Chapter 3
Kinematics in Two Dimensions; Vectors
• Vectors and Scalars
• Addition of Vectors – Graphical Methods (One and Two-Dimension)
• Multiplication of a Vector by a Scalar
• Subtraction of Vectors – Graphical Methods
• Adding Vectors by Components
• Projectile Motion
• Projectile Motion Is Parabolic
• Relative Velocity
Tutorial every **Wednesday** from **4:30pm** to **5:30pm** in **CL-127**.
Recalling Last Lecture
Adding Vectors by Components

\[ |\vec{V}| = V \]  \hspace{2cm} (3.5)

\[ |\vec{V}| = V = \sqrt{V_x^2 + V_y^2} \]  \hspace{2cm} (3.6)

\[ (\vec{V})_x = V_x \quad \quad (\vec{V})_y = V_y \]
Adding Vectors by Components

\[ \vec{V} = \vec{V}_1 + \vec{V}_2 \]

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Adding Vectors by Components

Using equations 3.8 – 3.10, we can write the components of the vector \( \vec{V} \) as:

\[
V_x = V \cos \theta \quad (3.11)
\]

\[
V_y = V \sin \theta \quad (3.12)
\]

\[
\theta = \sin^{-1} \left( \frac{V_y}{V} \right) \quad \theta = \tan^{-1} \left( \frac{V_y}{V_x} \right)
\]

\[
\theta = \cos^{-1} \left( \frac{V_x}{V} \right)
\]

\[
V^2 = V_x^2 + V_y^2 \quad (3.10)
\]
Adding Vectors by Components

Note about arc-tan (\(\tan^{-1}\)):

The sign of \(\theta\) in the expression below depends on the signs of \(V_y\) and \(V_x\):

\[
\theta = \tan^{-1}\left(\frac{V_y}{V_x}\right)
\]

- \(\theta < 0\) if: \(V_y < 0\) and \(V_x > 0\), or \(V_y > 0\) and \(V_x < 0\)
- \(\theta > 0\) if: \(V_y < 0\) and \(V_x < 0\), or \(V_y > 0\) and \(V_x > 0\)

The quadrant where the vector is defined is given by its coordinates:

- \(V_y < 0\) and \(V_x > 0\) \(\rightarrow\) 4\(^{th}\) quadrant
- \(V_y < 0\) and \(V_x < 0\) \(\rightarrow\) 3\(^{rd}\) quadrant
- \(V_y > 0\) and \(V_x < 0\) \(\rightarrow\) 2\(^{nd}\) quadrant
- \(V_y > 0\) and \(V_x > 0\) \(\rightarrow\) 1\(^{st}\) quadrant

<table>
<thead>
<tr>
<th>2(^{nd}) quadrant</th>
<th>1(^{st}) quadrant</th>
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<tbody>
<tr>
<td>3(^{rd}) quadrant</td>
<td>4(^{th}) quadrant</td>
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Projectile Motion

We can then analyze the motion in the $x$ and $y$ direction separately using the equations of motion we obtained in chapter 2:

**horizontal motion**

\[
\begin{align*}
  \mathbf{v} &= \mathbf{v}_0 + \mathbf{a}t \\
  x &= x_0 + \mathbf{v}_0 t + \frac{1}{2} \mathbf{a} t^2 \\
  v^2 &= v_0^2 + 2a(x - x_0)
\end{align*}
\]

(2.10) (2.13) (2.14)

**vertical motion**

\[
\begin{align*}
  \mathbf{v} &= \mathbf{v}_0 + \mathbf{g}t \\
  y &= y_0 + \mathbf{v}_0 t + \frac{1}{2} \mathbf{g} t^2 \\
  v^2 &= v_0^2 + 2g(y - y_0)
\end{align*}
\]

(2.16) (2.17) (2.18)

**Note:** The sign of the acceleration $\mathbf{g}$ in Eqs. 2.16-2.17 depends on your choice of orientation of the axis $y$. You will have to rewrite the above equations using $-\mathbf{g}$ instead of $\mathbf{g}$ if you decide to choose $+y$ pointing upwards as in the figure.
**Projectile Motion**

### y direction

We want to find the position $y$ and velocity $v_y$ (in the $y$-direction) at a given instant of time “$t$”.

In the vertical direction, we have the acceleration, $g$, of gravity. The initial velocity is zero ($V_{y0} = 0$). The coordinate $y$ is chosen to be positive upward and zero at the origin selected as where the point the ball starts experiencing vertical motion as depicted in the figure. Then, at $t = 0$ we have:

- $y_0 = 0$
- $V_{y0} = 0$
- $a = -g$  
  ("$a$" points downward)

Using equation 2.17 we have:

$$y = y_0 + v_0 t + \frac{1}{2} (-g) t^2 = y = 0 + (0) t + \frac{1}{2} (-g) t^2$$

$$y = -\frac{1}{2} gt^2 \quad (3.13)$$

Using eq. 2.16:

$$v_y = v_{y0} + (-g)t = 0 - gt \quad v_y = -gt \quad (3.14)$$
**Projectile Motion**

**x direction**

In the horizontal direction, there is no acceleration. The initial velocity is $V_{x0}$. The coordinate $x$ is chosen to be eastward and zero at the origin selected as the point where the ball starts experiencing vertical motion as depicted in the figure.

Then, at $t = 0$ we have:

$x_0 = 0$

*initial velocity = $V_{x0}$

$a = 0$

Using equation 2.13 we have:

\[
x = x_0 + v_{x0} t + \frac{1}{2} a t^2 = 0 + v_{x0} t + \frac{1}{2} (0) t^2
\]

\[
x = v_{x0} t
\]

(3.15)

Using eq. 2.10:

\[
v_x = v_{x0} + at = v_{x0} + (0) t
\]

\[
v_x = v_{x0}
\]

(3.16)

The velocity is constant in the horizontal direction.
**Example 3.5:**
Show that an object projected horizontally will reach the ground in the same time as an object dropped vertically from the same height (use the figure below).
**Example 3.5:**
Show that an object projected horizontally will reach the ground in the same time as an object dropped vertically from the same height (use the figure below).

The only motion that matters here is the vertical (displacement in the y direction).

Equation 3.13 gives the y displacement for both balls since they are subject to similar initial conditions, namely:

\[ y_0 = 0 \]
\[ V_{y0} = 0 \]
\[ a = -g \quad (a \text{ points downward}) \]

Therefore

\[ y = -\frac{1}{2} gt^2 \]

applies in both cases at any instant of time.

**Thus, the two ball reach the ground together.**
Projectile Motion

If an object is launched at an initial angle of \( \theta_0 \) with the horizontal, the analysis is similar except that the initial velocity has a vertical component.

You can use equations 3.11 and 3.12 and then develop the problem in the very same way as on the previous slides.

\[ V_{x0} = V_0 \cos \theta_0 \]

\[ V_{y0} = V_0 \sin \theta_0 \]

\( \theta \) is the angle the vector velocity makes with the x direction at a given instant of time \( t \).
Problem 3.21 (textbook): A ball is thrown horizontally from the roof of a building 45.0 m tall and lands 24.0 m from the base. What was the ball’s initial velocity?

Solution developed on the blackboard
Problem 3.21 (textbook): A ball is thrown horizontally from the roof of a building 45.0 m tall and lands 24.0 m from the base. What was the ball’s initial speed?

Choose downward to be the positive y direction. The origin will be at the point where the ball is thrown from the roof of the building. In the vertical direction,

\[
\begin{align*}
    v_{y0} &= 0 \\
    a_y &= 9.80 \text{ m/s}^2 \\
    y_0 &= 0
\end{align*}
\]

and the displacement is 45.0 m. The time of flight is found from applying Eq. 2.17 to the vertical motion.

\[
y = y_0 + v_{y0} t + \frac{1}{2} a_y t^2 
\rightarrow \quad 45.0 \text{ m} = \frac{1}{2} (9.80 \text{ m/s}^2) t^2
\rightarrow \quad t = \sqrt{\frac{2 (45.0 \text{ m})}{9.80 \text{ m/s}^2}} = 3.03 \text{ sec}
\]

The horizontal speed (which is the initial speed) is found from the horizontal motion at constant velocity:

\[
\Delta x = v_x t 
\rightarrow \quad v_x = \Delta x / t = 24.0 \text{ m} / 3.03 \text{ s} = 7.92 \text{ m/s}
\]
**Problem 3.31 (textbook)**: A projectile is shot from the edge of a cliff 125 m above ground level with an initial speed of 65.0 m/s at an angle of 37.0° with the horizontal, as shown in Fig. 3–35.

(a) Determine the time taken by the projectile to hit point P at ground level.
(b) Determine the range $X$ of the projectile as measured from the base of the cliff.

At the instant just before the projectile hits point P:

(c) Find the horizontal and the vertical components of its velocity
(d) Find the magnitude of the velocity
(e) Find the angle made by the velocity vector with the horizontal.
(f) Find the maximum height above the cliff top reached by the projectile.

**Solution developed on the blackboard**
Choose the origin to be at ground level, under the place where the projectile is launched, and upwards to be the positive $y$ direction. For the projectile,

\[
v_0 = 65.0 \text{ m/s} \\
\theta_0 = 37.0^\circ \\
a_y = -g \\
y_0 = 125 \text{ m} \\
v_{y0} = v_0 \sin \theta_0
\]

The time taken to reach the ground is found from Eq. 2.17, with a final height of 0.

\[
y = y_0 + v_{y0} t + \frac{1}{2} a_y t^2 
\rightarrow \quad 0 = 125 + v_0 \sin \theta_0 t - \frac{1}{2} g t^2 
\rightarrow
\]

\[
t = \frac{-v_0 \sin \theta_0 \pm \sqrt{v_0^2 \sin^2 \theta_0 - 4 \left(-\frac{1}{2} g\right) (125)}}{2 \left(-\frac{1}{2} g\right)} = \frac{-39.1 \pm 63.1}{-9.8} = 10.4 \text{ s} , \ -2.45 \text{ s} = [10.4 \text{ s}]
\]

Choose the positive sign since the projectile was launched at time $t = 0$. 

\[

\]
Choose the origin to be at ground level, under the place where the projectile is launched, and upwards to be the positive $y$ direction. For the projectile,

\[ v_0 = 65.0 \text{ m/s} \]
\[ \theta_0 = 37.0^\circ \]
\[ a_y = -g \]
\[ y_0 = 125 \]
\[ v_{y0} = v_0 \sin \theta_0 \]

(b)

The horizontal range is found from the horizontal motion at constant velocity.

\[ \Delta x = v_x t = (v_0 \cos \theta_0) t = (65.0 \text{ m/s}) \cos 37.0^\circ \times (10.4 \text{ s}) = 541 \text{ m} \]
Choose the origin to be at ground level, under the place where the projectile is launched, and upwards to be the positive $y$ direction. For the projectile,

\[ v_0 = 65.0 \text{ m/s} \]
\[ \theta_0 = 37.0^\circ \]
\[ a_y = -g \]
\[ y_0 = 125 \]
\[ v_{y0} = v_0 \sin \theta_0 \]

(c)

At the instant just before the particle reaches the ground, the horizontal component of its velocity is the constant

\[ v_x = v_0 \cos \theta_0 = (65.0 \text{ m/s}) \cos 37.0^\circ = 51.9 \text{ m/s} \]

The vertical component is found from Eq. 2.16:

\[ v_y = v_{y0} + at = v_0 \sin \theta_0 - gt = (65.0 \text{ m/s}) \sin 37.0^\circ - (9.80 \text{ m/s}^2)(10.4 \text{ s}) \]
\[ = -63.1 \text{ m/s} \]
Choose the origin to be at ground level, under the place where the projectile is launched, and upwards to be the positive $y$ direction. For the projectile,

\[ v_0 = 65.0 \text{ m/s} \]
\[ \theta_0 = 37.0^\circ \]
\[ a_y = -g \]
\[ y_0 = 125 \]
\[ v_{y0} = v_0 \sin \theta_0 \]

(d)

The magnitude of the velocity is found from the $x$ and $y$ components calculated in part c) above.

\[
v = \sqrt{v_x^2 + v_y^2} = \sqrt{(51.9 \text{ m/s})^2 + (-63.1 \text{ m/s})^2} = 81.7 \text{ m/s}
\]
Choose the origin to be at ground level, under the place where the projectile is launched, and upwards to be the positive $y$ direction. For the projectile,

\[ v_0 = 65.0 \text{ m/s} \]
\[ \theta_0 = 37.0^\circ \]
\[ a_y = -g \]
\[ y_0 = 125 \]
\[ v_{y0} = v_0 \sin \theta_0 \]

The direction of the velocity is \textit{(see slide number 7 of the previous lecture)}

\[ \theta = \tan^{-1} \frac{v_y}{v_x} = \tan^{-1} \frac{-63.1}{51.9} = -50.6^\circ \]

and so the object is moving $50.6^\circ \text{ below the horizon}$.
Choose the origin to be at ground level, under the place where the projectile is launched, and upwards to be the positive $y$ direction. For the projectile,

\[ v_0 = 65.0 \text{ m/s} \]
\[ \theta_0 = 37.0^\circ \]
\[ a_y = -g \]
\[ y_0 = 125 \]
\[ v_{y_0} = v_0 \sin \theta_0 \]

(f)

The maximum height above the cliff top reached by the projectile will occur when the $y$-velocity is 0, and is found from Eq. 2.18.

\[ v_y^2 = v_{y_0}^2 + 2a_y(y - y_0) \rightarrow 0 = v_0^2 \sin^2 \theta_0 - 2gy_{\text{max}} \]
\[ y_{\text{max}} = \frac{v_0^2 \sin^2 \theta_0}{2g} = \frac{(65.0 \text{ m/s})^2 \sin^2 37.0^\circ}{2\left(9.80 \text{ m/s}^2\right)} = 78.1 \text{ m} \]
Important Note

You might consider a problem where an object is subject to an acceleration other than that of gravity. The problem is again solved using the method of components where now the acceleration vector can also be represented by its components on the x and y axes:

In this particular case:

\[ \vec{a} = \vec{a}_x + \vec{a}_y \quad ; \quad (\vec{a})_x = a_x \quad ; \quad (\vec{a})_y = a_y \]

The resultant acceleration is the addition of \( \vec{a} \) and \( \vec{g} \)

\[ \vec{A} = \vec{a} + \vec{g} \]

\[ \vec{A}_y = \vec{a}_y + \vec{g} \quad \quad \vec{A}_x = \vec{a}_x \]

\[ (\vec{A})_x = (\vec{a})_x + (\vec{g})_x = a_x + 0 = a_x \]

\[ (\vec{A})_y = (\vec{a})_y + (\vec{g})_y = a_y - g \]
Projectile Motion

**Important Note**

The equations of motion in x and y are then written as:

**horizontal motion**

\[ \begin{align*}
    v_x &= v_{x0} + a_xt \\
    x &= x_0 + v_{x0}t + \frac{1}{2}a_xt^2 \\
    v_x^2 &= v_{x0}^2 + 2a_x(x - x_0)
\end{align*} \]

**vertical motion**

\[ \begin{align*}
    v_y &= v_{y0} + (a_y - g)t \\
    y &= y_0 + v_{y0}t + \frac{1}{2}(a_y - g)t^2 \\
    y - y_0 &= v_{y0}t + \frac{1}{2}(a_y - g)t^2
\end{align*} \]

Note that the minus sign in \((\vec{A})_y = (\vec{a})_y + (\vec{g})_y = a_y - g\) is due to choice of having the y axis pointing upwards.

If there is **NO gravity**, consider \(g = 0\) in the above equations.
It is not difficult to show that whenever we assume the acceleration of gravity to be constant and ignore the air resistance, then the vertical displacement $y$ can be written as a function of the horizontal displacement $x$. Using the equations of motion 2.10, 2.13, 2.14 and 2.16-2.18, we can show that:

$$y = Ax - Bx^2$$

I leave for you to obtain the values of $A$ and $B$ in terms of the initial angle $\theta_0$, initial velocity $v_0$ and acceleration of gravity $g$ (assume $y$ axis pointing upwards).
Relative Velocity

We have seen that it is important to choose a reference frame where we can define quantities such as velocity, position, etc.

An example was that of a person walking along a train which is moving relative to the ground. The velocity you measure of this person depends on whether you are on the train or standing on the ground.

It is common to place yourself at the origin of your coordinate system (reference frame) when you are the one performing the measurement.
Relative Velocity

This is an example of relative velocity. If the train moves with velocity $\vec{v}_{TG}$ relative to the ground, and the person moves with velocity $\vec{v}_{PT}$ relative to the train, then, you will measure the velocity $\vec{v}_{PY}$ relative to you as:

You are on the train:

$$\vec{v}_{PY} = \vec{v}_{PT} + \vec{v}_{TY} = \vec{v}_{PT} + 0 = \vec{v}_{PT}$$

Where $\vec{v}_{TY}$ is the velocity of the train relative to you (zero in this case)

(ground is moving this way relative to you)
You are on the ground:

\[
\vec{v}_{PY} = \vec{v}_{PT} + \vec{v}_{TY} = \vec{v}_{PT} + \vec{v}_{TG}
\]
Relative Velocity

In a more general case, you could be moving in a car with velocity $\vec{v}_{YG}$ relative to the ground. In this case you will measure:

First note that: $\vec{v}_{YG} = -\vec{v}_{GY}$ \hfill (3.17)

Then: $\vec{v}_{PY} = \vec{v}_{PT} + \vec{v}_{TG} + \vec{v}_{GY}$ \hfill \text{or} \hfill $\vec{v}_{PY} = \vec{v}_{PT} + \vec{v}_{TG} - \vec{v}_{YG}$ \hfill (3.18)
Relative Velocity

The previous example addressed a one dimensional problem. However, it can be easily generalized to two (or three) dimensional problems thanks to the fact that we have been using vectors as a method (tool) to analyze motion.

Let $\vec{v}_{BW}$ be the velocity of a boat relative to the river water and $\vec{v}_{WS}$ the velocity of the water relative to the shore. Then the velocity $\vec{v}_{BS}$ of the boat relative to the shore will be given by:

$$\vec{v}_{BS} = \vec{v}_{BW} + \vec{v}_{WS} \quad (3.19)$$

You can then use the component method for vector addition to obtain $\vec{v}_{BS}$.
Relative Velocity

Note that like in the one dimensional case, you can consider the addition of more than three relative velocities to obtain the velocity of an object relative to you or to any other reference frame.

For example, consider a person walking on the boat with velocity $\vec{v}_{PB}$ relative to the boat. His velocity relative to the shore will be given by:

$$\vec{v}_{PS} = \vec{v}_{PB} + \vec{v}_{BW} + \vec{v}_{WS}$$

recalling previous slide:

$$\vec{v}_{BS} = \vec{v}_{BW} + \vec{v}_{WS} \quad (3.19)$$

Then

$$\stackrel{(Eq. 3.19)}{\longrightarrow} \quad \vec{v}_{PS} = \vec{v}_{PB} + \vec{v}_{BS}$$
Relative Velocity

**Notes:**

In general, if \( \vec{v}_{AB} \) is the velocity of \( A \) relative to \( B \), then the velocity of \( B \) relative to \( A \), \( \vec{v}_{BA} \) will be in the same line as \( \vec{v}_{AB} \) but in opposite direction:

\[
\vec{v}_{BA} = -\vec{v}_{AB} \quad \text{(3.20)}
\]

If \( \vec{v}_{CA} \) is the velocity of \( C \) relative to \( A \), then \( \vec{v}_{CB} \), the velocity of \( C \) relative to \( B \), will be given by:

\[
\vec{v}_{CB} = \vec{v}_{CA} + \vec{v}_{AB} \quad \text{(3.21)}
\]
Relative Velocity

We will solve some problems in the next lecture
Assignment 2

Textbook (Giancoli, 6th edition). Due on Oct. 6th.


3. A projectile, fired with unknown initial velocity, lands 20.0 s later on the side of a hill, 3000 m away horizontally and 450 m vertically above its starting point. (Ignore any effects due to air resistance.)

   (a) What is the vertical component of its initial velocity?
   (b) What is the horizontal component of its initial velocity?
   (c) What was its maximum height above its launch point?
   (d) As it hit the hill, what speed did it have and what angle did its velocity make with the vertical?