AN INTRODUCTION TO FLAG MANIFOLDS

Notes[1] for the Summer School on Combinatorial Models in Geometry and Topology of Flag Manifolds, Regina 2007

1. The manifold of flags

The *(complex) full flag manifold* is the space F_n consisting of all sequences

$$V_1 \subset V_2 \subset \ldots \subset V_n = \mathbb{C}^n$$

where V_j is a complex linear subspace of $\mathbb{C}^n, \dim V_j = j$, for all $j = 1, \ldots, n$ (such sequences are sometimes called *flags in* $\mathbb{C}^n$). The manifold structure of F_n arises from the fact that the general linear group $GL_n(\mathbb{C})$ acts transitively on it: by this we mean that for any sequence $V_\bullet$ like above, there exists an $n \times n$ matrix g which maps the standard flag

$$\mathrm{Span}_{\mathbb{C}}(e_1) \subset \mathrm{Span}_{\mathbb{C}}(e_1, e_2) \subset \ldots \subset \mathbb{C}^n$$

to the flag $V_\bullet$ (here $e_1, \ldots, e_n$ is the standard basis of $\mathbb{C}^n$). That is, we have

$$g(\mathrm{Span}_{\mathbb{C}}(e_1, \ldots, e_j)) = V_j, \tag{1}$$

for all $j = 1, 2, \ldots, n-1$. In fact, if we return to (1) we see that we can choose the matrix g such that $ge_1, ge_2, \ldots, ge_n$ is an orthonormal basis of $\mathbb{C}^n$ with respect to the standard Hermitean product $\langle \ , \ \rangle$ on $\mathbb{C}^n$; the latter is given by

$$\langle \sum_j \zeta_j e_j, \sum_j \xi_j e_j \rangle := \sum_j \zeta_j \overline{\xi_j},$$

for any two vectors $\zeta = \sum_j \zeta_j e_j$ and $\xi = \sum_j \xi_j e_j \in \mathbb{C}^n$. In other words, g satisfies

$$\langle g\zeta, g\xi \rangle = \langle \zeta, \xi \rangle$$

for all $\zeta, \xi \in \mathbb{C}^n$; so g is an element of the *unitary group* $U(n)$.

In conclusion: flags can be identified with elements of the group $GL_n(\mathbb{C})$, respectively $U(n)$, modulo their subgroup which leave the standard flag fixed. The two subgroups can be described as follows:

- for the $GL_n(\mathbb{C})$ action, the group B_n of all upper triangular (invertible) matrices with coefficients in $\mathbb{C}$
- for the $U(n)$ action, the group T^n of all diagonal matrices in $U(n)$; more specifically, T^n consists of all matrices of the form

  $$\mathrm{Diag}(z_1, z_2, \ldots, z_n)$$

  where $z_j \in \mathbb{C}$, $|z_j| = 1$, $j = 1, 2, \ldots, n$, which means that T^n can be identified with the direct product $(S^1)^{\times n}$; we say that T is an n-torus.

This implies that F_n can be written as

$$F_n = GL_n(\mathbb{C})/B_n = U(n)/T^n$$

so it is what we call a homogeneous manifold (for more explanations, see Section 2, the paragraph after Definition 2.4).

Another related space is the *Grassmannian* $Gr_k(\mathbb{C}^n)$ which consists of all complex linear subspaces

$$V \subset \mathbb{C}^n$$

[1] I am indebted to Leonardo Mihalcea and Matthieu Willems for helping me prepare these notes.

with $\dim V = k$. The spaces F_n and $Gr_k(\mathbb{C}^n)$ are the two extremes of the following class of spaces: fix $0 \le k_1 < k_2 < \ldots < k_p \le n$ and consider the space $F_{k_1,\ldots,k_p}$ which consists of all sequences

$$V_1 \subset V_2 \subset \ldots \subset V_p \subset \mathbb{C}^n$$

with $\dim V_j = k_j$, for all $j = 1, \ldots, p$. These spaces are called *flag manifolds*. Each of them has a transitive action of $U(n)$ (the arguments we used above for F_n go easily through), so they are homogeneous manifolds.

There exist also Lie theoretical generalizations of those manifolds. The goal of these notes is to define them and give some basic properties of them. First, we need some background in Lie theory. This is what the next section presents.

2. Lie groups, Lie algebras, and generalized flag manifolds

This is an informal introduction to Lie groups. It is intended to make the topics of the School accessible to the participants who never took a course in Lie groups and Lie algebras (but do know some point-set topology and some basic notions about differentiable[2] manifolds).

Definition 2.1. *A* Lie group *is a subgroup of some general linear group $GL_n(\mathbb{R})$ which is a closed subspace.*

Note. For people who know Lie theory: the groups defined here are actually "matrix Lie groups". These are just a special class of Lie groups. However, unless otherwise specified, no other Lie groups will occur in these notes and probably also in the School's lectures.

We note that $GL_n(\mathbb{R})$ has a natural structure of a differentiable manifold (being open in the space $\mathrm{Mat}^{n\times n}(\mathbb{R}) \simeq \mathbb{R}^{n^2}$ of all $n \times n$ matrices – why?). Moreover, the basic group operations, namely multiplication

$$GL_n(\mathbb{R}) \times GL_n(\mathbb{R}) \ni (g_1, g_2) \mapsto g_1 g_2 \in GL_n(\mathbb{R})$$

and taking the inverse

$$GL_n(\mathbb{R}) \ni g \mapsto g^{-1} \in GL_n(\mathbb{R})$$

are differentiable maps (why?). Finally, consider the exponential map

$$\exp : \mathrm{Mat}^{n\times n}(\mathbb{R}) \to GL_n(\mathbb{R})$$

given by

$$\exp(X) = \sum_{k\ge 0} \frac{1}{k!} X^k. \tag{2}$$

The differential of exp at 0 is the identity map on $\mathrm{Mat}^{n\times n}(\mathbb{R})$; consequently exp is a local diffeomorphism at 0.

An important theorem says that any Lie group $G \subset GL_n(\mathbb{R})$ like in Definition 2.1 is a differentiable manifold, actually a submanifold[3] of $GL_n(\mathbb{R})$. Consequently, the maps

$$G \times G \ni (g_1, g_2) \mapsto g_1 g_2 \in G$$

and

$$G \ni g \mapsto g^{-1} \in G$$

[2]By "differentiable" we always mean "of class C^∞".

[3]For us, *submanifolds* are *closed embedded submanifolds*, unless otherwise specified.

are differentiable. A proof of this theorem can be found for instance in [Br-tD, Ch. I, Theorem 3.11] or [He, Ch. II, Section 2, Theorem 2.3]. The idea of the proof is to construct first a local chart around I: more precisely, one shows that

$$\mathfrak{g} := \{X \in \mathrm{Mat}^{n\times n}(\mathbb{R}) \ : \ \exp(tX) \in G \ \forall t \in \mathbb{R}\} \tag{3}$$

is a vector subspace of $\mathrm{Mat}^{n\times n}(\mathbb{R})$. Moreover, exp (see (2)) maps $\mathfrak{g}$ to G and is a local homeomorphism at 0: this homeomorphism gives the chart at I; a chart at an arbitrary point g is obtained by "multiplying" the previous chart by g. The map

$$\exp : \mathfrak{g} \to G$$

is called the *exponential map* of G. The subspace $\mathfrak{g}$ of $\mathrm{Mat}^{n\times n}(\mathbb{R})$ is called the *Lie algebra* of G. It is really an algebra (non-commutative though), with respect to the composition law defined in the following lemma.

Lemma 2.2. *Let G be a Lie group and $\mathfrak{g}$ its Lie algebra.*

(a) If g is in G and X in $\mathfrak{g}$ then

$$\mathrm{Ad}(g)X := gXg^{-1} \in \mathfrak{g}.$$

(b) If $X, Y \in \mathfrak{g}$, one has

$$\frac{d}{dt}|_0 \exp(tX)Y\exp(-tX) = XY - YX$$

Consequently, the commutator

$$[X, Y] := XY - YX$$

is in $\mathfrak{g}$.

Proof. Exercise. □

The composition law on $\mathfrak{g}$ we were referring at above is

$$\mathfrak{g} \times \mathfrak{g} \ni (X, Y) \mapsto [X, Y] \in \mathfrak{g}.$$

Note that the algebra $(\mathfrak{g}, [\ ,\ ])$ is not commutative and not associative; instead, the Lie bracket operation $[\ ,\ ]$ satisfies

$$[X, Y] = -[Y, X] \text{ (anticommutativity)}$$
$$[[X, Y], Z] + [[Z, X], Y] + [[Y, Z], X] = 0 \text{ (Jacobi identity)}$$

for any $X, Y, Z \in \mathfrak{g}$.

Note. Have you ever seen a composition law with the above properties before? I hope everyone remember that the space of vector fields on a manifold M can be equipped with a Lie bracket which satisfies the two properties from above. Namely, if u, v are two vector fields on M, then the vector field $[u, v]$ is defined by

$$[u, v]_p(\varphi) := u_p(v(\varphi)) - v_p(u(\varphi)), \tag{4}$$

for any $p \in M$ and any germ of real function φ (defined on an open neighborhood of p in M). This is called the *Lie algebra of vector fields* on M and is denoted by $\mathcal{X}(M)$.

There is a fundamental question we would like to address at this point: to which extent does the Lie group G depend on the surrounding space $GL_n(\mathbb{R})$? Because G can be imbedded in infinitely many ways in a general linear group: for instance, we have

$$G \subset GL_n(\mathbb{R}) \subset GL_{n+1}(\mathbb{R}) \subset \dots .$$

All notions defined above — the space $\mathfrak{g}$, the map $\exp : \mathfrak{g} \to G$, and the bracket $[\ ,\ ]$ — are defined in terms of $GL_n(\mathbb{R})$. In spite of their definition, they actually depend only on the manifold G and the group multiplication (in other words, they are intrinsically associated to the Lie group G). This is what the following proposition says. First, we say that a vector field u on G is a *G-invariant vector field* if

$$u_g = (dg)_I(u_I)$$

for all $g \in G$ (here we identify g with the map from G to itself give by left multiplication by g). We denote by $\mathcal{X}^G$ the space of all G-invariant vector fields on G.

Proposition 2.3. *Let G be a Lie group.*

(a) The space $\mathfrak{g}$ is equal to the tangent space T_IG of G at the identity element I. The map $\mathfrak{g} \to \mathcal{X}^G$ given by

$$X \mapsto [G \ni g \mapsto gX]$$

is a linear isomorphism.

(b) If X is in $\mathfrak{g}$, then the map

$$\mathbb{R} \ni t \mapsto \exp(tX)$$

is the integral curve associated to the (invariant) vector field

$$G \ni g \mapsto gX.$$

(c) The space $\mathcal{X}^G$ is closed under the Lie bracket $[\ ,\ ]$ given by (4).

(d) The map

$$\mathfrak{g} \ni X \mapsto [G \ni g \mapsto gX]$$

is a Lie algebra isomorphism (that is, a linear isomorphism which preserves the Lie brackets) between $\mathfrak{g}$ and $\mathcal{X}^G$.

Proof. To prove (a), we note that for any $X \in \mathfrak{g}$, the assignment $\mathbb{R} \ni t \mapsto \exp(tX)$ is a curve on G whose tangent vector at $t = 0$ is

$$\frac{d}{dt}|_0 \exp(tX) = X.$$

The equality follows by dimension reasons.

The only non-trivial remaining point is (d). Namely, we show that if X, Y are in $\mathfrak{g}$, than the Lie bracket of the vector fields

$$G \ni g \mapsto gX \text{ and } G \ni g \mapsto gY$$

is equal to

$$G \ni g \mapsto g[X, Y].$$

It is sufficient to check that the two vector fields are equal at I. Indeed, the Lie bracket of the two vector fields at I evaluated on a real function φ defined on a neighborhood of I in G equals to

$$\frac{d}{dt}|_0 \frac{d}{ds}|_0[\varphi(\exp(tX)\exp(sY))] - \frac{d}{dt}|_0 \frac{d}{ds}|_0[\varphi(\exp(tY)\exp(sX))],$$

which is the same as

$$(d\varphi)_I(XY - YX).$$

The proof is finished (the students are invited to check all missing details). $\square$

Consequently, if we denote by $GL(\mathfrak{g})$ the space of all linear automorphisms of $\mathfrak{g}$, then the map

$$\mathrm{Ad} : G \to GL(\mathfrak{g}), \ g \mapsto \mathrm{Ad}(g)$$

(see Lemma 2.2, (b)) is also independent of the choice of $GL_n(\mathbb{R})$ in Definition 2.1. The map Ad is called the *adjoint representation* of G. It can be interpreted as an action of G on $\mathfrak{g}$, that is, it satisfies[4]

$$\mathrm{Ad}(g_1g_2)X = \mathrm{Ad}(g_1)(\mathrm{Ad}(g_2)X)$$
$$\mathrm{Ad}(e)X = X,$$

for all $g, g_1, g_2 \in G$ and all $X \in \mathfrak{g}$.

In the category of Lie groups, a *morphism* is a group homomorphism which is also a differentiable map. An *isomorphism* is of course a bijective morphism whose inverse is also a morphism. Two Lie groups are isomorphic if there exists an isomorphism between them. Proposition 2.3 implies that for Lie groups that are isomorphic, there is a natural way to identify the adjoint orbits (by identifying first their Lie algebras).

Definition 2.4. *A* (generalized) flag manifold *is an adjoint orbit*

$$\mathrm{Ad}(G)X_0 := \{\mathrm{Ad}(g)X_0 \ : \ g \in G\},$$

where G is a compact Lie group.

Before seeing examples, we would like to explain the manifold structure of $\mathrm{Ad}(G)X_0$. Let G_{X_0} be the G-stabilizer of X_0, which consists of all $g \in G$ with

$$\mathrm{Ad}(g)X_0 = X_0.$$

This is also a closed subgroup of $GL_n(\mathbb{R})$, thus a Lie group. Set-theoretical, we can obviously identify

$$\mathrm{Ad}(G)X_0 = G/G_{X_0}, \tag{5}$$

where the latter is the space of all cosets gG_{X_0}, with $g \in G$. The quotient G/G_{X_0} is a *homogeneous manifold*: to define the manifold structure, we consider the Lie algebra $\mathfrak{g}_{X_0}$ of G_{X_0} and pick a direct complement of it in $\mathfrak{g}$, call it $\mathfrak{h}$; that is, we have

$$\mathfrak{g} = \mathfrak{g}_{X_0} \oplus \mathfrak{h}.$$

For a sufficiently small neighborhood U of 0 in $\mathfrak{h}$, the map

$$U \ni X \mapsto \exp(X)G_{X_0} \in G/G_{X_0}$$

is a local chart around the coset IG_{X_0}. A chart at an arbitrary point gG_{X_0} is obtained by "multiplying" the previous one by g.

Examples. 1. $(\mathbb{R}_{>0}, \cdot)$ and more generally $((\mathbb{R}_{>0})^n, \cdot)$ (the multiplication is componentwise) are Lie groups. Indeed, the latter can be identified with the subgroup of $GL_n(\mathbb{R})$ consisting of diagonal matrices with all entries strictly positive: the latter is a closed subspace of $GL_n(\mathbb{R})$.

2. $(\mathbb{R}, +)$ and more generally, $(\mathbb{R}^n, +)$ are Lie groups. Because the latter can be identified with the Lie group $((\mathbb{R}_{>0})^n, \cdot)$ from the previous example via the exponential map.

3. Consider the circle

$$S^1 := \{z \in \mathbb{C} \mid |z| = 1\}.$$

[4]In general, a (smooth) *action* of a Lie group G on a manifold M is a (differentiable) map $G \times M \to M$, $G \times M \ni (g, p) \mapsto gp \in M$ with the properties $Ip = p$ and $g_1(g_2p) = (g_1g_2)p$, for all $p \in M$, $g, g_1, g_2 \in G$. If $p \in M$, then *orbit* of p is $Gp := \{gp : g \in G\}$ and the *stabilizer* of p is $G_p := \{g \in G \ : \ gp = p\}$.

The group

$$T^n := (S^1)^n$$

is a Lie group, being identified with the subgroup of $GL_n(\mathbb{C})$ consisting of

$$\mathrm{Diag}(z_1, \ldots, z_n),$$

with $z_k \in S^1$, $1 \le k \le n$. The group T^n is called the n-dimensional <u>*torus*</u>. Note that the map

$$\mathbb{R}^n \to T^n, \ (\theta_1, \ldots, \theta_n) \mapsto (e^{i\theta_1}, \ldots, e^{i\theta_n})$$

is a group homomorhism, being at the same time a covering map (all pre-images can be identified with $\mathbb{Z}$).

4. Recall from Section 1 that the unitary group $U(n)$ consists of all $n \times n$ matrices g which preserve the canonical Hermitean product on $\mathbb{C}^n$. An easy exercise shows that

$$U(n) = \{g \in \mathrm{Mat}^{n\times n}(\mathbb{C}) \ : \ g \cdot \bar{g}^T = I_n\}.$$

Show that $U(n)$ is a compact topological subspace of $GL_n(\mathbb{C})$ (hint: what are the diagonal elements of the product $g\bar{g}^T$?). Then show that $U(n)$ is a connected topological subspace of $GL_n(\mathbb{C})$ (hint: consider first the canonical action of $U(n)$ on $\mathbb{C}^n$ and note that the orbit of e_1 is the unit sphere S^{2n-1} and the stabilizer of the same e_1 is $U(n-1)$; this implies that $S^{2n-1} = U(n)/U(n-1)$ is a connected space; use induction by n.) Via the identification[5]

$$\mathrm{Mat}^{n\times n}(\mathbb{C}) \ni a + ib = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \in \mathrm{Mat}^{2n\times 2n}(\mathbb{R}),$$

$U(n)$ becomes a closed subgroup of $GL_{2n}(\mathbb{R})$, thus a Lie group. By the definition given by (3), its Lie algebra is the space

$$\mathfrak{u}(n) := \{X \in \mathrm{Mat}^{n\times n}(\mathbb{C}) \ : \ X + \bar{X}^T = 0\}$$

of all skew-hermitean $n \times n$ matrices. We want to describe the corresponding adjoint orbits. A result in linear algebra says that the eigenvalues of a skew-hermitean matrix are purely imaginary; moreover, any skew-hermitean matrix is $U(n)$-conjugate to a matrix of the form

$$X_0 = \mathrm{Diag}(ix_1, ix_2, \ldots, ix_n),$$

where $x_j \in \mathbb{R}$, $j = 1, 2, \ldots, n$, such that

$$x_1 \le x_2 \le \ldots \le x_n.$$

So it is sufficient to describe the orbit of X_0. This is obviously the set of all skew-hermitean matrices with eigenvalues $ix_1, ix_2 \ldots, ix_n$. For sake of simplicity we assume that

$$x_1 < x_2 < \ldots < x_n.$$

The orbit of X_0 can be identified with the space of all n-tuples $(L_1, L_2, \ldots, L_n)$ of complex lines (one-dimensional complex subspaces) in $\mathbb{C}^n$ which are any two orthogonal: that is, to any such n-tuple corresponds the matrix X with eigenvalues $ix_1, ix_2, \ldots, ix_n$ and corresponding eigenspaces $L_1, L_2, \ldots, L_n$. Via

$$(L_1, L_2, \ldots, L_n) \mapsto L_1 \subset L_1 \oplus L_2 \subset \ldots \subset L_1 \oplus L_2 \oplus \ldots \oplus L_n = \mathbb{C}^n,$$

the latter space coincides with the manifold F_n defined in section 1. In conclusion, *the flag manifolds corresponding to $U(n)$ are the manifolds of flags defined in section 1.* The <u>*special unitary group*</u> is

$$SU(n) = \{g \in U(n) \mid \det(g) = 1\}.$$

[5] Any matrix $a + ib \in \mathrm{Mat}^{n\times n}(\mathbb{C})$ defines an $\mathbb{R}$-linear automorphism of $\mathbb{R}^{2n}$, via $(a + ib)(u + iv) = a(u) - b(v) + i(b(u) + a(v))$.

Its Lie algebra is

$$\mathfrak{su}(n) := \{X \in \mathrm{Mat}^{n\times n}(\mathbb{C}) \ : \ X + \bar{X}^T = 0, \mathrm{Tr}(X) = 0\},$$

as one can easily check. We can see by using the same arguments as above that the adjoint orbits of $SU(n)$ are the manifolds of flags as well.

5. The *orthogonal group* $O(n)$ consists of all $g \in GL_n(\mathbb{R})$ which leave the usual scalar product on $\mathbb{R}^n$ invariant. Concretely,

$$O(n) = \{g \in \mathrm{Mat}^{n\times n}(\mathbb{R}) \ : \ g \cdot g^T = I\},$$

so $O(n)$ is a Lie group. It is not connected: indeed, its subgroup

$$SO(n) := \{g \in O(n) \ : \ \det(g) = 1\}$$

is connected (use the same connectivity argument as for $U(n)$, see above) and $O(n)$ is the disjoint union of $SO(n)$ and

$$O^-(n) := \{g \in O(n) \ : \ \det(g) = -1\}.$$

The group $SO(n)$ is called the *special orthogonal group*. It is not difficult to see that it is a compact Lie group. There exist descriptions of its adjoint orbits in terms of flags, similar to those of $U(n)$. We will not use those in the School, so I just address the interested students to [Fu-Ha, p. 383] or [Bi-Ha, p. 7].

6. Let

$$\mathbb{H} = \{a + bi + cj + dk \mid a, b, c, d \in \mathbb{R}\}$$

denote the (non-commutative) ring of quaternions. This is a real algebra generated by i, j, and k, subject to the relations

$$i^2 = j^2 = k^2 = -1, \ ij = k, \ jk = i, \ ki = j.$$

The conjugate of an arbitrary quaternion $q = a + bi + cj + dk$ is by definition

$$\bar{q} = a - bi - cj - dk.$$

Note that we have

$$\overline{q_1 \cdot q_2} = \overline{q_2} \cdot \overline{q_1},$$

for any $q_1, q_2 \in \mathbb{H}$. For $n \geq 1$, we regard $\mathbb{H}^n$ as an $\mathbb{H}$-module with multiplication from the *right*. In this way, any matrix $A \in \mathrm{Mat}^{n\times n}(\mathbb{H})$ induces a $\mathbb{H}$-linear transformation of $\mathbb{H}^n$, by the usual matrix multiplication. Any $\mathbb{H}$-linear transformation of $\mathbb{H}^n$ is induced in this way. We identify

$$\mathbb{H}^n = \mathbb{C}^n \oplus j\mathbb{C}^n \simeq \mathbb{C}^{2n}$$

via

$$\mathbb{H}^n \ni u = x + jy \mapsto (x, y) \in \mathbb{C}^n \oplus \mathbb{C}^n, \tag{6}$$

where x, y are vectors in $\mathbb{C}^n$. The latter is a $\mathbb{C}$-linear isomorphism. Take $A \in \mathrm{Mat}(n \times n, \mathbb{H})$ and write it as

$$A = B + jC,$$

where $B, C \in \mathrm{Mat}(n \times n, \mathbb{C})$. The $\mathbb{C}$-linear endomorphism of $\mathbb{C}^{2n}$ induced by a is determined by

$$(B + jC)(x + jy) = Bx - \bar{C}y + j(Cx + \bar{B}y).$$

We deduce that

$$\mathrm{Mat}(n \times n, \mathbb{H}) = \{\begin{pmatrix} B & -\bar{C} \\ C & \bar{B} \end{pmatrix} \mid B, C \in \mathrm{Mat}(n \times n, \mathbb{C})\}. \tag{7}$$

We can easily see that if we denote

$$J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix},$$

then

$$\mathrm{Mat}(n \times n, \mathbb{H}) = \{A \in \mathrm{Mat}(2n \times 2n, \mathbb{C}) \mid AJ = J\bar{A}\}.$$

Now let us consider the pairing (,) on $\mathbb{H}^n$ given by

$$(u, v) := \sum_{\nu=1}^{n} u_\nu \bar{v}_\nu = u \cdot v*,$$

for all $u = (u_1, \ldots, u_n)$ and $v = (v_1, \ldots, v_n)$ in $\mathbb{H}^n$. The symplectic group $Sp(n)$ is defined as

$$Sp(n) = \{A \in \mathrm{Mat}^{n\times n}(\mathbb{H}) \mid (Au, Av) = (u, v)\ \forall u, v \in \mathbb{H}^n\}.$$

The condition which characterizes A in $Sp(n)$ can be written as

$$A \cdot u \cdot v^* \cdot A^* = u \cdot v^*$$

for for all $q \in \mathbb{H}$. This is equivalent to

$$A \cdot q \cdot A^* = q,$$

for all $q \in \mathbb{H}$. We deduce that an alternative presentation of $Sp(n)$ is

$$Sp(n) = \{A \in \mathrm{Mat}^{n\times n}(\mathbb{H}) \mid A^* \cdot A = I_n\}.$$

If we use the identification (6), we can express the pairing of $u = x + jy$ with itself as

(8) $\|u\|^2 = (u, u) = (x+jy)\cdot(x^* - jy^T) = xx^* + \bar{y}y^T + j(yx^* - \bar{x}y^T) = xx^* + yy^* = \|x\|^2 + \|y\|^2.$

An element A of $\mathrm{Mat}(n \times n, \mathbb{H})$ is in $Sp(n)$ if and only if

$$(Au, Au) = (u, u),$$

for any $u \in \mathbb{H}^n$. If we write again

$$A = B + jC = \begin{pmatrix} B & -\bar{C} \\ C & \bar{B} \end{pmatrix}$$

as above, then

$$\|A(u)\|^2 = \|(B + jC)(x + jy)\|^2 = \|Bx - \bar{C}y\|^2 + \|Cx + \bar{B}y\|^2 = \| \begin{pmatrix} B & -\bar{C} \\ C & \bar{B} \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} \|^2.$$

We deduce that $Sp(n) \subset U(2n)$. In fact, we have

$$Sp(n) = \{\begin{pmatrix} B & -\bar{C} \\ C & \bar{B} \end{pmatrix} \in U(2n)\} = \{g \in U(2n) \mid g = J\bar{g}J^{-1}\} \tag{9}$$

Thus $Sp(n)$ is a Lie group. Like in Example 5, descriptions of its adjoint orbits in terms of flags can be found in [Fu-Ha, p. 383] or [Bi-Ha, p. 7].

The groups $U(n)$, $SU(n)$, $O(n)$, $SO(n)$ and $Sp(n)$ are called the classical groups.

3. The flag manifold of $(\mathfrak{g}, X_0)$

By Definition 2.4, a flag manifold is determined by a compact Lie group G and the choice of an element X_0 in its Lie algebra $\mathfrak{g}$. In this section we discuss the classification of compact Lie groups with the same Lie algebra $\mathfrak{g}$. We will see that for all such Lie groups the adjoint orbits of $X \in \mathfrak{g}$ are the same. The treatment will be sketchy: the details can be found for instance in [Kn, Chapter I, Section 11] and/or [He, Chapter II, Section 6].

Definition 3.1. *A* Lie algebra *is a vector subspace* $\mathfrak{g}$ *of some* $\mathrm{Mat}^{n\times n}(\mathbb{R})$ *which is closed under the operation*

$$\mathrm{Mat}^{n\times n}(\mathbb{R}) \ni X, Y \mapsto X \cdot Y - Y \cdot X \in \mathrm{Mat}^{n\times n}(\mathbb{R}).$$

We also say that $\mathfrak{g}$ *is a Lie subalgebra of* $\mathrm{Mat}^{n\times n}(\mathbb{R})$.

We have seen that to any Lie group $G \subset GL_n(\mathbb{R})$ corresponds a Lie algebra $\mathfrak{g} \subset \mathrm{Mat}^{n\times n}(\mathbb{R})$. Related to this, we have the following result.

Theorem 3.2. *(a) Any Lie algebra* $\mathfrak{g}$ *is the Lie algebra of a Lie group* G*. More specifically if* $\mathfrak{g} \subset \mathrm{Mat}^{n\times n}(\mathbb{R})$ *is a Lie subalgebra, then there exists a unique connected immersed subgroup* $G \subset GL_n(\mathbb{R})$ *whose Lie algebra is* $\mathfrak{g}$.

(b) More generally, if the Lie group G *has Lie algebra* $\mathfrak{g}$ *and* $\mathfrak{h} \subset \mathfrak{g}$ *is a Lie subalgebra (that is, a linear subspace closed under* $[\ ,\ ]$*), then there exists a unique connected immersed subgroup* $H \subset G$ *with Lie algebra* $\mathfrak{h}$.

Note. The groups G and H arising at point (a), respectively (b), are not necessarily Lie groups, as defined in Definition 2.1. For example, the Lie algebra of the torus $T^2 = S^1 \times S^1$ is $\mathbb{R}^2$ (abelian). The subspace

$$\mathfrak{h} := \{(\alpha t, \beta t) \ : \ t \in \mathbb{R}\}$$

is a Lie subalgebra (here α and β are two real numbers). The subgroup H prescribed by point (b) above is

$$H = \{(e^{i\alpha t}, e^{i\beta t}) \ : \ t \in \mathbb{R}\}.$$

One can show that if the ratio α/β is irrational, then H is dense in T^2. So there is no way of making H into a (closed) matrix Lie group — see also the note following Definition 2.1. In fact such situations occur "rarely". This is why we will always assume (without proof!) that the subgroups G and H arising at point (a), respectively (b) above are (closed matrix) Lie groups.

In general, there is more than one G with Lie algebra $\mathfrak{g}$, as the following example shows.

Example. The center of $SU(n)$ is

$$Z = \{\zeta^k I \mid 0 \le k \le n-1\},$$

where $\zeta := e^{\frac{2\pi i}{n}}$. Then $SU(n)/Z$ is a Lie group (the map $Ad : SU(n) \to GL(\mathfrak{su}(n))$ has kernel equal to Z). Its Lie algebra coincides with the tangent space at the identity; but one can easily see that the map $SU(n) \to SU(n)/Z$ is a local diffeomorphism around I and a group homomorphism, thus the two spaces have the same tangent space at I, with the same Lie bracket. Finally, note that Z can be chosen to be an arbitrary subgroup of $Z(G)$ (we call it a central subgroup).

In fact, we can describe exactly all Lie groups with given Lie algebra $\mathfrak{g}$, as follows.

Theorem 3.3. *(a) If $\mathfrak{g}$ is a Lie algebra, then there exists a unique simply connected Lie group, call it $\tilde{G}$, of Lie algebra $\mathfrak{g}$.*

(b) Any Lie group of Lie algebra $\mathfrak{g}$ is of the form

$$G = \tilde{G}/Z,$$

where Z is a discrete central subgroup of $\tilde{G}$. The group $\tilde{G}$ is called the <u>universal cover</u> *of G.*

For us, this result is important for the following application:

Corollary 3.4. *Let $\mathfrak{g}$ be a Lie algebra and X_0 an element of $\mathfrak{g}$. Then for any Lie group G with Lie algebra $\mathfrak{g}$, the adjoint orbits $\mathrm{Ad}(G)X_0$ and $\mathrm{Ad}(\tilde{G})X_0$ are equal.*

Proof. We have $G = \tilde{G}/Z$, where $Z \subset \tilde{G}$ is a central subgroup. For $g \in \tilde{G}$ we denote by $[g]$ its coset in G. We have

$$\mathrm{Ad}([g])X = \frac{d}{dt}|_0[g][\exp(tX)][g^{-1}] = \frac{d}{dt}|_0[g\exp(tX)g^{-1}] = \frac{d}{dt}|_0 g\exp(tX)g^{-1} = \mathrm{Ad}(g)X$$

where we have identified $\tilde{G}$ and G locally around I. $\square$

If we want to restrict ourselves to *compact* Lie groups, we may face the situation where G is compact but $\tilde{G}$ is not. For example, the universal cover of $G = T^n$ (see Example 3, Section 2) is $\tilde{G} = \mathbb{R}^n$ (see Example 2, Section 2). Note however that the torus is not an interesting example for us, as we have

$$\mathrm{Ad}(T)X_0 = \{X_0\},$$

for any X_0 in the Lie algebra of T, thus an adjoint orbit consists of only one point. By contrary, if $\tilde{G}$ is compact, then any $G = \tilde{G}/Z$ like in the previous proposition is compact as well. The following result shows that we do not lose any generality by considering only adjoint orbits of groups that are compact and simply connected.

Theorem 3.5. *If G is a compact Lie group, then there exists a compact simply connected Lie group $\tilde{H}$ and a torus T such that*

$$G = (\tilde{H} \times T)/Z$$

where Z is a central discrete subgroup of $\tilde{H} \times T$.

Examples. 1. The group $U(n)$ is not simply connected, for any $n \geq 1$. This follows by using again the inductive argument from Example 4, Section 2: since S^{2n-1} is simply connected for any $n \geq 2$, this shows that the fundamental group of $U(n)$ is the same as the one of $U(1)$. But the latter is nothing but the circle S^1 in $\mathbb{C}$, so it's not simply connected.

2. Like in the previous example, the group $SU(n)$ is simply connected, for any $n \geq 1$. This time we note that $SU(1)$ is a point, thus simply connected.

3. Like in the previous example, $Sp(n)$ is simply connected, for any $n \geq 1$.

4. The group $SO(n)$ is not simply connected. Again, we rely on the inductive argument and the fact that the fundamental group of $SO(3)$ is $\mathbb{Z}/2$. The same will be the fundamental group of $SO(n)$ for any $n \geq 3$. The case $n = 2$ is special, since

$$SO(2) = \{\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \mid \theta \in \mathbb{R}\},$$

which can be identified with S^1; the fundamental group is $\mathbb{Z}$.

5. We can describe $\tilde{H}$ in Theorem 3.5 for $U(n)$ and $SO(n)$. For $U(n)$, we consider the isomorphism

$$U(n) \simeq S^1 \times SU(n)$$

given by

$$SU(n) \times S^1 \ni (A, z) \mapsto zA \in U(n).$$

As about $SO(n)$, the situation is very different: namely, there exists a simply connected group, denoted by $Spin(n)$, whose centre Z is isomorphic to $\mathbb{Z}/2$ such that

$$Spin(n)/Z \simeq SO(n).$$

We wonder now which are the Lie algebras which correspond via Theorem 3.2 to simply connected Lie groups which are compact. There exists an algebraic description of such Lie algebras, which is given in the following theorem. First, if $\mathfrak{g}$ is an arbitrary Lie algebra, and X is in $\mathfrak{g}$, by $\mathrm{ad}X$ we denote the linear endomorphism of $\mathfrak{g}$ given by

$$Y \ni \mathfrak{g} \mapsto \mathrm{ad}X(Y) := [X, Y].$$

The <u>*Killing form*</u> of $\mathfrak{g}$ is the bilinear form B on $\mathfrak{g}$ given by

$$\kappa(X, Y) := \mathrm{Tr}(\mathrm{ad}X \circ \mathrm{ad}Y),$$

for $X, Y \in \mathfrak{g}$.

Theorem 3.6. *A simply connected Lie group G is compact if and only if its Lie algebra has strictly negative definite Killing form.*

The conclusion of the section can be formulated as follows:

Any flag manifold is an adjoint orbit of a Lie group whose Lie algebra $\mathfrak{g}$ has strictly negative definite Killing form. Any such group is automatically compact and has finite centre. The adjoint orbit of $X \in \mathfrak{g}$ is the same for any Lie group of Lie algebra $\mathfrak{g}$.

4. Structure and classification of Lie algebras with $\kappa < 0$

Throughout this section $\mathfrak{g}$ will denote a Lie algebra whose Killing form κ is strictly negative definite. The pairing

$$\mathfrak{g} \ni X, Y \mapsto \langle X, Y \rangle := -\kappa(X, Y)$$

makes $\mathfrak{g}$ into a Euclidean space. Moreover, we have

$$\langle [X, Y], Z \rangle = \langle X, [Y, Z] \rangle, \tag{10}$$

for all $X, Y, Z \in \mathfrak{g}$ (check this!). We pick $\mathfrak{t} \subset \mathfrak{g}$ a maximal abelian subspace (by this we mean that $[X, Y] = 0$, for all $X, Y \in \mathfrak{t}$, and if $\mathfrak{t}' \subset \mathfrak{g}$ has the same property and $\mathfrak{t} \subset \mathfrak{t}'$, then $\mathfrak{t}' = \mathfrak{t}$). From equation (10) we deduce that for any $X \in \mathfrak{t}$, the map denoted

$$\mathrm{ad}X : \mathfrak{g} \to \mathfrak{g}, \ \mathrm{ad}X(Y) := [X, Y]$$

is a skew-symmetric endomorphism of $\mathfrak{g}$. Consequently its eigenvalues are purely imaginary. To describe the eigenspaces, we need to complexify $\mathfrak{g}$, that is, to consider

$$\mathfrak{g}^{\mathbb{C}} := \mathfrak{g} \otimes \mathbb{C}.$$

The family $\mathrm{ad}X$, $X \in \mathfrak{t}$, consists of $\mathbb{C}$-linear endomorphisms of $\mathfrak{g}^{\mathbb{C}}$ which commute with each other. The eigenspace decomposition of the latter space is

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{t}^{\mathbb{C}} \oplus \bigoplus_{\alpha \in \Phi(\mathfrak{g}, \mathfrak{t})} \mathfrak{g}^{\mathbb{C}}_{\alpha}, \tag{11}$$

where $\mathfrak{t}^{\mathbb{C}} := \mathfrak{t} \otimes \mathbb{C}$, α are some non-zero linear functions $\mathfrak{t} \to \mathbb{R}$ and[6]

$$\mathfrak{g}_\alpha^{\mathbb{C}} := \{Y \in \mathfrak{g}^{\mathbb{C}} \ : \ [X,Y] = i\alpha(X)Y, \text{ for all } X \in \mathfrak{t}\}. \tag{12}$$

The functions α are called the *roots* of the pair $(\mathfrak{g}, \mathfrak{t})$. One can show that $\mathfrak{g}_\alpha^{\mathbb{C}}$ are complex vector subspaces of $\mathfrak{g}^{\mathbb{C}}$ of dimension 1, called *root spaces*. Equation (12) is called the *root space decomposition* of $\mathfrak{g}^{\mathbb{C}}$. As we will see shortly, the roots are the key towards the classification result we are heading to. Right now we would like to use this opportunity to define a few more notions which are directly related to the roots and will be needed later. The *root lattice* is the $\mathbb{Z}$-span (in $\mathfrak{t}^*$) of all the roots. There is a canonical isomorphism between $\mathfrak{t}$ and $\mathfrak{t}^*$, given by

$$\mathfrak{t} \ni X \mapsto \langle X, \cdot \rangle \in \mathfrak{t}^*. \tag{13}$$

To the root $\alpha \in \mathfrak{t}^*$ corresponds the vector in $\mathfrak{t}$ denoted by v_α. The *coroot* corresponding to α is

$$\alpha^\vee := \frac{2v_\alpha}{\langle \alpha, \alpha \rangle}.$$

The *coroot lattice* is the $\mathbb{Z}$-span (in $\mathfrak{t}$) of all the coroots. Finally, a *weight* is a an element λ of $\mathfrak{t}^*$ with the property that

$$\lambda(\alpha^\vee) \in \mathbb{Z}, \ \forall \alpha \in \Phi.$$

The set of all weights is a lattice (that is, a $\mathbb{Z}$-vector subspace of $\mathfrak{t}$), called the *weight lattice*.

The following theorem describes some properties of the roots. The scalar product induced on $\mathfrak{t}^*$ by the isomorphism (13) is denoted again by $\langle \ , \ \rangle$.

Theorem 4.1. *With the notations above the following holds.*

(a) $\Phi(\mathfrak{g}, \mathfrak{t})$ spans $\mathfrak{t}^*$ and $0 \notin \Phi(\mathfrak{g}, \mathfrak{t})$

(b) For any α in $\Phi(\mathfrak{g}, \mathfrak{t})$, the reflection about the hyperplane $\alpha^\perp$, call it s_α, maps $\Phi(\mathfrak{g}, \mathfrak{t})$ to itself.

(c) For any $\alpha, \beta \in \Phi(\mathfrak{g}, \mathfrak{t})$, the number

$$\frac{2\langle \alpha, \beta \rangle}{\langle \alpha, \alpha \rangle}$$

is in $\mathbb{Z}$. Otherwise expressed, the vector

$$s_\alpha(\beta) - \beta = -\frac{2\langle \alpha, \beta \rangle}{\langle \alpha, \alpha \rangle}\alpha$$

is an integer multiple of α.

(d) If $\alpha, \beta \in \Phi(\mathfrak{g}, \mathfrak{t})$ are proportional, then $\alpha = \beta$ or $\alpha = -\beta$.

Rather than proving this (see e.g. [Br-tD, Chapter V, Theorem 3.12]), we do some examples.

Examples. 1. The Lie algebra of $SU(n)$ is the space $\mathfrak{su}(n)$ of all $n \times n$ skew-hermitian matrices with trace equal to 0 (see Section 2, Example 4). A simple calculation shows that the Killing form is given by

$$B(X,Y) = 2n\mathrm{Trace}(XY),$$

for any $X, Y \in \mathfrak{su}(n)$. A maximal abelian subspace $\mathfrak{t} \subset \mathfrak{su}(n)$ is the one consisting of all diagonal matrices (whose entries are necessarily purely imaginary) with trace 0. We identify

$$\mathfrak{t} = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \ : \ x_1 + \ldots + x_n = 0\}.$$

[6]It is a fact that $\dim_{\mathbb{C}} \mathfrak{g}_\alpha^{\mathbb{C}} = 1$.

The restriction of the negative of the Killing form to $\mathfrak{t}$ is, up to a rescaling, given by the canonical product on $\mathbb{R}^n$. The complexification of $\mathfrak{su}(n)$ is isomorphic to the Lie algebra

$$\mathfrak{sl}(n, \mathbb{C}) = \{A \in \mathrm{Mat}^{n\times n}(\mathbb{C}) \ : \ \mathrm{Trace}(A) = 0.\}$$

The isomorphism is given by

$$\mathfrak{su}(n) \otimes \mathbb{C} \to \mathfrak{sl}(n, \mathbb{C}), \ X \otimes z \mapsto zX.$$

It turns out that the roots are of the form

$$\alpha_{ij} : \mathfrak{t} \to \mathbb{R}, \ \mathfrak{t} \ni x = (x_1, \ldots, x_n) \mapsto \alpha_{ij}(x) = x_i - x_j,$$

where $1 \leq i \neq j \leq n$. Consequently, the dual roots are

$$\alpha_{ij}^\vee = \frac{2(e_i - e_j)}{\langle e_i - e_j, e_i - e_j \rangle} = e_i - e_j,$$

$1 \leq i, j \leq n$, where $e_1, \ldots, e_n$ is the canonical basis of $\mathbb{R}^n$. Point (c) of the previous theorem implies that for general $\mathfrak{g}$, any root is a weight, so the root lattice is contained in the weight lattice. The converse is in general not true, as $\mathfrak{su}(2)$ shows: one can see (see for instance [Hu1, Section 13.1]) that the weight lattice is spanned by

$$\lambda_1 = \frac{1}{3}\alpha_1 + \frac{2}{3}\alpha_2 \text{ and } \lambda_2 = \frac{2}{3}\alpha_1 + \frac{1}{3}\alpha_2.$$

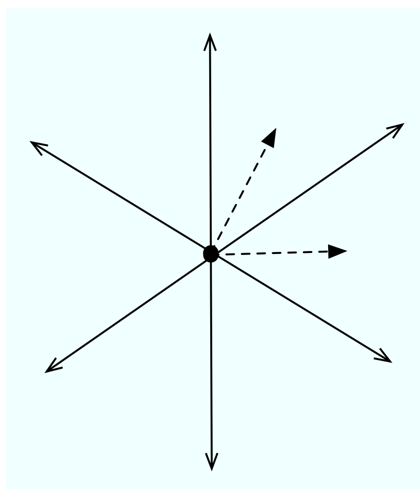

FIGURE 1. The roots of $\mathfrak{su}(2)$: we identify $\mathfrak{t}^*$ with the 2-plane $\mathfrak{t} = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \ : \ x_1 + x_2 + x_3 = 0\}$. The six continuous arrows are the roots and the two dotted arrows span the weight lattice.

2. The Lie algebra of $SO(n)$ (see Example 6, Section 2) is the space $\mathfrak{so}(n)$ of all skew-symmetric $n \times n$ matrices with real entries and trace 0 (prove this!). Its complexification

$$\mathfrak{so}(n, \mathbb{C}) := \mathfrak{so}(n) \otimes \mathbb{C}$$

consists of all skew-symmetric $n \times n$ matrices with complex entries and trace 0. There is a difference between the cases n even and n odd, due to the form of the elements of the maximal abelian subalgebra. To give the students a rough idea about this, we only mention that

- a maximal abelian subspace of $\mathfrak{so}(4)$ consists of all matrices of the form

$$\begin{pmatrix} 0 & h_1 & 0 & 0 \\ -h_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & h_2 \\ 0 & 0 & -h_2 & 0 \end{pmatrix}$$

  where $h_1, h_2 \in \mathbb{R}$
- a maximal abelian subspace of $\mathfrak{so}(5)$ consists of all matrices of the form

$$\begin{pmatrix} 0 & h_1 & 0 & 0 & 0 \\ -h_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & h_2 & 0 \\ 0 & 0 & -h_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

  where $h_1, h_2 \in \mathbb{R}$

For a complete discussion of the roots, one can see [Kn, Ch. II, Section 1] or [Fu-Ha, Section 18.1]. We will confine ourselves to describe the roots and the weights for $\mathfrak{so}(4, \mathbb{C})$ and $\mathfrak{so}(5, \mathbb{C})$, see Figures 2 and 3).

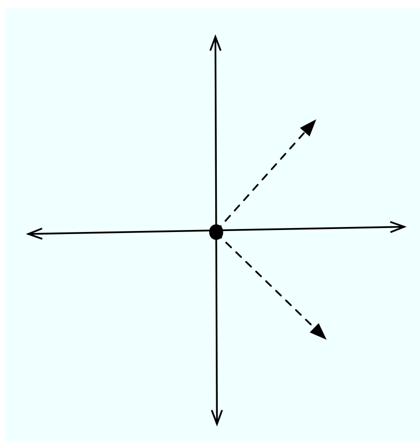

FIGURE 2. The roots of $\mathfrak{so}(4)$, which are represented by continuous lines. The dotted arrows span the weight lattice.

3. To determine the Lie algebra of $Sp(n)$, call it $\mathfrak{sp}(n)$, we use the description given in Example 7, Section 2, eq. (9): we deduce that $\mathfrak{sp}(n)$ consists of all $X \in \mathfrak{u}(2n)$ (that is, skew-hermitean $2n \times 2n$ matrices) with

$$X = J\bar{X}J^{-1}.$$

The set of all $2n \times 2n$ matrices which satisfy *only* the last condition is denoted by[7] $\mathfrak{sp}(n, \mathbb{C})$, so that we can write

$$\mathfrak{sp}(n) = \mathfrak{u}(2n) \cap \mathfrak{sp}(n, \mathbb{C}).$$

Otherwise expressed,

$$\mathfrak{sp}(n) = \{\begin{pmatrix} B & -\bar{C} \\ C & \bar{B} \end{pmatrix} \ : \ B, C \in \mathrm{Mat}^{n\times n}(\mathbb{C}), \bar{B}^T = -B, C^T = C\}.$$

[7]This notation is used for instance in [Kn] (see Chapter I, Section 8]); in [Fu-Ha], the same Lie algebra is denoted by $\mathfrak{sp}_{2n}(\mathbb{C})$ (see Lecture 16).

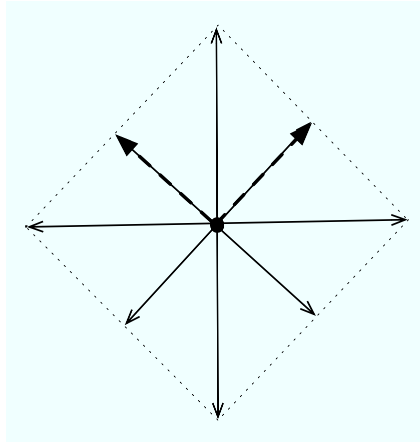

FIGURE 3. The roots of $\mathfrak{so}(5)$: it is interesting to note that here the root and the weight lattice coincide.

The complexification of this is just $\mathfrak{sp}(n, \mathbb{C})$ (this is because the complexification of $\mathfrak{u}(2n)$ is the space of all complex $2n \times 2n$ matrices, and $\mathfrak{sp}(n, \mathbb{C})$ is a complex vector space). A maximal abelian subalgebra of $\mathfrak{sp}(n, \mathbb{C})$ is

$$\mathfrak{t} = \{ \begin{pmatrix} D & 0 \\ 0 & \bar{D} \end{pmatrix} \ : \ D \text{ is a diagonal } n \times n \text{ matrix }, \bar{D} = -D \}.$$

A detailed description of the roots can be found for instance in [Kn, Ch. 2, Section 1]. We only present here the roots and the weights in the case $n = 2$ (see Figure 4 below)

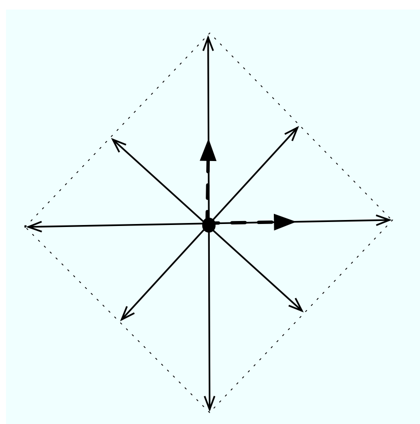

FIGURE 4. The roots and the generators of the weight lattice for $\mathfrak{sp}(2)$.

Now let us return to the context described in Theorem 4.1. Properties (a)-(c) say that $\Phi(\mathfrak{g}, \mathfrak{t})$ is an *abstract root system* in the euclidean space $(\mathfrak{t}^*, \langle \ , \ \rangle)$. There exists a theory of abstract root systems, see for instance [Kn, Ch. II, Section 5]. From this general theory one can deduce as follows:

- There exists a subset

$$\Delta = \{\alpha_1, \ldots, \alpha_\ell\}$$

of $\Phi(\mathfrak{g}, \mathfrak{t})$ which is a linear basis of $\mathfrak{t}^*$ and any $\alpha \in \Phi(\mathfrak{g}, \mathfrak{t})$ can be written as

$$\alpha = \sum_{k=1}^{\ell} m_k \alpha_k,$$

where $m_k \in \mathbb{Z}$ are all in $\mathbb{Z}_{\geq 0}$ or all in $\mathbb{Z}_{\leq 0}$. Here ℓ denotes the dimension of $\mathfrak{t}$. The elements of Δ are called simple roots. The roots α for which all m_k are in $\mathbb{Z}_{\geq 0}$ are called positive: the set of all those roots is denoted by Φ^+.

- The group of linear transformations of[8] $\mathfrak{t}$ generated by the reflections about the planes $\ker \alpha$, $\alpha \in \Phi(\mathfrak{g}, \mathfrak{t})$, is finite; this is called the Weyl group of $(\mathfrak{g}, \mathfrak{t})$. It is actually generated by the reflections $s_1, \ldots, s_\ell$ about $\ker \alpha_1, \ldots, \ker \alpha_\ell$.

An important result in the theory of abstract root systems gives their classification:

The A-G_2 classification of root systems. *Any abstract root system can be decomposed into (or rather recomposed from) finitely many "irreducible components"; each of those is of the form $\Phi(\mathfrak{g}, \mathfrak{t})$, where $\mathfrak{g}$ is one of the following:*

1. $\mathfrak{su}(n)$ — *the root system is called of* A_{n-1}
2. $\mathfrak{so}(2n+1)$ — *the root system is called of type* B_n
3. $\mathfrak{sp}(n)$ — *the root system is called of type* C_n
4. $\mathfrak{so}(2n)$ — *the root system is called of type* D_n
5. $\mathfrak{e}_k$, $k = 6, 7$ *or 8, each of them being the Lie algebra of a certain compact simply connected Lie group — the root system is called of type* E_6, E_7, *respectively* E_8
6. $\mathfrak{f}_4$, *the Lie algebra of a certain compact simply connected Lie group — the root system is called of type* F_4
7. $\mathfrak{g}_2$, *the Lie algebra of a certain compact simply connected Lie group — the root system is called of type* G_2.

There are a few comments which should be made concerning this classification. At the first sight, it may seem strange that we have considered $\mathfrak{so}(2n)$ and $\mathfrak{so}(2n+1)$ separately; we could have considered them together and obtain a classification in only six types. Also (but not only) to understand this, we mention that to any abstract root system corresponds what is called the *Dynkin diagram.* This is actually an oriented graph, constructed as follows:

- its vertices (sometimes called *nodes*) are the simple roots
- the vertices α and β are joined by a number of

$$\frac{4\langle \alpha, \beta \rangle^2}{\langle \alpha, \alpha \rangle \langle \beta, \beta \rangle}$$

  edges
- all of the latter edges have an arrow pointing from α to β if $\|\beta\| < \|\alpha\|$ (if the lengths are equal, there is no arrow)

To achieve the A-G_2 classification above, one classifies first the Dynkin diagrams of abstract root systems: only *after that*, one makes sure that any of the diagrams is coming from a Lie algebra. This also explains why $\mathfrak{so}(2n)$ and $\mathfrak{so}(2n+1)$ are considered separately (see Figure 5 and recall that $\mathfrak{so}(2n+1)$ has type B_n, whereas $\mathfrak{so}(2n)$ has type D_n).

Another question is which are the Lie algebras/groups of type E-G mentioned above. We will not discuss this question here: we address the interested students to [Ad2], [Ba], or

[8] We actually use the identification between $\mathfrak{t}^*$ and $\mathfrak{t}$ induced by the isomorphism described above: the hyperplane $\alpha^\perp$ becomes $\ker \alpha$.

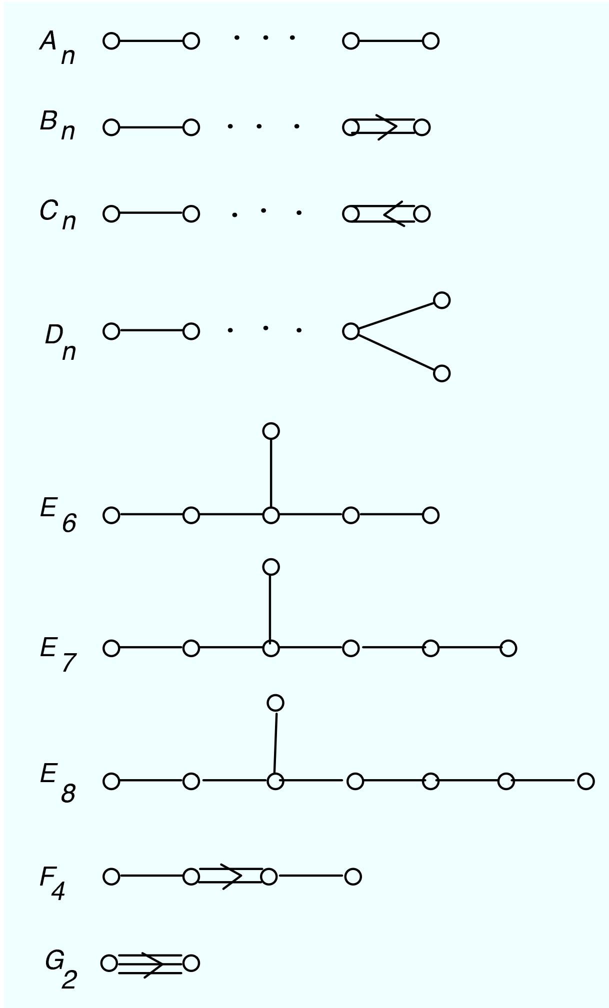

Figure 5. The Dynkin diagrams of the irreducible root systems.

[Fu-Ha, Lecture 22]. We only mention that the type A-D groups are called the *classical Lie algebras/groups*, and the type E-G ones are called the *exceptional Lie algebras/groups.*

It is now time to conclude the section. Theorem 4.1 assigns to each Lie algebra $\mathfrak{g}$ with $\kappa < 0$ a root system; this can be decomposed into several "irreducible" components (see the A-G_2 classification above); by a result which we didn't mention here, namely the one-to-one correspondence

$$\{\text{Lie algebras with } \kappa < 0\} \leftrightarrow \{\text{abstract root systems}\}$$

we deduce that:

Theorem 4.2. *Any Lie algebra $\mathfrak{g}$ with $\kappa < 0$ can be decomposed as*

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_r,$$

where $\mathfrak{g}_1, \ldots, \mathfrak{g}_r$ are Lie algebras belonging to the A-G_2 list above.

5. The flag manifold of a type A-G_2 Dynkin diagram with marked nodes

From Theorem 3.3 and the results stated at the end of section 3 we deduce that any compact simply connected Lie group G can be written as

$$G = G_1 \times \ldots \times G_r,$$

where $G_1, \ldots, G_r$ are compact simply connected belonging to the A-G_2 list. The result at the end of section 4 implies that any flag manifold is a product of adjoint orbits associated to $(\mathfrak{g}, X_0)$, where $\mathfrak{g}$ is in the A-G_2 list and $X_0 \in \mathfrak{g}$. Consequently, we do not lose any generality if we assume as follows:

Assumption 1. *The Lie group G is connected and compact and its Lie algebra $\mathfrak{g}$ belongs to the A-G_2 list in section 4.*

More assumptions can be imposed on the element X_0 of $\mathfrak{g}$, due to the following theorem. Recall first that $\mathfrak{t} \subset \mathfrak{g}$ denotes a maximal abelian subalgebra. By Theorem 3.2 (b), there exists a unique subgroup $T \subset G$ with Lie algebra $\mathfrak{t}$. One can show that T is an abelian group (one uses the fact that $\exp : \mathfrak{t} \to T$ is surjective). In fact T is isomorphic to a torus (see Example 3, Section 2). We call it a *maximal torus* in G (cf. [Kn, Ch. IV, Proposition 4.30]).

Theorem 5.1. *(a) The map* $\exp : \mathfrak{g} \to G$ *is surjective.*

(b) Any element of G is conjugate to an element of T. More precisely, for any $k \in G$ there exists $g \in G$ such that

$$gkg^{-1} \in T.$$

(c) Any $\mathrm{Ad}(G)$ *orbit intersects $\mathfrak{t}$. More precisely, for any $X \in \mathfrak{g}$ there exists $g \in G$ such that*

$$\mathrm{Ad}(g)X \in \mathfrak{t}.$$

We will not prove the theorem. Point (b) is known as the "Maximal Torus Theorem". Point (c) is related[9] to it via (a). For a nice geometric proof of all points of the theorem, we recommend [Bu, Theorems 16.3 and 16.4] (point (c) is not explicitly proved there, but it follows immediately from the proof); another useful reference is [Du-Ko, Ch. 3, Theorem 3.7.1]. It is worth trying to understand point (c) in the special case $G = SU(n)$ (recall that $\mathfrak{t}$ consists of diagonal matrices with purely imaginary entries): it says that any skew-hermitian matrix is $SU(n)$-diagonalizable.

Point (c) shows that we do not lose any generality if we study flag manifolds $\mathrm{Ad}(G)X_0$ such that

Assumption 2. *The element X_0 of $\mathfrak{g}$ is actually in $\mathfrak{t}$.*

In fact, as we will see shortly, we can refine this assumption. In Section 4 we have assigned to the pair $(\mathfrak{g}, \mathfrak{t})$ a root system, call it now Φ, and then the corresponding Weyl group, call it W. We recall that the latter is the group of linear transformations of $\mathfrak{t}$ generated by the reflections s_α, $\alpha \in \Phi$. Now let us consider the group

$$N(T) := \{g \in G \ : \ gTg^{-1} \subset T\},$$

that is, the normalizer of T in G. It is easy to see that for any $g \in N(T)$ we have

$$g\mathfrak{t}g^{-1} \subset \mathfrak{t},$$

[9] I don't see any obvious way to deduce (c) from (b), though.

and also that for any $g \in T$ we have

$$gXg^{-1} = X, \ \forall X \in \mathfrak{t}.$$

This means that the group $N(T)/T$ acts linearly on $\mathfrak{t}$. The action is effective, in the sense that the natural map $N(T)/T \to GL(\mathfrak{t})$ is injective.

Theorem 5.2. *The subgroups W and $N(T)/T$ of $GL(\mathfrak{t})$ are equal.*

For a proof one can see for instance [Br-tD, Ch. V, Theorem 2.12] or [Kn, Ch. IV, Theorem 4.54].

Exercise. Show that the Weyl group of $SU(n)$ acts on $\mathfrak{t}$ described in Example 1, Section 4, by permuting the coordinates $x_1, \ldots, x_n$. In other words, we can identify

$$W(\mathfrak{su}(n), \mathfrak{t}) = S_n.$$

Then find the element g_{ij} of $N(T)$ whose coset modulo T gives the reflection about the root hyperplane $\{x_i = x_j\}$ in $\mathfrak{t}$.

The components of

$$\mathfrak{t} \setminus \bigcup_{\alpha \in \Phi} \ker \alpha$$

are polyhedral cones, called <u>*Weyl chambers*</u>. Pick a simple root system $\Delta = \{\alpha_1, \ldots, \alpha_\ell\}$ for Φ. Then one of the Weyl chambers is

$$C(\Delta) = \{X \in \mathfrak{t} \ : \ \alpha_j(X) < 0, j = 1, \ldots, \ell\}.$$

This is called the <u>*fundamental Weyl chamber*</u>. The planes $\ker \alpha_j$, $1 \leq j \leq \ell$, which bound it are called the <u>*walls*</u>. The following result is relevant for our goals.

Theorem 5.3. *The Weyl group acts simply transitively on the set of Weyl chambers. That is, for any Weyl chamber C there exists a unique $w \in W$ such that*

$$wC = C(\Delta).$$

Consequently, for the element X_0 in Assumption 2 one can find $w \in W$ such that wX_0 is in the closure $\overline{C(\Delta)}$. On the other hand, we can write

$$w = nT \in N(T)/T,$$

thus

$$wX_0 = nX_0n^{-1}$$

is in the adjoint orbit of X_0. This shows that the next assumption is not restrictive.

> **Assumption 3.** *The element X_0 of $\mathfrak{g}$ is actually in $\overline{C(\Delta)}$ (the fundamental chamber $C(\Delta)$ together with its walls) .*

Our next goal is to understand how the diffeomorphism type of the flag manifold $\mathrm{Ad}(G)X_0$ changes when X_0 moves inside $\overline{C(\Delta)}$. First we need to return to the root decomposition of $\mathfrak{g}^{\mathbb{C}}$ given by (12). This induces a similar decomposition of $\mathfrak{g}$ itself, as follows:

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_\alpha, \tag{14}$$

where

$$\mathfrak{g}_\alpha = (\mathfrak{g}_\alpha^{\mathbb{C}} \oplus \mathfrak{g}_{-\alpha}^{\mathbb{C}}) \cap \mathfrak{g} = \{Y \in \mathfrak{g} \ : \ [X, [X, Y]] = -\alpha(X)^2 Y, \ \forall X \in \mathfrak{t}\}.$$

Proposition 5.4. *Let the walls of $\overline{C(\Delta)}$ to which X_0 belongs be* $\ker \alpha_{i_1}, \dots, \ker \alpha_{i_k}$*, where* $i_1, \dots, i_k \in \{1, \dots, \ell\}$*. Then we have the diffeomorphism*

$$\mathrm{Ad}(G)X_0 \simeq G/G_{i_1,\dots,i_k}$$

where $G_{i_1,\dots,i_k}$ denotes the connected closed subgroup of G of Lie algebra

$$\mathfrak{t} \oplus \bigoplus_{\alpha \in \Phi^+(X_0)} \mathfrak{g}_\alpha.$$

Here

$$\Phi^+(X_0) := \{\alpha \in \Phi^+ \ : \ \alpha(X_0) = 0\} = \Phi^+ \cap \mathrm{Span}_{\mathbb{Z}}\{\alpha_{i_1}, \dots, \alpha_{i_k}\}. \tag{15}$$

In particular, if X_0 is in $C(\Delta)$ (that is, contained in no wall), then

$$\mathrm{Ad}(G)X_0 \simeq G/T.$$

Proof. We only need to show that the stabilizer of X_0 is

$$G_{X_0} = G_{i_1,\dots,i_k}, \tag{16}$$

and use the homogeneous structure of $\mathrm{Ad}(G)X_0$ described by equation (5). To prove 16, we check as follows:

1. The Lie algebra of G_{X_0} is the same as the one of $G_{i_1,\dots,i_k}$
2. The group G_{X_0} is connected

and then we use Theorem 3.2 (b).

Let us start with 1. The Lie algebra of G_{X_0} is

$$\{X \in \mathfrak{g} \ : \ [X, X_0] = 0\}.$$

Can you justify this? One can now easily see that we have

$$\{X \in \mathfrak{g}^{\mathbb{C}} \ : \ [X, X_0] = 0\} = \mathfrak{t}^{\mathbb{C}} \oplus \bigoplus_{\alpha \in \Phi, \alpha(X_0)=0} \mathfrak{g}^{\mathbb{C}}_\alpha.$$

To obtain the Lie algebra of G_{X_0} we intersect the latter space with $\mathfrak{g}$ and obtain the desired description; it only remains to justify the second equation in (15) — we leave it as an exercise for the students.

As about 2., we rely on the fact that the flag manifold $\mathrm{Ad}(G)X_0 = G/G_{X_0}$ is simply connected, a fact which will be proved later on (see Corollary 7.2). The long exact homotopy sequence of the bundle

$$G_{X_0} \to G \to G/G_{X_0}$$

implies that G_{X_0} is connected. For a more direct argument, one can see for instance [Du-Ko, Chapter 3, Theorem 3.3.1]. $\square$

The School will be concerned with the study of the flag manifolds

$$G/G_{i_1,\dots,i_k}$$

defined in Proposition 5.4, where G is a compact connected Lie group whose Lie algebra $\mathfrak{g}$ is one of the seven in the A-G_2 list and $i_1, \dots, i_k$ are labels of certain simple roots, that is, certain nodes of the Dynkin diagram of $\mathfrak{g}$.

Note. By Theorem 3.3 and Corollary 3.4, G can be *any* connected Lie group with Lie algebra $\mathfrak{g}$. We will often assume that G is *the* simply connected one: the benefit is described in the following theorem (see for instance [Br-tD, Chapter V, Section 7]).

Theorem 5.5. *Let $\mathfrak{g}$ be in the A-G_2 list and G a connected Lie group with Lie algebra $\mathfrak{g}$. Then the fundamental group of G is*

$$\pi_1(G) \simeq \ker(\exp : \mathfrak{t} \to T) \ / \ \mathrm{Span}_{\mathbb{Z}}(\text{coroots}).$$

In particular, if G is simply connected, then we have

$$\ker(\exp : \mathfrak{t} \to T) \ = \ \mathrm{Span}_{\mathbb{Z}}(\text{coroots}).$$

Exercise. Describe the flag manifold F_n and the Grassmannian $Gr_k(\mathbb{C}^n)$ in terms of Dynkin digrams with marked nodes.

6. Flag manifolds as complex projective varieties

A useful observation in the study of flag manifolds is that they can be realized as complex submanifolds of some complex projective space. We recall that if V is a finite dimensional complex vector space, then the corresponding projective space

$$\mathbb{P}(V) := \{\ell \ : \ \ell \text{ is a } 1-\text{dimensional vector subspace of } V\}$$

is a complex manifold[10]. One way to see this is by identifying $V = \mathbb{C}^n$ and constructing a holomorphic atlas on $\mathbb{P}(V) = \mathbb{P}(\mathbb{C}^n)$. There is another way, which is more instructive for us. Namely, we note first that $\mathbb{P}(\mathbb{C}^n)$ is just the grassmannian $Gr_1(\mathbb{C}^n)$, so a flag manifold. As noticed in the first section, the group $GL_n(\mathbb{C})$ acts transitively on $Gr_1(\mathbb{C}^n)$ and the stabilizer of $\mathbb{C}e_1$ is $\mathbb{C}^* \times GL_{n-1}(\mathbb{C})$ (the latter is naturally embedded in $GL_n(\mathbb{C})$). Consequently we have

$$\mathbb{P}(V) = \mathbb{P}(\mathbb{C}^n) = Gr_1(\mathbb{C}^n) = GL_n(\mathbb{C})/(\mathbb{C}^* \times GL_{n-1}(\mathbb{C})).$$

The latter quotient space, regarded as a homogeneous manifold, is complex, since both $GL_n(\mathbb{C})$ and $\mathbb{C}^* \times GL_{n-1}(\mathbb{C})$ are open subspaces of $\mathrm{Mat}^{n\times n}(\mathbb{C}) \simeq \mathbb{C}^{n^2}$. A similar argument shows that any of the manifolds of flags defined in Section 1 is a complex manifold. It is not clear though why are they *projective* complex manifolds, that is, how can we embed them (like F_n or $Gr_k(\mathbb{C}^n)$) in some $\mathbb{P}(V)$: this will be explained shortly.

Before going any further we discuss a few basic things about complex Lie groups. First recall (see footnote page 6) that the space $\mathrm{Mat}^{n\times n}(\mathbb{C})$ is a closed subspace of $\mathrm{Mat}^{2n\times 2n}(\mathbb{R})$ via the embedding

$$\mathrm{Mat}^{n\times n}(\mathbb{C}) \ni A + iB \mapsto \begin{pmatrix} A & -B \\ B & A \end{pmatrix}.$$

We deduce that $GL_n(\mathbb{C})$ is a closed subspace (actually subgroup) of $GL_{2n}(\mathbb{R})$.

Definition 6.1. *A* <u>complex Lie group</u> *is a subgroup G of some general linear group $GL_n(\mathbb{C})$ such that*

1. *G is a closed subspace of $GL_n(\mathbb{C})$*
2. *the Lie algebra of G is a complex vector subspace of* $\mathrm{Mat}^{n\times n}(\mathbb{C})$.

Note that condition 1 implies that G is a Lie group (because it's closed in $GL_{2n}(\mathbb{R})$) and also that its Lie algebra is contained in $\mathrm{Mat}^{n\times n}(\mathbb{C})$ as a *real* (but not necessarily *complex*) vector space.

Another notion we need is that of complex Lie algebra:

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[10]By definition, this is a manifold with an atlas whose charts are of the form (U, φ) where U is open in some $\mathbb{C}^N$ (same N for all charts) and for any two charts (U, φ), (V, ψ) the change of coordinates map $\varphi \circ \psi^{-1}$ is holomorphic.

Definition 6.2. *A* complex Lie algebra *is a complex vector subspace of* $\mathrm{Mat}^{n\times n}(\mathbb{C})$ *for some* $n \geq 1$*, which is closed under the Lie bracket* $[\ ,\ ]$ *given by*

$$\mathrm{Mat}^{n\times n}(\mathbb{C}) \ni X, Y \mapsto [X, Y] := XY - YX \in \mathrm{Mat}^{n\times n}(\mathbb{C}).$$

We list here a few properties of complex Lie groups/Lie algebras:

- Any complex Lie group is a complex manifold.
- If G is a complex Lie group of Lie algebra $\mathfrak{g}$ and $\mathfrak{h} \subset \mathfrak{g}$ is a complex Lie subalgebra (that is, a complex vector subpace which is closed under the bracket $[\ ,\ ]$), then there exists a unique closed and connected subgroup $H \subset G$ with Lie algebra $\mathfrak{h}$ (H is necessarily a complex Lie group, although it is worth looking again at the note following Theorem 3.2). The quotient space
$$G/H = \{gH\ :\ g \in G\}$$
has a natural structure of a complex manifold.

The complex Lie groups corresponding to the Lie algebras $\mathfrak{g} \otimes \mathbb{C}$ in examples 1, 2, 3 in Section 4 are as follows.

Examples of complex Lie groups. 1. The Lie algebra of

$$SL(n, \mathbb{C}) := \{g \in GL_n(\mathbb{C})\ :\ \det(g) = 1\}$$

is $\mathfrak{sl}(n, \mathbb{C})$. Thus $SL(n, \mathbb{C})$ is a complex Lie group.

2. The Lie algebra of

$$SO(n, \mathbb{C}) = \{g \in GL_n(\mathbb{C})\ :\ g \cdot g^T = I_n\}$$

is $\mathfrak{so}(n, \mathbb{C})$. Thus $SO(n, \mathbb{C})$ is a complex Lie group.

3. The Lie algebra of

$$Sp(n, \mathbb{C}) := \{g \in GL_n(\mathbb{C})\ :\ g = J\bar{g}J^{-1}\}.$$

is $\mathfrak{sp}(n, \mathbb{C})$. Thus $Sp_n(\mathbb{C})$ is a complex Lie group. We also note the obvious identity

$$Sp(n) = U(2n) \cap Sp_n(\mathbb{C}).$$

In general, whenever G is a compact Lie group, say $G \subset GL_n(\mathbb{R})$, one considers its Lie algebra $\mathfrak{g} \subset \mathrm{Mat}^{n\times n}(\mathbb{R})$ and its complexification $\mathfrak{g} \otimes \mathbb{C}$ which is a Lie subalgebra of $\mathrm{Mat}^{n\times n}(\mathbb{C})$. The complex Lie group corresponding to $\mathfrak{g} \otimes \mathbb{C}$ is called the[11] *complexification* of G and is usually denoted by $G^{\mathbb{C}}$. The complexifications of $SU(n)$, $SO(n)$ and $Sp(n)$ are $SL_n(\mathbb{C})$, $SO(n, \mathbb{C})$, respectively $Sp(n, \mathbb{C})$.

From now on in this section, G will be a compact connected Lie group of Lie algebra $\mathfrak{g}$ in the A-G_2 list, and $\mathfrak{g}^{\mathbb{C}} = \mathfrak{g} \otimes \mathbb{C}$, $G^{\mathbb{C}}$ will be their complexifications. As usually, we take $\mathfrak{t} \subset \mathfrak{g}$ a maximal abelian subalgebra, together with the corresponding root system $\Phi = \Phi(\mathfrak{g}, \mathfrak{t})$. We also pick a simple root system $\Delta \subset \Phi$ and denote by Φ^+ the corresponding set of positive roots. The space

$$\mathfrak{b} := \mathfrak{t}^{\mathbb{C}} \oplus \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_\alpha^{\mathbb{C}}$$

is a Lie subalgebra of $\mathfrak{g}^{\mathbb{C}}$.

[11] Of course one can ask if $G^{\mathbb{C}}$ depends on the embedding $G \subset GL_n(\mathbb{R})$: the answer is "no, it doesn't" (see e.g. [Du-Ko, p. 297]). This is not important for us, as we know exactly which are the complexifications for G of type A-D and for the other (exceptional) types, we may just say that we take an arbitrary embedding and the corresponding complexification.

Exercise. Prove the previous claim. First show that

$$[\mathfrak{g}^{\mathbb{C}}_\alpha, \mathfrak{g}^{\mathbb{C}}_\beta] = \begin{cases} \mathfrak{g}^{\mathbb{C}}_{\alpha+\beta}, \text{ if } \alpha+\beta \in \Phi \\ \{0\}, \text{ if contrary.} \end{cases}$$

Let B denote the (complex) connected Lie subgroup of $G^{\mathbb{C}}$ whose Lie algebra is $\mathfrak{b}$. We call it a *Borel subgroup*. More generally, let us pick some nodes of the Dynkin diagram, say $i_1, \ldots, i_k$. Consider

$$\mathfrak{p} = \mathfrak{p}_{i_1,\ldots,i_k} := \mathfrak{t}^{\mathbb{C}} \oplus \bigoplus_{\alpha} \mathfrak{g}^{\mathbb{C}}_\alpha,$$

where the sum runs this time over

$$\alpha \in \Phi^+ \bigcup (\Phi^- \cap \mathrm{Span}_{\mathbb{Z}}\{\alpha_{i_1}, \ldots, \alpha_{i_k}\}).$$

Again one can show that $\mathfrak{p}$ is a Lie subalgebra of of $\mathfrak{g}$. The corresponding Lie subgroup of $G^{\mathbb{C}}$, call it

$$P = P_{i_1,\ldots,i_k}$$

is called a *parabolic subgroup*[12].

The next proposition tells us that the flag manifolds are $G^{\mathbb{C}}/P$, where P is parabolic.

Proposition 6.3. *Denote $P = P_{i_1,\ldots,i_k}$, $\mathfrak{p} = \mathfrak{p}_{i_1,\ldots,i_k}$.*

(a) With the notations at the end of Section 5 we have

$$G_{i_1,\ldots,i_k} \subset P.$$

The natural map

$$G/G_{i_1,\ldots,i_k} \to G^{\mathbb{C}}/P$$

is a diffeomorphism.

(b) The normalizer

$$N_{G^{\mathbb{C}}}(\mathfrak{p}) := \{g \in G^{\mathbb{C}} \;:\; g\mathfrak{p}g^{-1} = \mathfrak{p}\}$$

is equal to P. Consequently the only Lie subgroup of $G^{\mathbb{C}}$ with Lie algebra $\mathfrak{p}$ is P.

(c) We have

$$P \cap G = G_{i_1,\ldots,i_k}.$$

In particular,

$$B \cap G = T.$$

Proof (for $P = B$). Set

$$B' := \{g \in G^{\mathbb{C}} \mid \mathrm{Ad}g(\mathfrak{p}) \subset \mathfrak{p}\}.$$

The latter is a closed subgroup of $G^{\mathbb{C}}$ of Lie algebra $\mathfrak{b}$ (Exercise: check this!)

We show that the natural map

$$G/T \ni gT \mapsto gB' \in G^{\mathbb{C}}/B' \tag{17}$$

is a diffeomorphism. We identify the tangent space to $G^{\mathbb{C}}/B'$ at IB' as

$$T_{IB'}G^{\mathbb{C}}/B' = \mathfrak{g}^{\mathbb{C}}/\mathfrak{b} \simeq \sum_{\alpha\in\Phi^+} \mathfrak{g}^{\mathbb{C}}_\alpha.$$

[12]It is interesting (although not useful to us) that any subgroup P of $G^{\mathbb{C}}$ with $B \subset P$ is automatically parabolic (see e.g. [Fu-Ha, Claim 23.48]).

Similarly, the tangent space to IT at G/T is

$$T_{IT}G/T = \mathfrak{g}/\mathfrak{t} \simeq \sum_{\alpha\in\Phi^+} \mathfrak{g}_\alpha.$$

The differential at IT of the map (17) is

$$\mathfrak{g}/\mathfrak{t} \to \mathfrak{g}^{\mathbb{C}}/\mathfrak{b}$$

induced by the inclusions; its kernel consists of the cosets in $\mathfrak{g}/\mathfrak{t}$ of $\mathfrak{b}\cap\mathfrak{g}$. Since the latter is $\mathfrak{t}$, the kernel is 0, thus the differential is an isomorphism (surjectivity follows from a dimension argument). Consequently the map (17) is open, so its image is open in $G^{\mathbb{C}}/B'$. Since G is compact, its image is closed, so (17) is surjective. Injectivity follows from the fact that

$$G \cap B' = T.$$

To justify the last equation, we only need to check that $G \cap B' \subset T$. To this end we write

$$\mathfrak{b} = \mathfrak{t}^{\mathbb{C}} + \mathfrak{n},$$

where

$$\mathfrak{n} := \sum_{\alpha\in\Phi^-} \mathfrak{g}_\alpha^{\mathbb{C}}.$$

Pick $X_0 \in \mathfrak{t}$ which is not in $\ker\alpha$, for any $\alpha \in \Phi$. This implies that

$$\{X \in \mathfrak{g} \ : \ [X, X_0] = 0\} = \mathfrak{t}.$$

If g is in $G \cap B'$ then

$$\mathrm{Ad}g(X_0) = Y + H,$$

where $Y \in \mathfrak{t}^{\mathbb{C}}$ and $H \in \mathfrak{n}$. On the other hand, $\mathrm{Ad}g(X_0)$ is in $\mathfrak{g}$, which implies that $Y \in \mathfrak{t}$ and $H \in \mathfrak{g}$ (see eq. (14)). Since $\mathfrak{g} \cap \mathfrak{n} = \{0\}$, we deduce that

$$\mathrm{Ad}g(X_0) \in \mathfrak{t}.$$

Due to the choice of X_0, this implies that

$$\mathrm{Ad}g(\mathfrak{t}) \subset \mathfrak{t},$$

which means that $g \in N_G(T)$. We have to show that actually $g \in T$. Suppose it isn't: this implies that

$$g \mathrm{mod} T = w$$

which is an element of the Weyl group $W = N_G(T)/T$, is different from the identity in the latter group. One can show that

$$\mathrm{Ad}g(\mathfrak{g}_\alpha^{\mathbb{C}}) = \mathfrak{g}_{w\alpha}^{\mathbb{C}},$$

for any $\alpha \in \Phi$ (can you justify this?). There is a result in the theory of root systems which says that for any $w \in W$, $w \neq 1$, there exists a positive root α such that $w\alpha$ is a negative root (see e.g. [Kn, Chapter II, Section 5, Proposition 2.70]). This contradicts the fact that $\mathrm{Ad}(g)\mathfrak{b} = \mathfrak{b}$.

Now since $G^{\mathbb{C}}/B'$ is diffeomorphic to G/T, the former has to be simply connected. We deduce that B' is connected, thus $B' = B$ and the proof is finished. The same arguments work for general parabolic P. □

We are now ready to accomplish the goal mentioned at the beginning of the section:

Proposition 6.4. *The flag manifold $G^{\mathbb{C}}/P_{i_1,\ldots,i_k}$ is a projective manifold, in the sense that it is a complex submanifold of some $\mathbb{P}(V)$, where V is a complex vector space.*

Proof (main ideas). Let $\lambda \in \mathfrak{t}^*$ be a weight with the property that

$$\lambda(\alpha_i^\vee) \geq 0, \ \forall 1 \leq i \leq \ell$$
$$\lambda(\alpha_i^\vee) = 0 \text{ iff } i \in \{i_1, \ldots, i_k\}.$$

From the general representation theory, there exists a complex vector space V acted on by $G^{\mathbb{C}}$ via complex linear automorphisms such that the action of $\mathfrak{t}^{\mathbb{C}}$ on V given by

$$\mathfrak{t}^{\mathbb{C}} \times V \ni (X, v) \mapsto X.v := \frac{d}{dt}|_0 \exp(tX).v \in V$$

has the following properties[13]:

1. We have the "weight space" decomposition
$$V = \bigoplus_\mu V_\mu,$$
where μ are weights and $V_\mu = \{0\}$ unless $\mu \in \lambda - \mathrm{Span}_{\mathbb{Z}_{\geq 0}}\{\alpha_{i_1}, \ldots, \alpha_{i_\ell}\}$.
2. $\dim V_\lambda = 1$.
3. $\mathfrak{g}^{\mathbb{C}}_\alpha.V_\lambda \subset V_{\lambda+\alpha}$, for any $\alpha \in \Phi$.
4. For any $\alpha \in \Phi^-$ we have $V_{\lambda-\alpha} = \{0\}$ (this is a direct consequence of 1.).
5. For $\alpha \in \Phi^+$ we have $V_{\lambda-\alpha} = \{0\}$ if and only if $\lambda(\alpha^\vee) = 0$.

The group $G^{\mathbb{C}}$ acts naturally on $\mathbb{P}(V)$. In the latter space we consider the point x represented by V_λ (see point 2. above). We show that the $G^{\mathbb{C}}$-stabilizer of x is $P_{i_1,\ldots,i_k}$. The Lie algebra of the stabilizer consists of all $X \in \mathfrak{g}^{\mathbb{C}}$ with

$$X.V_\lambda \subset V_\lambda.$$

Is this hard to understand? From points 4. and 5. above we deduce that the Lie algebra of the stabilizer of x is $\mathfrak{p}$. From Proposition 6.3 (b), we deduce that the stabilizer is equal to P, so the orbit $G^{\mathbb{C}}.x$ is just the flag manifold $G^{\mathbb{C}}/P$. □

Example. Let $\mathcal{V} = \mathbb{C}^n$ be the standard representation of $SL_n(\mathbb{C})$. The latter group acts linearly also on the vector space $\bigwedge^k \mathcal{V}$, where $k \geq 1$. It turns out that this representation arises in the way described in the proof above (see [Fu-Ha, Section 15.2]). The $SL_n(\mathbb{C})$ orbit on $\mathbb{P}(\bigwedge^k \mathcal{V})$ consists of points $\mathbb{C}(v_1 \wedge \ldots \wedge v_k)$, with $v_1, \ldots, v_k \in \mathcal{V}$. One can see that this is a projective embedding of the Grassmannian $Gr_k(\mathbb{C}^n)$: in each k-plane V one picks a basis $v_1, \ldots, v_k$ and one assigns to V the point $\mathbb{C}(v_1 \wedge \ldots \wedge v_k)$. This is called the *Plücker embedding* of the Grassmannian. For more details, see for instance [Fu-Ha, Section 23.3].

7. The Bruhat decomposition

Let $\mathfrak{g}$ be a (real) Lie algebra belonging to the A-G list and a (compact) connected Lie group G of Lie algebra $\mathfrak{g}$. We pick some nodes $i_1, \ldots, i_k$ in the Dynkin diagram and denote

$$P := P_{i_1,\ldots,i_k}.$$

We are interested in the topology of the flag manifold $G^{\mathbb{C}}/P$. In this section we describe it as a CW complex. The idea is simple to describe: the cells are just the orbits of B, which acts on $G^{\mathbb{C}}/P$ by left multiplication. The main instrument is the Bruhat decomposition of $G^{\mathbb{C}}$, presented in the following theorem. We recall that the Weyl group of G is $W = N_G(T)/T$.

[13] The space V is the representation of $\mathfrak{g}^{\mathbb{C}}$ of *highest weight* λ, see for instance [Hu1, Chapter VI].

For each $w \in W$ we pick $n_w \in N_G(T)$ whose coset modulo T is w. The subspace Bn_wB of $G^{\mathbb{C}}$ is independent of the choice of n_w (because $T \subset B$); we denote

$$BwB := Bn_wB.$$

We also recall that W can also be regarded as a group of linear transformations of $\mathfrak{t}$: it is generated by the reflections $s_1, \ldots, s_\ell$ about the hyperplanes $\ker \alpha_1, \ldots, \ker \alpha_\ell$ which bound the fundamental Weyl chamber. To the parabolic subgroup $P = P_{i_1,\ldots,i_k}$ we attach

$$W_P := \langle s_{i_1}, \ldots, s_{i_k} \rangle$$

which is a subgroup of W. One can show that if $w \in W_P$ is of the form $w = n \mathrm{mod} T$, then[14] $n \in P$. Consequently, the subset BwP of $G^{\mathbb{C}}$ depends only on the class of w in W/W_P.

Theorem 7.1. *(a) The group $G^{\mathbb{C}}$ is the disjoint union*

$$G^{\mathbb{C}} = \bigsqcup_{w \in W} BwB.$$

If P is parabolic then

$$G^{\mathbb{C}} = \bigsqcup_{w \mathrm{mod} W_P \in W/W_P} BwP.$$

(b) In the disjoint union

$$G^{\mathbb{C}}/B = \bigsqcup_{w \in W} BwB/B$$

any BwB/B is a subvariety of $G^{\mathbb{C}}/B$ isomorphic to $\mathbb{C}^{\ell(w)}$. Here

$$\ell(w) := \min\{p \ : \ w \ \text{ is the product of } p \text{ elements of } \{s_1, \ldots, s_\ell\}\}$$

is the length of w. In the disjoint union

$$G^{\mathbb{C}}/P = \bigsqcup_{w \mathrm{mod} W_P \in W/W_P} BwP/P$$

any BwP/P is a locally closed subvariety of $G^{\mathbb{C}}/P$ isomorphic to $\mathbb{C}^{\ell(w')}$, where w' is the element of the coset $w \mathrm{mod} W_P$ whose length is minimal.

We will not prove the theorem (point (a) is proved for instance in [Ku, Theorem 5.1.3], [He, Ch. IX, Theorem 1.4] or [Hu2, Theorem 28.3]; for point (b), one can see [Ku, Theorem 5.1.5]). Instead, we will discuss an example — see below. Point (b) says that $G^{\mathbb{C}}/P$ is a CW complex, the (real) dimension of each cell being an even number. From a general result about the homology of CW complexes (see for instance [Br, Ch. IV, Section 10] or [Bo-Tu, Proposition 17.12]) we deduce as follows:

Corollary 7.2. *Any flag manifold $G^{\mathbb{C}}/P$ is simply connected. Moreover, all cohomology groups $H^{2k+1}(G^{\mathbb{C}}/P; \mathbb{Z})$ are zero (so we can have non-vanishing homology groups only in even dimensions).*

The orbit BwP/P is called a <u>Bruhat cell</u>. Its closure

$$X_w := \overline{BwP/P}$$

[14] A proof goes as follows: assume $w = s_{i_j}$; from $\mathrm{Ad} n(\mathfrak{g}_\alpha^{\mathbb{C}}) = \mathfrak{g}^{\mathbb{C}}_{s_{i_j}\alpha}$, for any $\alpha \in \Phi$, we deduce that $\mathrm{Ad} n(\mathfrak{p}_{i_1,\ldots,i_k}) = \mathfrak{p}_{i_1,\ldots,i_k}$, thus $n \in P$.

is called a <u>*Schubert variety*</u>. Results in algebraic geometry imply that X_w is a closed subvariety[15] of $G^{\mathbb{C}}/P$ (recall from our Section 6 that the latter is a complex projective variety. It is also irreducible[16]: indeed, the B-orbit is a (locally closed) subvariety isomorphic to $\mathbb{C}^{\ell(w)}$, thus it is irreducible (w.r.t. Zariski topology); in general, the closure of an irreducible topological subspace is irreducible.

We will use the following general facts concerning the cohomology of algebraic varieties (the details can be found in [Fu, Section B3]).

Fact 1. If Y is a d-codimensional (possibly singular) irreducible closed subvariety of a smooth projective variety X, then we can attach to it the <u>*fundamental cohomology class*</u>

$$[Y] \in H^{2d}(X; \mathbb{Z}).$$

Fact 2. Let X be a smooth projective variety with a filtration

$$X = X_0 \supset X_1 \supset \ldots \supset X_s = \phi,$$

where each X_j, $0 \le j \le s$, is a closed subvariety. Assume that for each j we have

$$X_i \setminus X_{i+1} = \bigsqcup_j U_{ij}$$

where U_{ij} is a (locally closed) subvariety isomorphic to some $\mathbb{C}^{n(i,j)}$. Then the classes $[\overline{U_{ij}}]$ are a basis of $H^*(X, \mathbb{Z})$.

These two facts will be used to prove that the fundamental classes of Schubert varieties are a basis of the cohomology group of $G^{\mathbb{C}}/P$. We first give a result which describes X_w; this is B invariant, thus a union of B-orbits (that is, cells) and the point is to say which are those cells. For simplicity we only consider the case $P = B$.

Proposition 7.3. *The following two statements are equivalent:*

(a) $BvB/B \subset X_w$

(b) There exists simple roots $\alpha_{i_1}, \ldots, \alpha_{i_k}$ *such that*

- $w = vs_{i_1} \ldots s_{i_k}$
- $\ell(w) = \ell(v) + k.$

If so, we denote $v \le w$. *The resulting ordering on the Weyl group* W *is called the* <u>Bruhat ordering</u> *(this is of course not a total ordering).*

The "standard" proof can be found for instance in [Ku, Theorem 5.1.5] (this is actually a very abstract proof, using only the axioms of a Tits system) . For the classical flag manifold F_n, the result is proved in [Fu, Section 10.5].

Theorem 7.4. *The classes* $[X_w]$, $wW_P \in W/W_P$, *are a basis of* $H^*(G^{\mathbb{C}}/P)$, *which is called the Schubert basis.*

Proof. (for $P = B$) Set

$$X_d := \bigcup_{\ell(w) \le d} BwB/B.$$

[15] Here are a few more details: First, $G^{\mathbb{C}}$ is an algebraic group (see [Fu-Ha, p. 374]) and P and B are algebraic subgroups. The orbits of the B action on the algebraic variety $G^{\mathbb{C}}/P$ are locally closed (see [Hu2, Proposition 8.3]). Consequently, the closures of such an orbit with respect to the Zariski and the usual topologies are the same (see [Mu, Theorem 2.3.4]).

[16] An topological space is called *irreducible* if it cannot be written as a union of two closed proper subspaces.

These are a filtration of $G^{\mathbb{C}}/P$. The theorem follows from Fact 2: we can easily see that the filtration X_d, $d \geq 0$, satisfies the hypotheses in Fact 2 (we use Proposition 7.3). $\square$

In the following we will discuss in more detail an example.

Schubert varieties in the Grassmannian.[17] For $G = SU(n)$ we have $G^{\mathbb{C}} = SL_n(\mathbb{C})$, $W = S_n$ (the symmetric group), B consists of all upper triangular matrices in $SL_n(\mathbb{C})$. We take the parabolic group

$$P = \{\begin{pmatrix} A & C \\ 0 & B \end{pmatrix} \ : \ A \in GL_k(\mathbb{C}), B \in GL_{n-k}(\mathbb{C})\} \cap SL_n(\mathbb{C}).$$

The Grassmannian is

$$Gr_k(\mathbb{C}^n) = SL_n(\mathbb{C})/P.$$

This is the same as the set of all k-planes in $\mathbb{C}^n$: indeed, to any such plane V we attach the coset of the element g of $SL_n(\mathbb{C})$ with the property that

$$V = g\mathrm{Span}_{\mathbb{C}}\{e_1, \ldots, e_k\}.$$

The Bruhat cells are the B orbits of W/W_P, which is embedded in $Gr_k(\mathbb{C}^n)$. Let us first understand the latter embedding. We have

$$W_P = S_k \times S_{n-k},$$

which is embedded in S_n via

$$S_k \times S_{n-k} \ni (\mu, \tau) \mapsto (\mu(1), \ldots, \mu(k), \tau(1) + k, \ldots, \tau(n-k) + k) \in S_n$$

(exercise: try to justify this). Otherwise expressed, $S_k \times S_{n-k}$ consists of permutations σ with

$$\sigma(1), \ldots, \sigma(k) \leq k \text{ and } k+1 \leq \sigma(k+1), \ldots, \sigma(n)$$

The elements of minimal length of $W/W_P = S_n/S_k \times S_{n-k}$ (which label the Schubert varieties, see Theorem 7.1, (b)) are permutations $\sigma \in S_n$ with the property that

$$\sigma(1) < \ldots < \sigma(k) \text{ and } \sigma(k+1) < \ldots < \sigma(n).$$

To each such σ we assign the k-plane

$$V_\sigma := \mathrm{Span}_{\mathbb{C}}\{e_{\sigma(1)}, \ldots, e_{\sigma(k)}\}.$$

We have $\binom{n}{k}$ such elements of $Gr_k(\mathbb{C}^n)$, to each of them corresponding a Schubert variety. We first describe the Bruhat cell of V_σ. It is not hard to describe the elements of $B.V_\sigma$ and see (exercise) that the latter can be identified with $\mathbb{C}^{d(\sigma)}$, where

$$d(\sigma) = (\sigma(1) - 1) + (\sigma(2) - 2) + \ldots + (\sigma(k) - k).$$

We denote

$$E_j := \mathrm{Span}_{\mathbb{C}}\{e_1, \ldots, e_j\},$$

for $1 \leq j \leq n$. One can show (exercise) that the Bruhat cell of V_σ is[18]

$$\Omega_\sigma := B.V_\sigma = \{U \in Gr_k(\mathbb{C}^n) \ : \ \dim(U \cap E_{\sigma(j)}) = j, \text{ for all } 1 \leq j \leq k\}.$$

The corresponding Schubert variety is

$$\overline{\Omega}_\sigma = \{U \in Gr_k(\mathbb{C}^n) \ : \ \dim(U \cap E_{\sigma(j)}) \geq j, \text{ for all } 1 \leq j \leq k\}.$$

[17] It is worth noticing that it was actually the Grassmannian manifold that Hermann Schubert, the founder of what is nowadays called "Schubert calculus", investigated: he realized that problems in incidence geometry can be solved by counting "intersection numbers" for the subspaces of $Gr_k(\mathbb{C}^n)$ which are now called after his name. His works were published between 1886 and 1903 (more on the history can be found for instance in [Kl]).

[18] The traditional notation is Ω, and not X, as above.

We can justify this as follows: if $(U_r)_r$ is a sequence of k-planes with $\lim_{r\to\infty} U_r = U$ and $\dim(U_r \cap E) = m$ (same for all r), then we have $\lim_{r\to\infty}(U_r \cap E) \subset U \cap E$, thus

$$\dim(U \cap E) \geq m.$$

There is a more convenient presentation of the Bruhat cells and Schubert varieties, which we give in what follows. The change is purely formal: we replace the data

$$1 < \sigma(1) < \sigma(2) < \ldots < \sigma(k) < n$$

by

$$n - k \geq \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k \geq 0$$

where λ_i are given by

$$0 \leq (\sigma(1)-1 =: n-k-\lambda_1) \leq (\sigma(2)-2 =: n-k-\lambda_2) \leq \ldots \leq (\sigma(k)-k =: n-k-\lambda_k) \leq n-k.$$

The same data can be arranges in a diagram as follows: in a rectangle of size $k(\text{height}) \times n - k(\text{width})$ we choose λ_1 boxes in the first row, λ_2 boxes in the second one, ..., λ_k boxes in the last one; the region inside the rectangle is a <u>*Young diagram*</u> (see Figure 6). With this new convention, the Schubert variety is

$$\overline{\Omega}_\lambda = \{U \in Gr_k(\mathbb{C}^n) \ : \ \dim(U \cap E_{n-k-\lambda_j+j}) \geq j, \text{ for all } 1 \leq j \leq k\}. \tag{18}$$

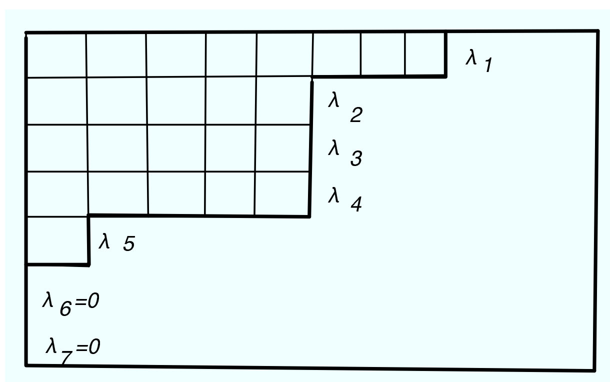

Figure 6. A Young diagram.

Let us record as follows:

Theorem 7.5. *Pick an arbitrary flag*

$$\{0\} \subset E_1 \subset \ldots \subset E_n = \mathbb{C}^n$$

where $\dim E_k = k$, $1 \leq k \leq n$. *Then the cohomology classes* $[\overline{\Omega}_\lambda]$, λ *a Young diagram fitting a* $k \times n - k$ *box, are a basis of* $H^*(G_k(\mathbb{C}^n); \mathbb{Z})$. *Here* $\overline{\Omega}_\lambda$ *is described by equation (18).*

It should be explained why did we start with an *arbitrary* flag $E_\bullet$, rather than the standard one, like in the previous discussion. The reason is that the cohomology class $[\overline{\Omega}_\lambda]$ does not depend on the choice of $E_\bullet$, even though the variety $\overline{\Omega}_\lambda$ does depend on it: if necessary, we will even use the notation $\overline{\Omega}_\lambda(E_\bullet)$.

As you know, the cohomology $H^*(Gr_k(\mathbb{C}^n); \mathbb{Z})$ has a ring structure (roughly, the product of two classes $[\overline{\Omega}_\lambda]$ and $[\overline{\Omega}_\mu]$ is "equal" to the class of the suvbariety $\overline{\Omega}_\lambda \cap \overline{\Omega}_\mu$). The main problem we will deal with in this context is:

The Littlewood-Richardson Problem. *Find combinatorial formulae for the structure constants of the ring $H^*(Gr_k(\mathbb{C}^n))$ with respect to the Schubert basis. In other words, determine the numbers $c_{\lambda\mu}^{\nu}$ from*

$$[\overline{\Omega}_\lambda][\overline{\Omega}_\mu] = \sum_\nu c_{\lambda\mu}^{\nu}[\overline{\Omega}_\nu].$$

Solutions to this problem are known and will be presented in the class lectures. The same problem can be posed in the general context of $G^{\mathbb{C}}/P$ (see Theorem 7.4): find the structure constants of the ring $H^*(G^{\mathbb{C}}/P)$ with respect to the Schubert basis. This question will also be addressed in the class lectures, probably mostly (only?) for the case of $G^{\mathbb{C}}/B$.

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