Calculus 2302A - Intermediate Calculus I
Fall 2013
§13.4: Tangential and normal acceleration

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Notation 1. Let us denote by:

- $\overrightarrow{r}(t)$ the position of a particle at time t.
- $\vec{v}(t) := \overrightarrow{r}'(t)$ the velocity.
- $v(t) := |\vec{v}(t)|$ the speed.
- $\vec{a}(t) := \overrightarrow{r}''(t)$ the acceleration.
- $\vec{T}(t) := \frac{\overrightarrow{r}'(t)}{|\overrightarrow{r}'(t)|}$ the unit tangent vector, whenever $\overrightarrow{r}'(t) \neq \vec{0}$.
- $\vec{N}(t) := \frac{\vec{T}'(t)}{|\vec{T}'(t)|}$ the principal unit normal vector, whenever $\vec{T}'(t) \neq \vec{0}$.
- $\kappa(t) = \frac{|\vec{T}'(t)|}{|\overrightarrow{r}'(t)|}$ the curvature.

The acceleration vector $\vec{a}(t)$ is always in the "osculating plane", spanned by $\vec{T}(t)$ and $\vec{N}(t)$.

Notation 2. Let us write the acceleration vector as

$$\vec{a}(t) = a_T\,\vec{T}(t) + a_N\,\vec{N}(t)$$

where a_T denotes the **tangential component** of $\vec{a}(t)$ and a_N denotes the **normal component** of $\vec{a}(t)$.

Note that $a_N \geq 0$ holds for all t, by our choice of the normal vector $\vec{N}(t)$.

Proposition 3. *The tangential component of the acceleration satisfies*

$$a_T = v' = \frac{\vec{v} \cdot \vec{a}}{v} = \frac{\overrightarrow{r}'(t) \cdot \overrightarrow{r}''(t)}{|\overrightarrow{r}'(t)|}.$$

The normal component of the acceleration satisfies

$$a_N = \kappa v^2 = \frac{|\vec{v} \times \vec{a}|}{v} = \frac{|\overrightarrow{r}'(t) \times \overrightarrow{r}''(t)|}{|\overrightarrow{r}'(t)|}.$$

Proposition 4. *The following conditions for a parametrized curve $\vec{r}(t)$ are equivalent.*

1. *The acceleration is always orthogonal to the velocity, i.e., $\vec{r}'(t) \cdot \vec{r}''(t) = 0$ holds for all t.*
2. *The tangential component a_T of the acceleration is identically zero, i.e., $a_T = 0$ holds for all t.*
3. *The speed $|\vec{r}'(t)|$ is constant.*

Example 5. Consider a particle moving along a circle of radius R increasingly fast, with position function

$$\vec{r}(t) = \left(R\cos t^2, R\sin t^2\right)$$

for $t \geq 0$. Let us find the tangential component a_T and normal component a_N of the acceleration.

The velocity is

$$\begin{aligned}\vec{r}'(t) &= R\left(-2t\sin t^2, 2t\cos t^2\right)\\ &= 2Rt\left(-\sin t^2, \cos t^2\right)\end{aligned}$$

whose magnitude is

$$|\vec{r}'(t)| = 2Rt.$$

The acceleration is

$$\begin{aligned}\vec{r}''(t) &= 2R\left(-\sin t^2, \cos t^2\right) + 2Rt\left(-2t\cos t^2, -2t\sin t^2\right)\\ &= 2R\left(-\sin t^2, \cos t^2\right) - 4Rt^2\left(\cos t^2, \sin t^2\right).\end{aligned}$$

Then we have:

$$\begin{aligned}\vec{r}'(t)\cdot\vec{r}''(t) &= 2Rt\left(-\sin t^2, \cos t^2\right)\cdot\left(2R\left(-\sin t^2, \cos t^2\right) - 4Rt^2\left(\cos t^2, \sin t^2\right)\right)\\ &= 4R^2t\left(-\sin t^2, \cos t^2\right)\cdot\left(-\sin t^2, \cos t^2\right) - 8R^2t^3\left(-\sin t^2, \cos t^2\right)\cdot\left(\cos t^2, \sin t^2\right)\\ &= 4R^2t(1) - 8R^2t^3(0)\\ &= 4R^2t\end{aligned}$$

and therefore the tangential component of the acceleration is

$$\begin{aligned}a_T &= \frac{\vec{r}'(t)\cdot\vec{r}''(t)}{|\vec{r}'(t)|}\\ &= \frac{4R^2t}{2Rt}\\ &= \boxed{2R}.\end{aligned}$$

We also have (viewing $\mathbb{R}^2$ as the xy-plane in $\mathbb{R}^3$):

$$\begin{aligned}\vec{r}'(t)\times\vec{r}''(t) &= 2Rt\left(-\sin t^2, \cos t^2, 0\right)\times\left(2R\left(-\sin t^2, \cos t^2, 0\right) - 4Rt^2\left(\cos t^2, \sin t^2, 0\right)\right)\\ &= -8R^2t^3\left(-\sin t^2, \cos t^2, 0\right)\times\left(\cos t^2, \sin t^2, 0\right)\\ &= -8R^2t^3\left(-\sin^2 t^2\,\vec{i}\times\vec{j} + \cos^2 t^2\,\vec{j}\times\vec{i}\right)\\ &= -8R^2t^3\left(-\sin^2 t^2 - \cos^2 t^2\right)\vec{k}\\ &= 8R^2t^3\vec{k}\end{aligned}$$

and therefore the normal component of the acceleration is

$$a_N = \frac{|\overrightarrow{r}'(t) \times \overrightarrow{r}''(t)|}{|\overrightarrow{r}'(t)|}$$

$$= \frac{8R^2t^3}{2Rt}$$

$$= \boxed{4Rt^2}.$$

Remark 6. We can explicitly verify Proposition 3 in Example 5.

The speed function is $v(t) = 2Rt$ and its derivative is $v'(t) = 2R$, so that the equality $\boxed{a_T = v'}$ holds indeed.

The curvature of a circle of radius R is the constant function $\kappa(t) = \frac{1}{R}$. Thus, we have

$$\kappa v^2 = \frac{1}{R}(2Rt)^2$$

$$= \frac{4R^2t^2}{R}$$

$$= 4Rt^2$$

so that the equality $\boxed{a_N = \kappa v^2}$ holds indeed.