e., on all of $\mathbb{R}^2$.

Example 4.3 (Exercise # 46b)**.** The function

$$f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

from Example 2.4 has partial derivatives everywhere, in particular near $(0, 0)$. However, we know that f is *not* differentiable at $(0, 0)$. By Theorem 4.1, this means that the partial derivatives f_x and f_y *cannot* both be continuous at $(0, 0)$.

Exercise 4.4. With that function f, show *directly* that f_x is discontinuous at $(0, 0)$ by first computing f_x.

Exercise 4.5. Same exercise, but with the function

$$f(x,y) = \begin{cases} \frac{y^3}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

from Example 2.5.