A CONJUGACY CRITERION FOR PURE
$E_0$-SEMIGROUPS

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Abstract. We show that the conjugacy class of an arbitrary pure $E_0$-semigroup is completely determined by its class of normal cocycles. If the $E_0$-semigroup admits a normal invariant state, then its conjugacy class is determined by the class of cocycles that stabilize density lists.

Introduction

Introduced by R.T. Powers in [7], $E_0$-semigroups are $\sigma$-weak continuous one-parameter semigroups $\rho = \{\rho_t\}_{t \geq 0}$ of unital $*$-endomorphisms of the von Neumann algebra $\mathcal{B}(H)$ of all bounded linear operators on a separable Hilbert space $H$ (see ref. [3] for a comprehensive treatment of the subject).

Two central results of the theory of $E_0$-semigroups assert that (i) every spatial $E_0$-semigroup is cocycle conjugate to an $E_0$-semigroup in standard form [8], and; (ii) the conjugacy classes of the $E_0$-semigroups in standard form within the cocycle conjugacy class of a spatial $E_0$-semigroup $\rho$ correspond to the orbits of the action of the local unitary $\rho$-cocycles on the set of semigroups of intertwining isometries of $\rho$ [1].

Powers’ reduction result (i) was extended in whole generality in [6], [4], where it was shown that every $E_0$-semigroup is cocycle conjugate to a pure $E_0$-semigroup, i.e., an $E_0$-semigroup $\rho = \{\rho_t\}_{t \geq 0}$ of which the tail algebra

$$T_\rho = \bigcap_{t \geq 0} \rho_t(\mathcal{B}(H))$$

reduces to scalars.

Our main purpose, in this paper, is to present a necessary and sufficient criterion for determining conjugacy within the class of cocycle conjugate, pure $E_0$-semigroups. This criterion captures and extends to full generality Alevras’ result (ii).

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As mentioned before, the present article concerns the relationship between conjugacy and cocycle conjugacy. We recall that two $E_0$-semigroups $\rho = \{\rho_t\}_{t \geq 0}$ and $\sigma = \{\sigma_t\}_{t \geq 0}$ of $\mathcal{B}(H)$ are conjugate, if there exists an automorphism $\theta$ of $\mathcal{B}(H)$ such that $\sigma_t \circ \theta = \theta \circ \rho_t$, for every $t \geq 0$. They are said to be cocycle conjugate, if $\sigma$ is conjugate to a cocycle perturbation $\{\text{Ad}(u_t) \circ \rho_t\}_{t \geq 0}$ of $\rho$, where $u = \{u_t\}_{t \geq 0}$ is a $\rho$-cocycle, i.e., a strongly continuous family of unitary operators $u_t$ of $\mathcal{B}(H)$ satisfying the cocycle relation $u_{t+s} = u_t \rho_t(u_s)$, $t, s \geq 0$.

This paper is structured as follows. In Section 1, we associate to each $E_0$-semigroup $\rho = \{\rho_t\}_{t \geq 0}$ of $\mathcal{B}(H)$, a $C^*$-semi-dynamical system $(\mathcal{A}_\rho, \rho \upharpoonright \mathcal{A}_\rho)$, where $\mathcal{A}_\rho$ is the $C^*$-algebra generated by the tower of relative commutants of $\rho$. The conjugacy class of $(\mathcal{A}_\rho, \rho \upharpoonright \mathcal{A}_\rho)$ is a cocycle conjugacy invariant for $\rho$, and this is used to prove the main result of this paper (Theorem 1.5), which shows that the conjugacy class of a pure $E_0$-semigroup corresponds to its normal cocycles. In Section 2, we discuss this conjugacy criterion within the class of pure $E_0$-semigroups that admit normal invariant states.

1. CONJUGACY OF PURE $E_0$-SEMIGROUPS

Let $\rho = \{\rho_t\}_{t \geq 0}$ be an $E_0$-semigroup of $\mathcal{B}(H)$. For every $t \geq 0$, we consider the relative commutant

$$\mathcal{A}_\rho(t) = \rho_t(\mathcal{B}(H))' \cap \mathcal{B}(H)$$

of the von Neumann algebra $\rho_t(\mathcal{B}(H))$ in $\mathcal{B}(H)$. The family $\{\mathcal{A}_\rho(t)\}_{t \geq 0}$, referred in [3] as the tower of the $E_0$-semigroup $\rho$, is an increasing family of type $I_\infty$ factors which gives rise to the local $C^*$-algebra

$$\mathcal{A}_\rho = \bigcup_{t \geq 0} \mathcal{A}_\rho(t).$$

Lemma 1.1. The $C^*$-algebra $\mathcal{A}_\rho$ is simple, and is irreducible in $\mathcal{B}(H)$ if and only if the $E_0$-semigroup $\rho$ is pure. The restriction $\rho \upharpoonright \mathcal{A}_\rho = \{\rho_t \upharpoonright \mathcal{A}_\rho\}_{t \geq 0}$ of the $E_0$-semigroup $\rho$ to $\mathcal{A}_\rho$ is a semigroup of unital $*$-endomorphisms of $\mathcal{A}_\rho$.

Proof. Since the relative commutant of $\mathcal{A}_\rho(s)$ in $\mathcal{A}_\rho(t)$, $0 < s < t$, is the type $I_\infty$ factor $\rho_s(\mathcal{A}_\rho(t-s))$, the simplicity of the $C^*$-algebra $\mathcal{A}_\rho$ follows from [5, Prop. 10]. The statement regarding irreducibility follows from the fact that the von Neumann algebra generated by the tower $\{\mathcal{A}_\rho(t)\}_{t \geq 0}$ coincides with the relative commutant $T'_\rho$ of the tail algebra $T_\rho$ of $\rho$, while the last statement follows from the fact that the tower $\{\mathcal{A}_\rho(t)\}_{t \geq 0}$ is shifted by the action of $\rho$, i.e., $\rho_s(\mathcal{A}_\rho(t)) \subseteq \mathcal{A}_\rho(s+t)$, for all $s, t \geq 0$. □
Definition 1.2. The ensemble \((\mathcal{A}_\rho, \rho \upharpoonright_{\mathcal{A}_\rho})\) is called the \(C^*\)-semiflow of the \(E_0\)-semigroup \(\rho\).

A central property of the \(C^*\)-semiflow of an \(E_0\)-semigroup is that its conjugacy class is a cocycle conjugacy invariant.

Proposition 1.3. If \(\rho = \{\rho_t\}_{t \geq 0}\) and \(\sigma = \{\sigma_t\}_{t \geq 0}\) are cocycle conjugate \(E_0\)-semigroups of \(\mathcal{B}(H)\), then their associated \(C^*\)-semiflows \((\mathcal{A}_\rho, \rho \upharpoonright_{\mathcal{A}_\rho})\) and \((\mathcal{A}_\sigma, \sigma \upharpoonright_{\mathcal{A}_\sigma})\) are conjugate.

Proof. Without loss of generality, we may assume that \(\sigma\) is a cocycle perturbation of \(\rho\), i.e., \(\sigma_t = \text{Ad}(u_t) \circ \rho_t\), \(t \geq 0\), where \(u = \{u_t\}_{t \geq 0}\) is a \(\rho\)-cocycle.

For every \(t \geq 0\), the automorphism \(\text{Ad}(u_t)\) of \(\mathcal{B}(H)\) restricts to a \(*\)-isomorphism \(\gamma_t := \text{Ad}(u_t) \upharpoonright_{\mathcal{A}_\rho(t)}: \mathcal{A}_\rho(t) \to \mathcal{A}_\sigma(t)\), (1.2) and the family \(\{\gamma_t\}_{t \geq 0}\) is coherent with respect to inclusion, i.e.,

\[\gamma_t \upharpoonright_{\mathcal{A}_\rho(s)} = \gamma_s, \quad 0 \leq s < t,\]

as one can easily see. We deduce from the universal property of the \(C^*\)-inductive limit that there exists a unique \(*\)-isomorphism \(\gamma_u: \mathcal{A}_\rho \to \mathcal{A}_\sigma\) satisfying the property

\[\gamma_u \upharpoonright_{\mathcal{A}_\rho(t)} = \gamma_t, \quad t \geq 0.\]

Moreover, since

\[\sigma_t \upharpoonright_{\mathcal{A}_\sigma(s)} \circ \gamma_s = \gamma_{t+s} \circ \rho_t \upharpoonright_{\mathcal{A}_\rho(s)}, \quad s, t \geq 0,\]

we infer that \(\sigma_t \upharpoonright_{\mathcal{A}_\sigma} \circ \gamma_u = \gamma_u \circ \rho_t \upharpoonright_{\mathcal{A}_\rho}\), for every \(t \geq 0\). Therefore the \(C^*\)-semiflows \((\mathcal{A}_\rho, \rho \upharpoonright_{\mathcal{A}_\rho})\) and \((\mathcal{A}_\sigma, \sigma \upharpoonright_{\mathcal{A}_\sigma})\) are conjugate, the conjugacy being implemented by \(\gamma_u\). \(\square\)

Definition 1.4. Let \(u = \{u_t\}_{t \geq 0}\) be a \(\rho\)-cocycle, and \(\sigma\) be the cocycle perturbation of \(\rho\) by \(u\). The \(*\)-isomorphism \(\gamma_u: \mathcal{A}_\rho \to \mathcal{A}_\sigma\) that satisfies

\[\gamma_u \upharpoonright_{\mathcal{A}_\rho(t)} = \gamma_t, \quad t \geq 0.\]

is called the quasi-free isomorphism induced by \(u\). The cocyle \(u\) is said to be normal, if the quasi-free isomorphism \(\gamma_u\) is normal, i.e., continuous with respect to the \(\sigma\)-weak operator topology of \(\mathcal{B}(H)\).

The main result of this paper asserts that the conjugacy classes of a pure \(E_0\)-semigroup \(\rho\) correspond to the set of all normal \(\rho\)-cocycles.

Theorem 1.5. Let \(\rho = \{\rho_t\}_{t \geq 0}\) and \(\sigma = \{\sigma_t\}_{t \geq 0}\) be pure \(E_0\)-semigroups of \(\mathcal{B}(H)\). The following two conditions are equivalent:

(i) \(\rho\) and \(\sigma\) are conjugate;
(ii) there exists a normal \(\rho\)-cocycle \(u = \{u_t\}_{t \geq 0}\) such that \(\sigma_t = \text{Ad}(u_t) \circ \rho_t\), for every \(t \geq 0\).
Proof. (i)⇒(ii). Suppose $\rho$ and $\sigma$ are conjugate, and let $u \in B(H)$ be a unitary operator such that $\sigma_t = \text{Ad}(u) \circ \rho_t \circ \text{Ad}(u^*)$, $t \geq 0$. It then easily follows that the exact $\rho$-cocycle $uuu(u) =\{u\rho_t(u^*)\}_{t \geq 0}$ has the required properties.

(ii)⇒(i). Let $uuu =\{u_t\}_{t \geq 0}$ be a normal $\rho$-cocycle such that $\sigma_t = \text{Ad}(u_t) \circ \rho_t$, for every $t \geq 0$. By Proposition 1.3, the $C^*$-semiflows $(A_\rho, \rho \restriction A_\rho)$ and $(A_\sigma, \sigma \restriction A_\sigma)$ are conjugate, the conjugacy being implemented by the quasi-free isomorphism $\gamma_{uuu}$. We claim that $\gamma_{uuu}$ admits a unique extension to an automorphism of $B(H)$, and this automorphism implements the conjugacy between the $E_0$-semigroups $\rho$ and $\sigma$.

The proof of this claim is based on the following standard arguments. Since $\gamma_{uuu}$ is normal, the linear functional $a \mapsto \text{tr}(\Omega \gamma_{uuu}(a))$, $a \in A_\rho$, is also normal on $A_\rho$, for every trace class operator $\Omega \in L^1(H)$. Moreover, since the $C^*$-algebra $A_\rho$ is irreducible in $B(H)$, this functional admits a unique extension to a normal linear functional on $B(H)$. Therefore there exists a unique trace class operator $\tilde{\Omega} \in L^1(H)$ such that

$$\text{tr}(\Omega \gamma_{uuu}(a)) = \text{tr}(\tilde{\Omega}a), \quad a \in A_\rho.$$ 

The resulting mapping $\Gamma : L^1(H) \to L^1(H)$, $\Gamma(\Omega) = \tilde{\Omega}$, is a bounded linear operator of the Banach space $L^1(H)$, of which adjoint $\Gamma^* : B(H) \to B(H)$ satisfies the relation

$$\text{tr}(\Omega \gamma_{uuu}(a)) = \text{tr}(\Omega \Gamma^*(a)), \quad a \in A_\rho, \quad \Omega \in L^1(H).$$

We infer that $\Gamma^*$ is an extension of $\gamma_{uuu}$ to $B(H)$. Moreover, by applying the same arguments to the inverse of the quasi-free isomorphism $\gamma_{uuu}$, we deduce that $\Gamma^*$ is a $*$-isomorphism of $B(H)$. Finally, since $A_\rho$ and $A_\sigma$ are irreducible in $B(H)$, and since $\gamma_{uuu}$ implements the conjugacy between $\rho \restriction A_\rho$ and $\sigma \restriction A_\rho$, we conclude that $\Gamma^*$ is the unique extension of $\gamma_{uuu}$, and that it implements a conjugacy between the $E_0$-semigroups $\rho$ and $\sigma$. □

2. CONJUGACY OF PURE $E_0$-SEMIGROUPS WITH NORMAL IN Variant STATES

In this section, we focus our attention on describing conjugacy classes within the class of cocycle conjugate pure $E_0$-semigroups that admit normal invariant states. We recall that if a normal state $\omega_\rho$ of $B(H)$ is invariant for a pure $E_0$-semigroup $\rho =\{\rho_t\}_{t \geq 0}$ of $B(H)$, i.e., $\omega_\rho \circ \rho_t = \omega_\rho$, $t \geq 0$, then $\omega_\rho$ is the unique normal invariant state for $\rho$ (see ref. [2] for a complete description).

The following result enhances the conjugacy criterion obtained in Theorem 1.5
Semigroups of

Let $\rho = \{\rho_t\}_{t \geq 0}$ and $\sigma = \{\sigma_t\}_{t \geq 0}$ be two pure $E_0$-semigroups of $\mathcal{B}(H)$ with invariant normal states $\omega_\rho$, respectively $\omega_\sigma$. The following conditions are equivalent:

(i) $\rho$ and $\sigma$ are conjugate $E_0$-semigroups;
(ii) there exists a $\rho$-cocycle $u = \{u_t\}_{t \geq 0}$ such that $\sigma_t = \text{Ad}(u_t) \circ \rho_t$,
    \( t \geq 0 \), and $\omega_\rho = \omega_\sigma \circ \gamma_u$ on $\mathcal{A}_\rho$.

Proof. (i) $\Rightarrow$ (ii). As in the proof of Theorem 1.5, we consider the exact $\rho$-cocycle $u = \{u\rho_t(u^*)\}_{t \geq 0}$, where $u \in \mathcal{B}(H)$ is a unitary operator such that $\sigma_t = \text{Ad}(u) \circ \rho_t \circ \text{Ad}(u^*)$, $t \geq 0$. Then for every $t \geq 0$, $\omega_\sigma \circ \gamma_u \mid \mathcal{A}_\rho(t) = \omega_\sigma \circ \text{Ad}(u) \mid \mathcal{A}_\rho(t)$, hence $\omega_\sigma \circ \gamma_u = \omega_\sigma \circ \text{Ad}(u)$. We then deduce from the uniqueness of the invariant states $\omega_\sigma$ and $\omega_\rho$ that $\omega_\sigma \circ \gamma_u = \omega_\rho$.

(ii) $\Rightarrow$ (i). Let $u = \{u_t\}_{t \geq 0}$ be a $\rho$-cocycle satisfying the conditions of (ii). We claim that $u$ is normal. Indeed, since $\omega_\rho = \omega_\sigma \circ \gamma_u$, and the $C^*$-algebras $\mathcal{A}_\rho$ and $\mathcal{A}_\sigma$ are simple, we deduce that the quasi-free isomorphism $\gamma_u$ admits an extension to a $^*$-isomorphism from $\pi_{\omega_\rho}(\mathcal{A}_\rho)^\prime\prime$ to $\pi_{\omega_\sigma}(\mathcal{A}_\rho)^\prime\prime$, where $\pi_{\omega_\rho}$ and $\pi_{\omega_\sigma}$ are the GNS representations of $\mathcal{A}_\rho$, respectively $\mathcal{A}_\sigma$, with respect to the states $\omega_\rho$ and $\omega_\sigma$. Moreover, since the states $\omega_\rho$ and $\omega_\sigma$ are normal, by using the irreducibility of the $C^*$-algebras $\mathcal{A}_\rho$ and $\mathcal{A}_\sigma$, we infer that the representation $\pi_{\omega_\rho}$, respectively $\pi_{\omega_\sigma}$, is quasi-equivalent to the identity representation of $\mathcal{A}_\rho$, respectively $\mathcal{A}_\sigma$, on $\mathcal{B}(H)$. Consequently, $\gamma_u$ admits an extension to a $^*$-automorphism of $\mathcal{B}(H)$, hence $u$ is a normal $\rho$-cocycle. The conclusion follows then from Theorem 1.5. $\square$

Let $\rho = \{\rho_t\}_{t \geq 0}$ be a pure $E_0$-semigroup of $\mathcal{B}(H)$ with normal invariant state $\omega_\rho$, and let $\mathcal{E}_\rho = \{\mathcal{E}_\rho(t)\}_{t \geq 0}$ be the concrete product system of $\rho$, i.e., $\mathcal{E}_\rho \subset (0, \infty) \times \mathcal{B}(H)$ is the Borel bundle of which fibers $\mathcal{E}_\rho(t)$ are the Hilbert spaces of intertwining operators

$$
\mathcal{E}_\rho(t) = \{a \in \mathcal{B}(H) \mid \rho_t(x)a = ax, \text{ for all } x \in \mathcal{B}(H)\}, \quad t > 0,
$$

with inner product $\langle a, b \rangle_{\mathcal{E}_\rho(t)} \cdot 1_H = b^*a$.

The state $\omega_\rho \mid \mathcal{A}_\rho$ is completely determined by a family $\Omega_\rho = \{\Omega_{t, \rho}\}_{t \geq 0}$ of positive trace-class operators $\Omega_{t, \rho} \in \mathcal{L}^1(\mathcal{E}_\rho(t))$, henceforth referred to as the density list of $\rho$, where each $\Omega_{t, \rho}$ uniquely characterized by the relation

$$
\langle \Omega_{t, \rho}x, y \rangle_{\mathcal{E}_\rho(t)} = \omega_\rho(xy^*), \quad x, y \in \mathcal{E}_\rho(t).
$$

Indeed, $\omega_\rho \mid \mathcal{A}_\rho$ is determined by the family of states $\{\omega_\rho \mid \mathcal{A}_\rho(t)\}_{t \geq 0}$, and each normal state $\omega_\rho \mid \mathcal{A}_\rho(t)$ has the form (see [3])

$$
\omega_\rho(a) = \text{tr}(\Omega_{t, \rho} \varphi_t^{-1}(a)), \quad a \in \mathcal{A}_\rho(t),
$$
where \( \vartheta_t : \mathcal{B}(\mathcal{E}_\rho(t)) \to \mathcal{A}_t(\rho) \) is the \(^*\)-isomorphism

\[
\vartheta_t(a) = \text{Ad}(W_{t,\rho})(a \otimes 1_H), \quad a \in \mathcal{B}(\mathcal{E}_\rho(t)),
\]

that is implemented by the unitary operator \( W_{t,\rho} : \mathcal{E}_\rho(t) \otimes H \to H, \)

\[
W_{t,\rho}(x \otimes \xi) = x\xi, \quad x \in \mathcal{E}_\rho(t), \quad \xi \in H.
\]

By regarding \( \rho \)-cocycles as isomorphisms of concrete product systems [3, Th. 2.4.10], we deduce from the previous theorem that the conjugacy class of \( \rho = \{\rho_t\}_{t \geq 0} \) corresponds to the class of all \( \rho \)-cocycles that stabilize density lists.

**Corollary 2.2.** Let \( \rho = \{\rho_t\}_{t \geq 0} \) and \( \sigma = \{\sigma_t\}_{t \geq 0} \) be two pure \( E_0 \)-semigroups of \( \mathcal{B}(H) \) with invariant normal states \( \omega_\rho \), respectively \( \omega_\sigma \), and corresponding density lists \( \Omega_\rho = \{\Omega_{t,\rho}\}_{t > 0}, \) respectively \( \Omega_\sigma = \{\Omega_{t,\sigma}\}_{t > 0} \).

The following conditions are equivalent:

(i) \( \rho \) and \( \sigma \) are conjugate \( E_0 \)-semigroups;

(ii) there exists a \( \rho \)-cocycle \( u = \{u_t\}_{t \geq 0} \) such that \( \sigma_t = \text{Ad}(u_t) \circ \rho_t, \)

\[ t \geq 0, \quad \text{and} \quad \text{Ad}(u_t)(\Omega_{t,\rho}) = \Omega_{t,\sigma}, \quad t > 0. \]

If the normal invariant state of a pure \( E_0 \)-semigroup \( \rho = \{\rho_t\}_{t \geq 0} \) of \( \mathcal{B}(H) \) is a vector state \( \omega_\xi, \xi \in H \), then the \( E_0 \)-semigroup \( \rho \) is said to be in standard form (see refs. [8][1]). An \( E_0 \)-semigroup in standard form \( \rho = \{\rho_t\}_{t \geq 0} \) admits a distinguished strongly continuous semigroup \( \{R_\rho(t)\}_{t \geq 0} \) of intertwining isometries \( R_\rho(t) \in \mathcal{E}_\rho(t) \), defined by the formula

\[ R_\rho(t)x\xi = \rho_t(x)\xi, \quad x \in \mathcal{B}(H). \]

Using (2.1) and the fact that the unit vector \( \xi \) is a common eigenvector of the operators \( a^*, \quad a \in \mathcal{E}_\rho(t), \quad t > 0 \) (see [1]), we deduce that the density list of \( \rho \) consists on the family of projections \( \{\psi^{-1}_t(R_\rho(t)R_\rho(t)^*)\}_{t > 0} \).

The next corollary, which follows from the previous one, is equivalent to Alevras’ criterion mentioned in the introduction (see [1, Th. 4.1]).

**Corollary 2.3.** Let \( \rho = \{\rho_t\}_{t \geq 0} \) and \( \sigma = \{\sigma_t\}_{t \geq 0} \) be \( E_0 \)-semigroups in standard form acting on \( \mathcal{B}(H) \) with invariant vector states \( \omega_\xi \), respectively \( \omega_\eta \). The following conditions are equivalent:

(i) \( \rho \) and \( \sigma \) are conjugate \( E_0 \)-semigroups;

(ii) there exists a \( \rho \)-cocycle \( u = \{u_t\}_{t \geq 0} \) such that \( \sigma_t = \text{Ad}(u_t) \circ \rho_t \)

and \( u_tR_\rho(t) = R_\sigma(t), \quad t \geq 0, \) where \( \{R_\rho(t)\}_{t \geq 0} \) and \( \{R_\sigma(t)\}_{t \geq 0} \) are the distinguished semigroups of intertwining isometries constructed as in (2.3).
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