TORIC VARIETIES

1. Background material I. Affine varieties

Definition. An <u>*affine variety*</u> in $\mathbb{C}^n$ is the common zero locus of a finite set of polynomials $f_1, \ldots, f_p \in \mathbb{C}[X_1, \ldots, X_n]$. If I is the ideal of $\mathbb{C}[X_1, \ldots, X_n]$ generated by $f_1, \ldots, f_p$, then we denote

$$\mathcal{V}(I) := \{x \in \mathbb{C}^n \ : \ f(x) = 0, \forall f \in I\}.$$

There are two basic question which arise in this context: given two ideals $I_1, I_2 \subset \mathbb{C}[x_1, \ldots, x_n]$, decide whether the varieties are $\mathcal{V}(I_1)$ and $\mathcal{V}(I_2)$ are equal, respectively isomorphic (in an appropriate sense).

In order to answer the first question, we note that if we enlarge I to the ideal

$$\sqrt{I} := \{f \in \mathbb{C}[X_1, \ldots, X_n] \ : \ f^m \in I \text{ for some } m \in \mathbb{Z}, m \geq 0\},$$

then the zero locus doesn't change, i.e.

$$\mathcal{V}(I) = \mathcal{V}(\sqrt{I}).$$

Obviously $\sqrt{I_1} = \sqrt{I_2} \Rightarrow \mathcal{V}(I_1) = \mathcal{V}(I_2)$. The converse is also true, in view of *Hilbert's Nullstellensatz*: first, we attach to each variety $V \subset \mathbb{C}^n$ the ideal

$$\mathcal{I}(V) := \{f \in \mathbb{C}[x_1, \ldots, x_n] \ : \ f(x) = 0, \forall x \in V\};$$

then Hilbert's Nullstellensatz says that for any ideal $I \subset \mathbb{C}[X_1, \ldots, X_n]$ we have

$$\mathcal{I}(\mathcal{V}(I)) = \sqrt{I}.$$

In other words, algebraic varieties in $\mathbb{C}^n$ are in one-to-one correspondence with <u>*radical ideals*</u>, which are ideals I with the property $I = \sqrt{I}$.

Example. Consider the ideal $I = \langle X^2 Y \rangle$ of $\mathbb{C}[X, Y]$. Then $\sqrt{I} = \langle XY \rangle$.

To answer the second question, we define a <u>*morphism*</u> between two algebraic varieties $V_1 \subset \mathbb{C}^n$ and $V_2 \subset \mathbb{C}^m$ as a polynomial function $\varphi : \mathbb{C}^n \to \mathbb{C}^m$ such that $\varphi(V_1) \subset V_2$. We attach to each variety V the <u>*coordinate ring*</u>

$$\mathbb{C}[V] = \mathbb{C}[X_1, \ldots, X_n]/\mathcal{I}(V).$$

Every morphism $\varphi : V_1 \to V_2$ induces the $\mathbb{C}$-algebra homomorphism

$$\varphi^* : \mathbb{C}[V_2] \to \mathbb{C}[V_1], \quad \varphi^*([f]) = [f \circ \varphi].$$

Consequently, if V_1 and V_2 are isomorphic as varieties, then $\mathbb{C}[V_1]$ and $\mathbb{C}[V_2]$ are isomorphic as $\mathbb{C}$-algebras. The converse is also true, as the following proposition says.

Proposition 1.1. *(a) For any $\mathbb{C}$-algebra homomorphism $F : \mathbb{C}[V_2] \to \mathbb{C}[V_1]$ there is a unique variety homomorphism $\varphi : V_1 \to V_2$ such that $F = \varphi^*$.*

(b) Two affine varieties are isomorphic if and only if their coordinate rings $\mathbb{C}[V_1]$ and $\mathbb{C}[V_2]$ are isomorphic.

Proof. (a) We first prove the existence of φ. Consider the polynomials $\varphi_1, \ldots, \varphi_m \in \mathbb{C}[X_1, \ldots, X_n]$ such that the homomorphism

$$F : \mathbb{C}[Y_1, \ldots, Y_m]/\mathcal{I}(V_2) \to \mathbb{C}[V_1] = \mathbb{C}[X_1, \ldots, X_n]/\mathcal{I}(V_1)$$

has the form

$$F([f(Y_1, \ldots, Y_m)]) = [f(\varphi_1, \ldots, \varphi_m)].$$

Note that if $f(Y_1,\dots,Y_m)\in\mathcal{I}(V_2)$, then $f(\varphi_1,\dots,\varphi_m)\in\mathcal{I}(V_1)$. We claim that the polynomial function $\varphi:=(\varphi_1,\dots,\varphi_m):\mathbb{C}^n\to\mathbb{C}^m$ maps V_1 to V_2. Indeed, for any $g(Y_1,\dots,Y_m)\in\mathcal{I}(V_2)$ we have $g(\varphi_1,\dots,\varphi_m)\in\mathcal{I}(V_1)$. Consequently, if $x^0\in V_1$ then $(\varphi_1(x^0),\dots,\varphi_m(x^0))$ is in V_2, because it lies in the kernel of g, where g was chosen arbitrary in $\mathcal{I}(V_2)$. Finally, the equation $F=\varphi^*$ is obvious.

In order to prove the uniqueness, we assume that φ and ψ are both homomorphisms $V_1\to V_2$, such that $\varphi^*=\psi^*$, as maps $\mathbb{C}[V_2]=\mathbb{C}[Y_1,\dots,Y_m]/\mathcal{I}(V_2)\to\mathbb{C}[V_1]=\mathbb{C}[X_1,\dots,X_n]/\mathcal{I}(V_1)$. This implies that for any $f\in\mathbb{C}[Y_1,\dots,Y_m]$ we have $f\circ\varphi-f\circ\psi\in\mathcal{I}(V_1)$. Consequently, if we fix $x^0\in V_1$, then for any $f\in\mathbb{C}[Y_1,\dots,Y_m]$ we have $f(\varphi(x^0))=f(\psi(x^0))$. This implies $\varphi(x^0)=\psi(x^0)$.

(b) Because of the functoriality properties

$$\varphi^*\circ\psi^*=(\psi\circ\varphi)^*,\quad \mathrm{id}^*=\mathrm{id}$$

it is sufficient to show that if $\varphi:V\to V$ is a morphism such that $\varphi^*=\mathrm{id}$, then $\varphi=\mathrm{id}$. This follows from (a). $\square$

Example. The varieties defined by $xy=1$, respectively $x^2-y^2=1$ in $\mathbb{C}^2$ are isomorphic. Their coordinate rings are $\mathbb{C}[X,Y]/\langle XY-1\rangle$ and $\mathbb{C}[X,Y]/\langle X^2-Y^2-1\rangle$ are obviously isomorphic, e.g. via $x\mapsto X-Y, y\mapsto X+Y$.

Corollary 1.2. *There is a one-to-one correspondence between points in V and $\mathbb{C}$-algebra homomorphisms $\mathbb{C}[V]\to\mathbb{C}$.*

Proof. Let $x^0=(x_1^0,\dots,x_n^0)$ be a point of V. The affine variety in $\mathbb{C}^n$ described by all polynomials $\mathcal{I}(V)$ together with $X_1-x_1^0,\dots,X_n-x_n^0$ consists of the point x^0. Because the coordinate ring of the latter variety is

$$\mathbb{C}[X_1,\dots,X_n]/\langle\mathcal{I}(V),X_1-x_1^0,\dots,X_n-x_n^0\rangle=\mathbb{C},$$

the inclusion map $\{x^0\}\hookrightarrow V$ is a homomorphism of affine varieties, hence it induces a $\mathbb{C}$-algebra homomorphism[1] from $\mathbb{C}[V]$ to $\mathbb{C}$. Conversely, if $F:\mathbb{C}[V]\to\mathbb{C}$ is a $\mathbb{C}$-algebra homomorphism, we regard $\mathbb{C}$ as $\mathbb{C}[X_1,\dots,X_n]/\langle X_1,\dots,X_n\rangle$, namely the coordinate ring of the point 0 in $\mathbb{C}^n$, and we deduce from the previous proposition that $F=\varphi^*$, where $\varphi:\{0\}\to V$ is a morphism of varieties. The point induced by F is then $x^0:=\varphi(0)$. $\square$

This corollary gives a direct relationship between the coordinate ring $\mathbb{C}[V]$ and the affine variety V. Formally, we can express this by saying

$$V=\mathrm{Hom}_{\mathbb{C}-\mathrm{alg}}(\mathbb{C}[V],\mathbb{C}),\tag{1}$$

where $\mathrm{Hom}_{\mathbb{C}-\mathrm{alg}}$ stands for the set of all $\mathbb{C}$-algebra homomorphisms. In turn, the latter can be identified with the space $\mathrm{Specm}(\mathbb{C}[V])$ of all maximal ideals of $\mathbb{C}[V]$. This leads us to the abstract notion of *affine scheme* associated to an arbitrary commutative ring R. By definition, this is the space $\mathrm{Spec}(R)$ of all *prime* ideals of R. We will not need anything of this in our treatment of toric varieties. For more details one can consult for instance [Cox1, section 1] (especially eq. (1.1) and the references), as well as [Ga, chapter 5] or [Ha, chapter 2].

[1]Strictly speaking, this is just the natural map

$$\mathbb{C}[X_1,\dots,X_n]/\mathcal{I}(V)\to\mathbb{C}[X_1,\dots,X_n]/\langle\mathcal{I}(V),X_1-x_1^0,\dots,X_n-x_n^0\rangle$$

We will end up this section with an answer to the following question: among all $\mathbb{C}$-algebras characterize those which are coordinate rings of affine varieties. The answer is given in the following proposition (see [Su, Lecture 1, Proposition 1.1]).

Proposition 1.3. *A $\mathbb{C}$-algebra R is the coordinate ring of an affine variety if and only if R is a finitely generated algebra with no nonzero nilpotents (i.e. if $f \in R$ satisfies $f^m = 0$, then $f = 0$).*

Proof. First assume that $R = \mathbb{C}[X_1, \ldots, X_n]/\mathcal{I}(V)$ for some affine variety $V \subset \mathbb{C}^n$. If $f \in \mathbb{C}[X_1, \ldots, X_n]$ has the property that f^m is equal to 0 modulo $\mathcal{I}(V)$, then $f^m \in \mathcal{I}(V)$. Since $\mathcal{I}(V)$ is a radical ideal, we deduce that $f \in \mathcal{I}(V)$.

Conversely, if R is finitely generated, there exists a surjective homomorphism of $\mathbb{C}$-algebras $A : \mathbb{C}[X_1, \ldots, X_n] \to R$. One can prove that $I := \ker A$ is a radical ideal of $\mathbb{C}[X_1, \ldots, X_n]$. By Hilbert's Nullstellensatz, if we take the variety $V := \mathcal{V}(I)$, then $\mathcal{I}(V) = \sqrt{I}$, which is the same as I (since R has no nonzero nilpotents). Consequently, we can write R as $\mathbb{C}[X_1, \ldots, X_n]/I(V)$. $\square$

2. First examples of toric varieties

We will start by looking at a few examples of affine varieties. They will arise in the following naive way. We start with a finitely generated ring R without nonzero nilpotents (see Proposition 1.3) and construct the variety $V = \mathrm{Hom}_{\mathbb{C}-\mathrm{alg}}(R, \mathbb{C})$ whose coordinate ring is R by expressing

$$R = \mathbb{C}[Y_1, \ldots, Y_r]/I.$$

The variety we need is $V := \mathcal{V}(I) \subset \mathbb{C}^r$.

Examples. 1. Consider the ring $R = \mathbb{C}[X_1, X_1^{-1}, \ldots, X_n, X_n^{-1}]$ of all Laurent polynomials in the variables $X_1, \ldots, X_n$. We have the following presentation (in fact, ring isomorphism):

$$R \simeq \mathbb{C}[X_1, \ldots, X_n, X_{n+1}]/\langle X_1 \ldots X_{n+1} - 1\rangle,$$

where $\mathbb{C}[X_1, \ldots, X_{n+1}] \ni f(X_1, \ldots, X_{n+1}) \mapsto f(X_1, \ldots, X_n, X_1^{-1} \ldots X_n^{-1})$. We deduce that the corresponding variety is

$$\mathcal{V}(X_1 \ldots X_{n+1} - 1) \subset \mathbb{C}^{n+1}.$$

Note that there is a natural embedding of this variety into $\mathbb{C}^n$, given by

$$\mathcal{V}(X_1 \ldots X_{n+1} - 1) \ni (x_1, \ldots, x_n, x_{n+1}) \mapsto (x_1, \ldots, x_n).$$

The image of this embedding is actually $(\mathbb{C}^*)^n$. This is why[2] the variety $\mathcal{V}(I)$ is called the <u>*complex n-dimensional torus*</u>. We note that there are some other embeddings of the variety $\mathcal{V}(X_1 \ldots X_{n+1} - 1)$ into $\mathbb{C}^n$, whose image is also $(\mathbb{C}^*)^n$. For instance we have

$$\mathcal{V}(X_1 \ldots X_{n+1} - 1) \ni (x_1, \ldots, x_n, x_{n+1}) \mapsto (x_{i_1}^{\pm 1}, \ldots, x_{i_n}^{\pm 1}),$$

for a choice of the indeces $1 \leq i_1 < \ldots < i_n \leq n + 1$ and of the signs $\pm$ (see example 3 below).

Finally, it is useful to note that the ring R we started with can be expressed as

$$R = \mathbb{C}[S],$$

which means that R is the group ring of the semigroup S, where

$$S := \mathbb{Z}e_1 + \ldots + \mathbb{Z}e_n = \mathbb{Z}_{\geq 0}e_1 \oplus \mathbb{Z}_{\geq 0}(-e_1) \oplus \ldots \oplus \mathbb{Z}_{\geq 0}e_n \oplus \mathbb{Z}_{\geq 0}(-e_n).$$

[2] The point is that if we "complexify" the "real" torus $S^1 \times \ldots \times S^1$ as a Lie group, the result is just $\mathbb{C}^* \times \ldots \times \mathbb{C}^*$.

Here $e_1 = (1, 0 \ldots, 0), \ldots, e_n = (0, \ldots, 0, 1)$ is the standard basis in $\mathbb{R}^n$. Indeed, by definition, the ring $\mathbb{C}[S]$ consists of all elements χ^u, $u \in S$, equipped with the multiplication

$$\chi^u \cdot \chi^v := \chi^{u+v},$$

$u, v \in S$. We just set $\chi^{e_i} = X_i$, $1 \leq i \leq n$.

2. This time we start with the semigroup

$$S = \mathbb{Z}_{\geq 0} e_1 \oplus \ldots \oplus \mathbb{Z}_{\geq 0} e_k \oplus \mathbb{Z} e_{k+1} \oplus \ldots \oplus \mathbb{Z} e_n$$

and the corresponding group ring

$$R := \mathbb{C}[S] = \mathbb{C}[X_1, \ldots, X_k, X_{k+1}, X_{k+1}^{-1}, \ldots, X_n, X_n^{-1}].$$

In order to describe the affine variety V whose coordinate ring is this, we use again the presentation

$$\mathbb{C}[S] \simeq \mathbb{C}[X_1, \ldots, X_{n+1}]/\langle X_{k+1} \cdot \ldots \cdot X_{n+1} = 1\rangle$$

and obtain

$$V = \mathbb{C}^k \times (\mathbb{C}^*)^{n-k}.$$

Note that S can be described in terms of the cone

$$\sigma := \mathbb{R}_{\geq 0} e_1 \oplus \ldots \oplus \mathbb{R}_{\geq 0} e_k \subset \mathbb{R}^n$$

as follows: it is the intersection of the dual cone[3]

$$\sigma^\vee := \{v \in \mathbb{R}^n \ : \ \langle v, u\rangle \geq 0, \forall u \in \sigma\}$$

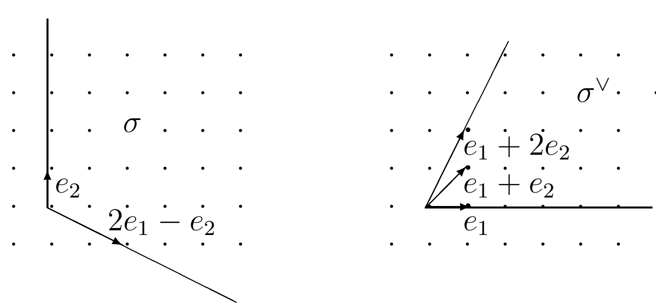
with the lattice $\mathbb{Z}^n$. In other words,

$$S = \sigma^\vee \cap \mathbb{Z}^n.$$

We write

$$U_\sigma = \mathbb{C}^k \times (\mathbb{C}^*)^{n-k}.$$

3. (see [Fu, section 1.1]) We consider the cone σ in $\mathbb{R}^2$ generated by e_2 and $2e_1 - e_2$.



The dual cone is

$$\sigma^\vee := \{v \in \mathbb{R}^2 \ : \ \langle v, u\rangle \geq 0, \forall u \in \sigma\}$$

A vector $v = v_1 e_1 + v_2 e_2$ leaves in $\sigma^\vee$ if and only if

$$\langle v_1 e_1 + v_2 e_2, 2y e_1 + (x - y) e_2\rangle = 2v_1 y + v_2(x - y) = v_2 x + (2v_1 - v_2) y \geq 0$$

[3] In this case we have $\sigma^\vee = \sigma$. But in general, this is not the case, as one can see in the next examples.

for any $x, y \geq 0$. This gives $v_2 \geq 0$ and $2v_1 - v_2 \geq 0$. Because

$$v_1e_1 + v_2e_2 = (2v_1 - v_2)\frac{1}{2}e_1 + v_2(\frac{1}{2}e_1 + e_2),$$

we deduce that $\sigma^\vee$ is the cone in $\mathbb{R}^2$ generated by e_1 and $e_1 + 2e_2$. The semigroup $\sigma^\vee \cap \mathbb{Z}^n$ is

$$S_\sigma := \sigma^\vee \cap \mathbb{Z}^n = \mathbb{Z}_{\geq 0}e_1 + \mathbb{Z}_{\geq 0}(e_1 + e_2) + \mathbb{Z}_{\geq 0}(e_1 + 2e_2).$$

In order to obtain the affine variety corresponding to the group ring $\mathbb{C}[S]$, we express the latter in terms of generators and relations. The generators are

$$U := \chi^{e_1}, V = \chi^{e_1+e_2}, W = \chi^{e_1+2e_2},$$

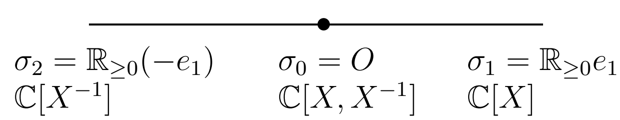
and the (obvious) relation is $V^2 = UW$. So

$$\mathbb{C}[S] = \mathbb{C}[U, V, W]/\langle V^2 - UW\rangle.$$

We deduce that the resulting affine variety, call it U_σ, is in $\mathbb{C}^3$, described by $v^2 = uw$.

The next examples are more complicated, in the sense that we will start from a collection *of cones, which is called a* fan, *then we will construct the varieties corresponding to each cone and we will glue those together in fairly natural way.*

4. (see [Fu, section 1.1]) A simple example of a fan is given in the figure from below.



We have a collection of three cones, σ_0, σ_1, and σ_2, with $\sigma_1 \cap \sigma_2 = \sigma_0$. To each of them we attach a variety by using the method exposed above. More precisely, we have $\sigma_1^\vee = \sigma_1$, so the semigroup S_{σ_1} is $\mathbb{Z}_{\geq 0}e_1$, hence $\mathbb{C}[S_{\sigma_1}]$ is the polynomial ring $\mathbb{C}[X]$, via

$$X = \chi^{e_1}.$$

If we use the latter convention, then $\mathbb{C}[S_{\sigma_2}] = \mathbb{C}[X^{-1}]$. Finally, $\sigma_0^\vee = \mathbb{R}$, hence the semigroup $S_{\sigma_0} := \sigma_0^\vee \cap \mathbb{Z}e_1 = \mathbb{Z}e_1$ is generated by e_1 and $-e_1$. Consequently, if we set $X_1 := \chi^{e_1}$, $X_2 := \chi^{-e_1}$, then we have

$$\mathbb{C}[S_{\sigma_0}] = \mathbb{C}[X_1, X_2]/\langle X_1X_2 - 1\rangle.$$

The corresponding varieties are

$$U_{\sigma_1} \simeq \mathbb{C}, \quad U_{\sigma_0} \simeq \mathbb{C}^*, \quad U_{\sigma_2} \simeq \mathbb{C}.$$

Now the obvious embedding $S_{\sigma_1} \hookrightarrow S_{\sigma_0}$ induces the natural inclusion

$$\mathbb{C}[X] = \mathbb{C}[S_{\sigma_1}] \hookrightarrow \mathbb{C}[S_{\sigma_0}] = \mathbb{C}[X_1, X_2]/\langle X_1X_2 - 1\rangle, \quad f(X) \mapsto f(X_1).$$

This gives the inclusion

$$\mathbb{C}^* = \mathcal{V}(X_1X_2 - 1) \hookrightarrow \mathbb{C}, \quad (x_1, x_2) \mapsto x_1.$$

Similarly, the embedding $S_{\sigma_2} \hookrightarrow S_{\sigma_0}$ induces

$$\mathbb{C}[X^{-1}] = \mathbb{C}[S_{\sigma_2}] \hookrightarrow \mathbb{C}[S_{\sigma_0}] = \mathbb{C}[X_1, X_2]/\langle X_1X_2 - 1\rangle, \quad f(X^{-1}) \mapsto f(X_2).$$

This gives the inclusion

$$\mathbb{C}^* = \mathcal{V}(X_1X_2 - 1) \hookrightarrow \mathbb{C}, \quad (x_1, x_2) \mapsto x_2 = x_1^{-1}.$$

We have obtained the following two embeddings of $\mathbb{C}^*$ into $\mathbb{C}$:

$$U_{\sigma_0} = \mathbb{C}^* \hookrightarrow U_{\sigma_1} = \mathbb{C} \text{ via } x \mapsto x,$$

and

$$U_{\sigma_0} = \mathbb{C}^* \hookrightarrow U_{\sigma_2} = \mathbb{C} \text{ via } x \mapsto x^{-1}.$$

We consider the space X obtained by gluing U_{σ_1} and U_{σ_2} along U_{σ_0}. This means

$$X = \mathbb{C}_1 \coprod \mathbb{C}_2 / \sim$$

where $\mathbb{C}_1$ and $\mathbb{C}_2$ are two different copies of $\mathbb{C}$ and the equivalence relation $\sim$ is given by

$$x \sim y \text{ if } x = y \text{ or } x \in \mathbb{C}_1, y \in \mathbb{C}_2 \text{ and } y = x^{-1}.$$

We note that the space X can be naturally identified with

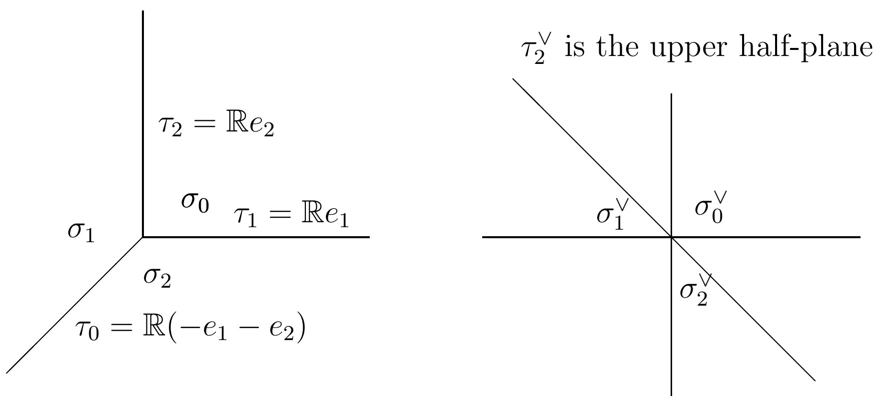
$$\mathbb{C}^2/\mathbb{C} := \{\mathbb{C}(z_1, z_2) \ : \ (z_1, z_2) \in \mathbb{C}^2\}.$$

More precisely, the identification is given by

$$\mathbb{C}_1 \ni x \mapsto \mathbb{C}(x, 1), \quad \mathbb{C}_2 \ni x \mapsto \mathbb{C}(1, x).$$

The map is well-defined, because $\mathbb{C}(1, x) = \mathbb{C}(x^{-1}, 1)$. Finally, note that $\mathbb{C}^2/\mathbb{C}$ represents the space of all complex lines in $\mathbb{C}^2$, which is just the complex projective space $\mathbb{P}^1$.

5. We consider the fan from below, consisting of the cones $\sigma_0, \sigma_1, \sigma_2$.



It is obvious that $\sigma_0^\vee = \sigma_0$. The cone σ_1 is generated by e_2 and $-e_1 - e_2$. A vector $v = v_1 e_1 + v_2 e_2$ leaves in $\sigma_1^\vee$ if and only if

$$\langle v_1 e_1 + v_2 e_2, (x - y)e_2 - y e_1 \rangle = v_2(x - y) - v_1 y = v_2 x - (v_1 + v_2) y \geq 0$$

for all $x, y \geq 0$. This gives $v_2 \geq 0$ and $v_1 + v_2 \leq 0$. Since

$$v_1 e_1 + v_2 e_2 = v_2(e_2 - e_1) - (v_1 + v_2)(-e_1),$$

we deduce that $\sigma_1^\vee$ is generated by $-e_1 + e_2$ and $-e_1$. Similarly (in fact, interchanging e_1 and e_2), we deduce that $\sigma_2^\vee$ is generated by $e_1 - e_2$ and $-e_2$.

The semigroups S_{σ_0}, S_{σ_1} and S_{σ_2} (i.e. the intersections of $\sigma_0^\vee$, $\sigma_1^\vee$ respectively $\sigma_2^\vee$ with $\mathbb{Z}^2$) are generated by $\{e_1, e_2\}$, $\{-e_1, -e_1 + e_2\}$ and $\{-e_2, -e_2 + e_1\}$. Set

$$X_1 = \chi^{e_1}, X_2 = \chi^{e_2}.$$

Then

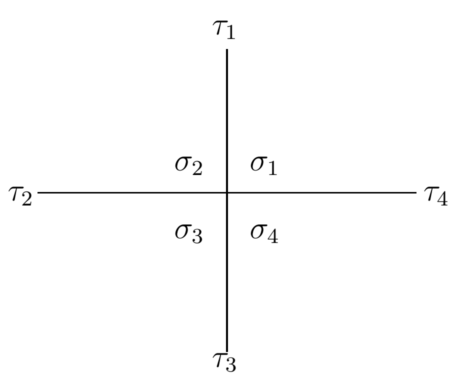
$$\mathbb{C}[S_{\sigma_0}] = \mathbb{C}[X_1, X_2], \quad \mathbb{C}[S_{\sigma_1}] = \mathbb{C}[X_1^{-1}, X_1^{-1} X_2], \quad \mathbb{C}[S_{\sigma_2}] = \mathbb{C}[X_1 X_2^{-1}, X_2^{-1}].$$

Each of the corresponding affine varieties is $\mathbb{C}^2$. Now we look at the affine variety corresponding to $\tau_2 = \mathbb{R}e_2$. The semigroup S_{τ_2} is generated by e_1, e_2, and $-e_1$. Set

$$X_3 = \chi^{-e_1}, X_4 = \chi^{-e_2}.$$

We have
$$\mathbb{C}[S_{\tau_2}] = \mathbb{C}[X_1, X_2, X_3]/\langle X_1X_3 - 1\rangle.$$
Similarly,
$$\mathbb{C}[S_{\tau_1}] = \mathbb{C}[X_1, X_2, X_4]/\langle X_1X_4 - 1\rangle$$
and
$$\mathbb{C}[S_{\tau_0}] = \mathbb{C}[X_1^{-1}, X_1^{-1}X_2, X_1X_2^{-1}]/\langle (X_1^{-1}X_2)(X_1X_2^{-1}) - 1\rangle.$$
Consider now the inclusion
$$\mathbb{C}[S_{\sigma_0}] \hookrightarrow \mathbb{C}[S_{\tau_2}], \quad f(X_1, X_2) \mapsto f(X_1, X_2, X_3).$$
This induces the inclusion
$$U_{\tau_2} = \mathcal{V}(X_1X_3 - 1) = \mathbb{C} \times \mathbb{C}^* \hookrightarrow \mathbb{C}^2 = U_{\sigma_0}, \quad (x_1, x_2, x_3) \mapsto (x_1, x_2).$$
Similarly, we obtain the inclusion
$$U_{\tau_2} = \mathcal{V}(X_1X_3 - 1) = \mathbb{C} \times \mathbb{C}^* \hookrightarrow \mathbb{C}^2 = U_{\sigma_1}, \quad (x_1, x_2, x_3) \mapsto (x_1^{-1}, x_1^{-1}x_2).$$
The other obvious inclusions can be described in a similar way. The induced gluings lead to the space
$$X = ((\mathbb{C}^2)_1 \coprod (\mathbb{C}^2)_2 \coprod (\mathbb{C}^2)_3)/\sim,$$
where $(\mathbb{C}^2)_1, (\mathbb{C}^2)_2, (\mathbb{C}^2)_3)$ are three different copies of $\mathbb{C}^2$ and the equivalence relation $\sim$ is defined as follows:
$$\begin{aligned}
(\mathbb{C}^* \times \mathbb{C})_1 \ni (x_1, x_2) &\sim (x_1^{-1}, x_1^{-1}x_2) \in (\mathbb{C}^* \times \mathbb{C})_2\\
(\mathbb{C} \times \mathbb{C}^*)_1 \ni (x_1, x_2) &\sim (x_1x_2^{-1}, x_2^{-1}) \in (\mathbb{C} \times \mathbb{C}^*)_3\\
(\mathbb{C} \times \mathbb{C}^*)_2 \ni (x_1, x_2) &\sim (x_2^{-1}, x_1x_2^{-1}) \in (\mathbb{C}^* \times \mathbb{C})_3.
\end{aligned}$$
Like in the previous example, there is a natural identification between X and the space
$$\mathbb{C}^3 \setminus \{0\}/\mathbb{C} = \{\mathbb{C}(z_1, z_2, z_3) \ : \ (z_1, z_2, z_3) \in \mathbb{C}^3 \setminus \{0\}\}.$$
This is given by
$$(\mathbb{C}^2)_1 \ni (x_1, x_2) \mapsto \mathbb{C}(x_1, x_2, 1), (\mathbb{C}^2)_2 \ni (x_1, x_2) \mapsto \mathbb{C}(1, x_2, x_1), (\mathbb{C}^2)_3 \ni (x_1, x_2) \mapsto \mathbb{C}(x_1, 1, x_2).$$
Finally we note that $\mathbb{C}^3 \setminus \{0\}/\mathbb{C}$ represents the space of all complex lines in $\mathbb{C}^3$, which is the complex projective plane $\mathbb{P}^2$.

6. We consider the fan in the figure from below.



We have $\sigma_i^\vee = \sigma_i$, $i = 1, 2, 3, 4$. If we set
$$X_1 = \chi^{e_1}, \quad X_2 = \chi^{e_2}, \quad X_3 = \chi^{-e_1}, \quad X_4 = \chi^{-e_2},$$

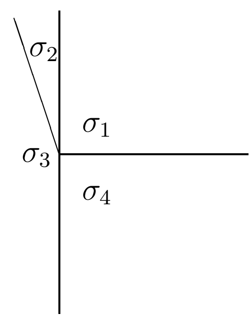
then

$$\mathbb{C}[S_{\sigma_1}] = \mathbb{C}[X_1, X_2], \quad \mathbb{C}[S_{\sigma_2}] = \mathbb{C}[X_2, X_3], \quad \mathbb{C}[S_{\sigma_3}] = \mathbb{C}[X_3, X_4], \quad \mathbb{C}[S_{\sigma_4}] = \mathbb{C}[X_4, X_1],$$

$$\mathbb{C}[S_{\tau_1}] = \mathbb{C}[X_1, X_2, X_3], \mathbb{C}[S_{\tau_2}] = \mathbb{C}[X_2, X_3, X_4], \mathbb{C}[S_{\tau_3}] = \mathbb{C}[X_3, X_4, X_1], \mathbb{C}[S_{\tau_4}] = \mathbb{C}[X_4, X_1, X_2].$$

This shows that all U_{σ_i} are isomorphic to $\mathbb{C}^2$ and all U_{τ_i} are isomorphic to $\mathbb{C}^* \times \mathbb{C}$. By gluing U_{σ_1} and U_{σ_2} we obtain $\mathbb{P}^1 \times \mathbb{C}$ (where the coordinate on $\mathbb{C}$ is X_2), and by gluing U_{σ_3} and U_{σ_4} we obtain again $\mathbb{P}^1 \times \mathbb{C}$ (this time the coordinate on $\mathbb{C}$ is X_4). The remaining two gluings — of U_{σ_1} and U_{σ_4}, respectively U_{σ_2} and U_{σ_3}) — are equivalent to the gluing of the two copies of $\mathbb{C}$ from above, which gives another $\mathbb{P}^1$. Consequently, the space we obtain after all gluings is $\mathbb{P}^1 \times \mathbb{P}^1$.

A slightly more complicated space is obtained from the following fan.



Here σ_2 is determined by $(0, 1)$ and $(-1, a)$, where a is a positive integer. The resulting space is called the *Hirzebruch surface*, usually denoted Σ_a, or $\mathbb{F}_a$. This turns out to be a smooth projective manifold, which has a natural structure of a $\mathbb{P}^1$ bundle over $\mathbb{P}^1$ (for more details, see [Fu, section 1.1] or [Gu, Example 2.2]).

3. Background material II: Gluing algebraic varieties

The gluing process we made use of in the previous section is a common procedure in the context of algebraic varieties. The goal of this section is to provide a few basic things needed to understand this contruction. It is worth mentioning in advance (see the upcoming section **??**) that a toric variety is in general obtained by gluing (affine) algebraic varieties, and the result will be an algebraic variety.

The main reference for this section are A. Gathmann's notes [Ga].

We return to the general setup from section 1. Let $V = \mathcal{V}(I) \subset \mathbb{C}^n$ be an affine variety. On V we can define the *Zariski topology*. By definition, the *closed* spaces in this topology are exactly the affine varieties contained in V. The sets

$$V_f := \{x \in V \ : \ f(x) \neq 0\}$$

where $f \in \mathbb{C}[X_1, \ldots, X_n]$ are called *principal open sets* and they generate the Zariski topology.

Remarks about the Zariski topology. 1. Any two nonempty open sets have a nonempty intersection (in particular, the Zariski topology is not Hausdorff). To see that, just take two open sets of the form $\mathbb{C}^n_f$ and $\mathbb{C}^n_g$ in $\mathbb{C}^n$.

2. Any open subset in V is dense. In order to understand the idea of the proof, let us just show that $\mathbb{C}^n_f$ is dense in $\mathbb{C}^n$. Take $x \in \mathbb{C}^n$ with $f(x) = 0$ and U an open set in $\mathbb{C}^n$ which contains x. The subspace U is open in the usual topology as well, because the latter is finer than the Zariski topology. Consequently, U has intersection points with $\mathbb{C}^n_f$, because otherwise f would vanish on U, hence f would be identically 0.

Definition. 1. We say that V is an *irreducible* variety if we cannot write $V = V_1 \cup V_2$, where V_1 and V_2 are closed subsets of V, non-empty and different from X.

2. We say that V is *connected* if we cannot write $V = V_1 \cup V_2$, where V_1 and V_2 are closed subsets of V, $V_1 \cap V_2 = \phi$, V_1 and V_2 different from X

Remark. If V is irreducible, then V is connected, but not conversely. For example, $V = \mathcal{V}(X_1X_2)$ in $\mathbb{C}^2$ is not irreducible, but it is connected. Indeed, the former assertion is clear; to prove the latter one, we just need to note that the Zariski topology is coarser than the usual topology, and V (the union of $\mathcal{V}(X_1)$ and $\mathcal{V}(X_2)$) is connected in the usual topology.

We have the following characterization of irreducible affine varieties.

Proposition 3.1. *Let $V \subset \mathbb{C}^n$ be an algebraic variety. The following assertions are equivalent.*

(i) *V is irreducible.*
(ii) *$\mathcal{I}(V) \subset \mathbb{C}[X_1, \dots, X_n]$ is a prime ideal*[4]*.*
(iii) *the coordinate ring $\mathbb{C}[V]$ is an (integral) domain.*

Proof. For the proof of the equivalence (i)⇔(ii) one can see for instance [Ga, Lemma 1.3.4]. The equivalence (ii)⇔(iii) is obvious. □

Examples. 1. The variety $\mathcal{V}(X^2 - Y^2) \subset \mathbb{C}^2$ is obviously not irreducible.

2. Consider the variety $V = \mathcal{V}(XY - ZV) \subset \mathbb{C}^4$. The polynomial $XY - ZV$ is prime, consequently the ideal $\langle XY - ZV \rangle$ is a radical ideal. This implies $\mathcal{I}(V) = \langle XY - ZV \rangle$. Again because the polynomial $XY - ZV$ is prime, we deduce that $\mathcal{I}(V)$ is a prime ideal, hence V is an irreducible variety.

Remark. One can show that any affine variety in $\mathbb{C}^n$ is a disjoint union of finitely many irreducible varieties (see [Ga, Proposition 1.3.8]).

Our next goal is to define the structure sheaf of an affine variety. First, we need a definition.

Definition. Let U be an open subspace of the affine variety V. A function $\varphi : U \to \mathbb{C}$ is called *regular* if for any $x_0 \in U$ there exists an open neighbourhood U_{x_0} of x_0 in U and two polynomials $f, g \in \mathbb{C}[X_1, \dots, X_n]$ such that for any $x \in U_{x_0}$ we have $g(x) \neq 0$ and $\varphi(x) = \frac{f(x)}{g(x)}$.

Example. Take $V = \mathcal{V}(X_1X_2 - X_3X_4) \subset \mathbb{C}^4$ and $U = V_{x_2} \cup V_{x_3}$. The function $\varphi : U \to \mathbb{C}$ given by

$$\varphi(x) = \begin{cases} \frac{x_1}{x_3}, & \text{if } x_3 \neq 0 \\ \frac{x_4}{x_2}, & \text{if } x_2 \neq 0 \end{cases}$$

is regular.

We denote by

$$\mathcal{O}_V(U) := \{\varphi : U \to \mathbb{C} \ : \ \varphi \text{ is regular}\}.$$

[4]By definition, this means that if $f_1, f_2 \in \mathbb{C}[X_1, \dots, X_n]$ have the property $f_1f_2 \in \mathcal{I}(V)$, then f_1 or f_2 are in $\mathcal{I}(V)$

This space has an obvious ring structure. It is called the <u>*ring of regular functions*</u> on U. We note that the assignment $\{U \subset V \ : \ U \text{ is open}\} \to \{\text{rings}\}$, $U \mapsto \mathcal{O}_V(U)$ is a <u>*sheaf of functions*</u>[5] on V, in the sense that it (obviously) satisfies the following properties.

S1. For any two open subspaces U_1, U_2 with $U_1 \subset U_2$, the set-theoretic restriction map $\varphi \mapsto \varphi|_{U_1}$ maps $\mathcal{O}_V(U_2)$ to $\mathcal{O}_V(U_1)$.
S2. If $U \subset V$ is an open subspace, $\{U_i\}_{i \in I}$ an open cover of U and $\varphi_i \in \mathcal{O}_V(U_i)$, $i \in I$, such that $\varphi_i|_{U_i \cap U_j} = \varphi_j|_{U_i \cap U_j}$ for all $i, j \in I$, then there exists a unique $\varphi \in \mathcal{O}_V(U)$ such that $\varphi|_{U_i} = \varphi_i$, for all $i \in I$.

In general, a topological space X with a sheaf $\mathcal{F}$ is called a <u>*ringed space*</u>. There exists a naturally defined notion of <u>*morphism of ringed space*</u>, which is a map satisfying the two conditions mentioned below in Proposition 3.2 (a). We can talk about the category of ringed spaces (for the details, see [Ga, Definition 2.3.1]).

The following result shows that an affine variety is determined uniquely by its structure sheaf.

Proposition 3.2. *(a) A map $f : V_1 \to V_2$ between the affine varieties V_1 and V_2 is a homomorphism in the sense defined in section 1 if and only if*

- *f is continuous, and*
- *for any open subset $U \subset V_2$ and any $\varphi \in \mathcal{O}_{V_2}(U)$, the function $f \circ \varphi : \varphi^{-1}(U) \to \mathbb{C}$ is in $\mathcal{O}_{V_1}(\varphi^{-1}(U))$.*

(b) Consequently, two affine varieties V_1 and V_2 are isomorphic in the sense defined in section 1 if and only if the pairs $(V_1, \mathcal{O}_{V_1})$ and $(V_2, \mathcal{O}_{V_2})$ are isomorphic as ringed spaces.

Proof. See [Ga, Lemma 2.3.7]. □

We are ready to define algebraic varieties.

Definition. A <u>*prevariety*</u> is a ringed space $(X, \mathcal{O}_X)$ such that X has an open cover $\{V_i\}_{i \in I}$ with the property that for any $i \in I$, the pair $(V_i, \mathcal{O}_X|_{V_i})$ is isomorphic as a ringed space to an affine variety.

Alternatively, we can say that a prevariety is constructed by <u>*gluing affine varieties*</u>. More precisely, it arises from a collection $(\{V_i\}_{i \in I}, \{U_{ij}\}_{i,j \in I}, \{g_{ij}\}_{i,j \in I})$, where V_i is an affine variety, $U_{ij} \subset V_i$ is open and $g_{ij} : (U_{ij}, \mathcal{O}_{V_i}|_{U_{ij}}) \to (U_{ji}, \mathcal{O}_{V_j}|_{U_{ji}})$ is an isomorphism with the properties that

- $g_{ii} = \mathrm{id}_{V_i}$,
- $g_{jk}|_{U_{jk} \cap U_{ji}} \circ g_{ij}|_{U_{ik} \cap U_{ij}} = g_{ik}|_{U_{ik} \cap U_{ij}}$ for all i, j, k

as follows. We set

$$X = \coprod_{i \in I} V_i / \sim, \tag{2}$$

where

$$V_i \ni x \sim y \in V_j \text{ if } x \in U_{ij} \text{ and } y = g_{ij}(x).$$

We equip X with the quotient topology and we patch the sheaves $\mathcal{O}_{V_i}$ to produce a sheaf $\mathcal{O}_X$ (for the details, see [Ga, section 2.4]).

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[5] There exists many examples of sheaves in mathematics: continuous functions on a topological space, smooth functions on a manifold, sections of a vector bundle (this is, by the way, not a sheaf *of functions*, but rather a sheaf *of sections*) etc.

The next goal is to define the notion of variety. To this end we have to define first the notion of *product* of two varieties. First, if $V = \mathcal{V}(I) \subset \mathbb{C}^n$ and $W = \mathcal{V}(J) \subset \mathbb{C}^m$ are two affine varieties, then we can see that the set theoretic cartesian product $V \times W$ is the zero locus in $\mathbb{C}^n \times \mathbb{C}^m = \mathbb{C}^{n+m}$ of the polynomials $f(X_1, \dots, X_n) + g(Y_1, \dots, Y_m)$, where $f \in I$ and $g \in J$. In other words, the product of the varieties V and W is

$$V \times W = \mathcal{V}(\langle I + J \rangle).$$

Note that the Zariski topology on $V \times W$ is *not* the product of the Zariski topologies on V and W. Now assume that X and Y are two prevarieties, obtained from the gluing of the affine varieties $\{V_i\}_{i \in I}$, respectively $\{W_\alpha\}_{\alpha \in A}$. There is a natural way of gluing the affine varieties $\{V_i \times W_\alpha\}_{(i,\alpha) \in I \times A}$. Set theoretical, the resulting space is the cartesian product $X \times Y$. The latter will come equipped with a structure of a prevariety.

Definition. A prevariety X is called a <u>*variety*</u> if the diagonal map

$$\Delta : X \mapsto X \times X, \quad x \mapsto (x, x),$$

is a closed map[6].

Remark. Any affine variety is a variety. To prove this, take $V = \mathcal{V}(I) \subset \mathbb{C}^n$. The image of Δ in $V \times V \subset \mathbb{C}^n \times \mathbb{C}^n$ is the joint zero locus of the polynomials

$$f(X_1, \dots, X_n), \text{ for } f \in I, X_1 - Y_1, \dots, X_n - Y_n.$$

The question is now, when is the prevariety X described by (2) a variety? The answer is given by the following proposition.

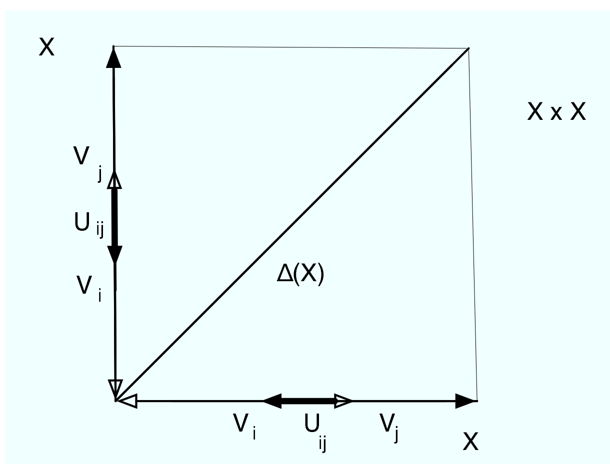


FIGURE 1. The product $X \times X$ is represented as a square.

Proposition 3.3. *If for any $i, j \in I$, the diagonal embedding*[7]

$$U_{ij} \hookrightarrow V_i \times V_j, \quad x \mapsto (x, x)$$

is closed[8]*, then the prevariety X given by (2) is actually a variety.*

[6]By this we just mean that the image of Δ is closed in $X \times X$ is a closed subset.

[7]Strictly speaking, this is the map $(x, x) \mapsto (x, g_{ij}(x))$; but we assume that U_{ij} and U_{ji} have been identified.

[8]It may be important to note that $V_i \times V_j$ is the product of the affine varieties V_i and V_j, in the sense defined above.

Proof. We need to show that the space

$$(X \times X) \setminus \Delta(X) = \bigcup (V_i \times V_j) \setminus \Delta(U_{ij})$$

is open. But a union of open spaces is always open, which finishes the proof. $\square$

Now let us try to understand the gluings in examples 4 and 5 from the previous section by using the formalism just exposed. In example 4, we have the affine varieties $U_{\sigma_0} = \mathbb{C}_1$ and $U_{\sigma_1} = \mathbb{C}_2$, which are two copies of $\mathbb{C}$. Inside each of them we have $\mathbb{C}^*$, which is (Zariski) open – prove this!. More precisely, $\mathbb{C}_1^* \subset \mathbb{C}_1$ and $\mathbb{C}_2^* \subset \mathbb{C}_2$. We consider the map

$$g_{12} : \mathbb{C}_1^* \to \mathbb{C}_2^*, \quad x \mapsto x^{-1},$$

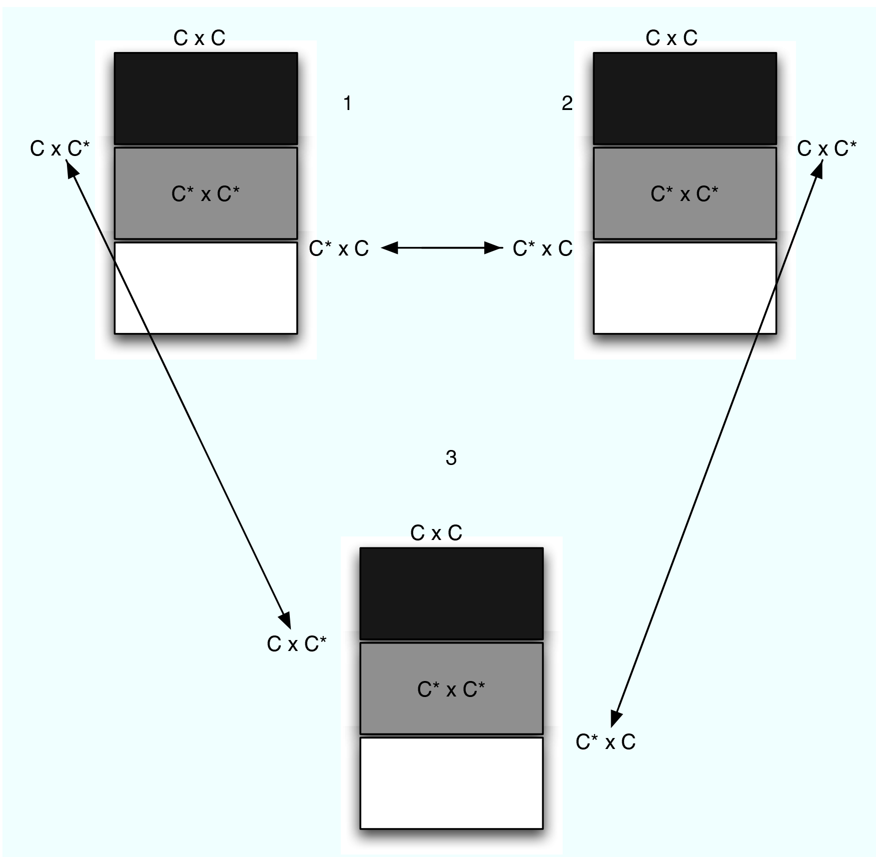
which is an isomorphism of varieties – prove this, too! The resulting space X is a prevariety. Now let us show that it is a variety, by applying Proposition 3.3. We need to show that the image of the map

$$\mathbb{C}^* \to \mathbb{C} \times \mathbb{C}, \quad x \mapsto (x, x^{-1})$$

is closed in $\mathbb{C} \times \mathbb{C}$ with respect to the direct product of the two Zariski topologies. This is true because the map

$$\mathbb{C} \times \mathbb{C} \to \mathbb{C}, \quad (x_1, x_2) \mapsto x_1 x_2$$

is continuous. Since the subspace $\{1\} \subset \mathbb{C}$ is closed, its preimage in $\mathbb{C} \times \mathbb{C}$ is closed as well.



As about example 5, let us just check that the maps

$$g_{12}(x_1, x_2) := (x_1^{-1}, x_1^{-1} x_2)$$
$$g_{13}(x_1, x_2) := (x_1 x_2^{-1}, x_2^{-1})$$
$$g_{23}(x_1, x_2) := (x_2^{-1}, x_1 x_2^{-1})$$

satisfy the compatibility condition $g_{12} = g_{13} \circ g_{23}$. This is an easy exercise. Also see the figure from above.

4. Convex polyhedral cones

The objects mentioned in the title are discussed in this section. We will be following [Fu, section 1.2] and Lecture 1 of [Su] (written by D. Cox).

Let V be a finite dimensional vector space. A *convex polyhedral cone* in V is a set

$$\sigma = \mathrm{Cone}(S) = \{\sum_{i=1}^{n} r_i v_i \ : \ r_i \geq 0\} \subset V,$$

where $S = \{v_1, \dots, v_s\}$ is a finite subset of V. We say that S generates σ. The *dimension* of σ is the dimension of the vector space $\mathbb{R}\sigma$. Let $V^* = \mathrm{Hom}(V, \mathbb{R})$ be the dual vector space and consider the evaluation pairing

$$V^* \times V \to \mathbb{R}, \quad (u, v) \mapsto \langle u, v \rangle := u(v).$$

For each $u \in V^*$ we consider the hyperplane $H_u := \ker u$ and the closed half-space

$$H_u^+ := \{v \in V \ : \ \langle u, v \rangle \geq 0\}.$$

Definition. If $u \in V^*$ such that $\sigma \subset H_u^+$, we say that $\sigma \cap H_u$ is a *face* of σ.

The biggest face (in the sense of inclusion) of σ is σ itself. The smallest face is $\sigma \cap (-\sigma)$. We summarize a few results about faces in the following lemma.

Lemma 4.1. *If σ is a convex polyhedral cone, then we have:*

a) *every face of σ is a convex polyhedral cone,*
b) *an intersection of two faces of σ is also a face of σ,*
c) *a face of a face of σ is also a face of σ.*

For a proof, see [Su, lecture 1, Lemma 3.2]. A face τ such that $\dim \tau = \dim \sigma - 1$ is called a *facet*. Let us just record the following result (which is property (6), page 10 of [Fu] or Lemma 3.3 in [Su, Lecture 1]).

Lemma 4.2. *If σ is a convex polyhedral cone, then any face is the intersection of all facets which contain it.*

If $\sigma \subset V$ is a convex polyhedral cone, we consider the set

$$\sigma^\vee := \{u \in V^* \mid \langle u, v \rangle \geq 0, \forall v \in \sigma\},$$

which is called the *dual* of σ. This is obviously a cone in V^*. The next theorem says that it is actually a convex polyhedral one.

Theorem 4.3. (Farkas' Theorem) *If σ is a convex polyhedral cone, then so is its dual $\sigma^\vee$.*

Sketch of the proof (for more details, see the references mentioned at the beginning of the section).

Step 1. (Duality Theorem) For any closed convex cone $\sigma \in V$ we have $(\sigma^\vee)^\vee = \sigma$.

Step 2. Assume that $\sigma \neq V$ and let the facets of σ be $\tau_i = H_{u_i} \cap \sigma$, for $u_i \in V^*$ such that $\sigma \subset H_{u_i}^+$, $1 \leq i \leq s$. Then one shows that

$$\sigma = \bigcap_{i=1}^{s} H_{u_i}^+.$$

In other words, σ is an intersection of closed half-spaces[9].

Step 3. The set $\mathrm{Cone}(u_1, \ldots, u_s)$ is a convex polyhedral cone in V^* and one shows that

$$\mathrm{Cone}(u_1, \ldots, u_s)^\vee = \bigcap_{i=1}^{s} H_{u_i}^+ = \sigma.$$

Final Step. From the duality theorem we deduce that

$$\sigma^\vee = \mathrm{Cone}(u_1, \ldots, u_s), \tag{3}$$

which is a convex polyhedral cone. □

It is important to characterize the faces of $\sigma^\vee$ in terms of the faces of σ. This is described in the following proposition. First for any $\tau \subset V$ we denote

$$\tau^\perp := \{u \in V^* \ : \ \langle u, v \rangle = 0, \forall v \in \tau\}.$$

Proposition 4.4. *Let $\sigma \subset V$ be a convex polyhedral cone.*

(i) If τ is a face of σ, then $\sigma^\vee \cap \tau^\perp$ is a face of $\sigma^\vee$.

(ii) The map $\tau \mapsto \sigma^\vee \cap \tau^\perp$ is a bijective, inclusion reversing, correspondence between the faces of σ and the faces of $\sigma^\vee$.

(iii) $\dim \tau + \dim(\sigma^\vee \cap \tau^\perp) = \dim V$.

$\boxed{\text{From here on we make the assumption that } V = N \otimes_{\mathbb{Z}} \mathbb{R}, \text{ where } N \text{ is a lattice.}}$

This means that N is a free abelian group of finite rank, i.e. isomorphic to $\mathbb{Z}^n$. If $M := \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$ is the dual lattice, then $V^* = M \otimes_{\mathbb{Z}} \mathbb{R}$. Alternatively, one can say that M consists of all $u \in V^*$ with the property that $u(N) \subset \mathbb{Z}$. The pairing $\langle\ ,\ \rangle$ from above is the $\mathbb{R}$-bilinear extension of the evaluation pairing $M \times N \to \mathbb{Z}$.

Definition. A convex polyhedral cone $\sigma \subset V$ is *rational* if $\sigma = \mathrm{Cone}(S)$, where S is a finite subset of N.

Lemma 4.5. *If $\sigma \subset V$ is a rational polyhedral cone, then we have:*

a) *any face of σ is rational,*
b) *the cone $\sigma^\vee \subset V$ is rational.*

Proof. a) If $\sigma = \mathrm{Cone}(S)$ and τ is a face of σ, then $\tau = \mathrm{Cone}(S \cap \tau)$.

b) First assume that $\dim \sigma = \dim V =: n$. By equation (3), we have to show that any facet of σ can be written as $H_u \cap \sigma$, where $u \in M$ (i.e. $u(N) \subset \mathbb{Z}$). Indeed, a facet is of the form $\mathrm{Cone}(S')$, where $S' \subset S \subset N$ has the property that $H := \mathrm{Span}(S')$ is a hyperplane in V. There exists $v \in N \setminus H$. Choose $u \in M$ such that $u|_H$ is identically 0 and $\langle u, v \rangle = 1$. Then u takes integer values on n linearly independent vectors in N, thus $u(N) \subset \mathbb{Z}$. Finally, we have $H = \mathrm{Span}(S') = H_u$, hence the facet is $H \cap \sigma = H_u \cap \sigma$.

Consider now the general case. Take $W := \mathrm{Span}(\sigma)$, which is a proper vector subspace of V. The set $N_W := N \cap W$ is a free abelian group of finite rank (because it is finitely generated, namely by S), and we have $N_W \otimes_{\mathbb{Z}} \mathbb{R} = W$. From the previous paragraph, we know that the dual of σ in W, call it $\sigma_W^\vee$, is a *rational* polyhedral cone in W^*. If $r : V^* \to W^*$ is the restriction map, then $\sigma_W^\vee = \mathrm{Cone}(S')$, where $S' \subset r(M) \subset W^*$ is finite. Assume that S'' is a subset of M which is mapped bijectively onto S' by r. One can easily see that

[9] This gives the following alternative definition: a convex polyhedral cone is an intersection of closed half-spaces.

$\sigma^\vee = r^{-1}(\sigma_W^\vee)$. This cone is obviously generated by $S'' \cup r^{-1}(0)$. Next, we can show that $\mathrm{Span}(r^{-1}(0) \cap M) = r^{-1}(0)$: the idea is that if $W \subset \mathbb{R}^n$ is a k-dimensional vector subspace which has a basis consisting of elements in $\mathbb{Z}^n$, then the orthogonal subspace $W^\perp$ also has a basis of that type[10]. We deduce that $\sigma^\vee$ is generated as a cone by $S'' \cup (r^{-1}(0) \cap M)$. It only remains to note that $r^{-1}(0) \cap M$ is a finitely generated group (because $r(M)$ is finitely generated, namely by S', and $r|_M : M \to r(M)$ is a group homomorphism). □

We will be especially interested in the lattice points in $\sigma^\vee$, namely $\sigma^\vee \cap M$, which is a semigroup.

Proposition 4.6. (Gordan's Lemma) *If σ is a rational convex polyhedral cone, then the semigroup $S_\sigma := \sigma^\vee \cap M$ is finitely generated.*

Proof. By Lemma 4.5, we know that there exists $u_1, \ldots, u_s \in \sigma^\vee \cap M$ which generate $\sigma^\vee$ as a cone. Take $K := \{\sum t_i u_i \ : \ 0 \le t_i \le 1\}$, which is compact. Cosequently the intersection $K \cap M$ is finite. We show that $K \cap M$ generates $\sigma^\vee \cap M$. Indeed, if $u \in \sigma^\vee \cap M$, we can write it as $u = \sum r_i u_i$, where $r_i \ge 0$. Write $r_i = m_i + t_i$, where $m_i \in \mathbb{Z}_{\ge 0}$ and $0 \le t_i < 1$. This gives

$$u = \sum m_i u_i + \sum t_i u_i. \tag{4}$$

Both u and the first sum are in M, hence the second sum is in M. This means that it is actually in $K \cap M$. We deduce that (4) gives a decomposition of u as a linear combination of elements in $K \cap M$ with coefficients in $\mathbb{Z}_{\ge 0}$ (note that u_i is obviously in $K \cap M$). □

The next result relates S_σ to S_τ, where $\tau \subset \sigma$ is a face.

Proposition 4.7. *If σ is a rational convex polyhedral cone in V and τ is a face of it, then there exists $u \in S_\sigma$ such that $\tau = H_u \cap \sigma$ and*

$$S_\tau = S_\sigma + \mathbb{Z}_{\ge 0}(-u).$$

Proof. By definition, we know that $\tau = H_u \cap \sigma$, for some $u \in V^*$. One can see that in fact u can be any vector in the relative interior of the dual face $\tau^\perp \cap \sigma^\vee$ (see Proposition 4.4).

Because that face is a rational cone, we can choose $u \in M$, and the first part of the statement is proved. To prove the second part, we note first that $-u \in \tau^\vee$, because $\langle u, \cdot \rangle$ vanishes on τ. Consequently, $-u \in \tau^\vee \cap M = S_\tau$, which implies $S_\sigma \cap \mathbb{Z}_{\ge 0}(-u) \subset S_\tau$.

To prove the inverse inclusion, we write $\sigma = \mathrm{Cone}(S)$, where $S \subset N$ is a finite set. Let w be an arbitrary element in S_τ. Set

$$C := \max\{|\langle w, v \rangle| \ : \ w \in S \setminus H_u\}.$$

Claim. $w + Cu \in S_\sigma$.

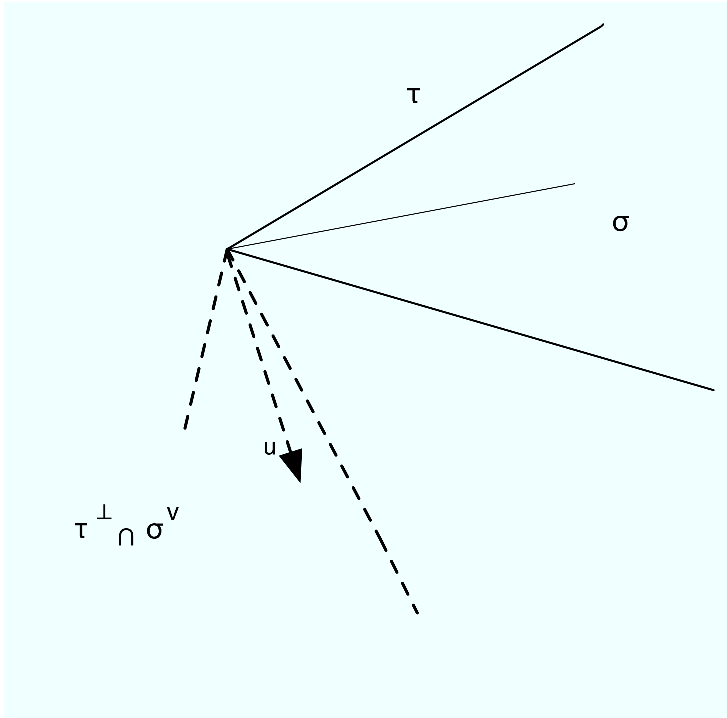
To prove the claim, we have to check that for any $v \in S$ it holds $\langle w + Cu, v \rangle \ge 0$. First we take $v \in S \setminus H_u$. We have

$$\langle w + Cu, v \rangle = \langle w, v \rangle + C\langle u, v \rangle \ge -C + C\langle u, v \rangle = C(\langle u, v \rangle - 1) \ge 0,$$

because $\langle u, v \rangle$ is an integer number which is positive (remember that $u \in S_\sigma$ and $v \in S$). Now, if $v \in H_u$, then $v \in H_u \cap \sigma = \tau$. It follows that

$$\langle w + Cu, v \rangle = \langle w, v \rangle \ge 0.$$

[10]Otherwise expressed, the space of solutions of a homogeneous system of linear equations with integer coefficients has a basis consisting of vectors with all entries integer



The claim implies that $w \in S_\sigma \cap \mathbb{Z}_{\geq 0}(-u)$, QED. □

Another result which will be needed later is the following.

Proposition 4.8. *Let σ and σ' be rational convex polyhedral cones in V such that $\sigma \cap \sigma'$ is a face of both of them. Then we have*

$$S_{\sigma\cap\sigma'} = S_\sigma + S_{\sigma'}.$$

Sketch of the proof. (see [Su, Lecture 1, Prop.5.6]).

Denote $\tau = \sigma \cap \sigma'$. The most difficult step is to show that there exists $u \in \sigma^\vee \cap (-\sigma')^\vee \cap M$ such that $\tau = H_u \cap \sigma$ (see the figure in order to get a feeling why is that so).

Because $S_\sigma \subset S_\tau$ and $S_{\sigma'} \subset S_\tau$, we deduce that $S_\sigma + S_{\sigma'} \subset S_\tau$. We now prove the opposite inclusion. From Proposition 4.7, we have that

$$S_\tau = S_\sigma + \mathbb{Z}_{\geq 0}(-u).$$

But $-u \in (\sigma')^\vee \cap M = S_{\sigma'}$, which implies that

$$S_\tau = S_\sigma + \mathbb{Z}_{\geq 0}(-u) \subset S_\sigma + S_{\sigma'},$$

and the proof is finished. □

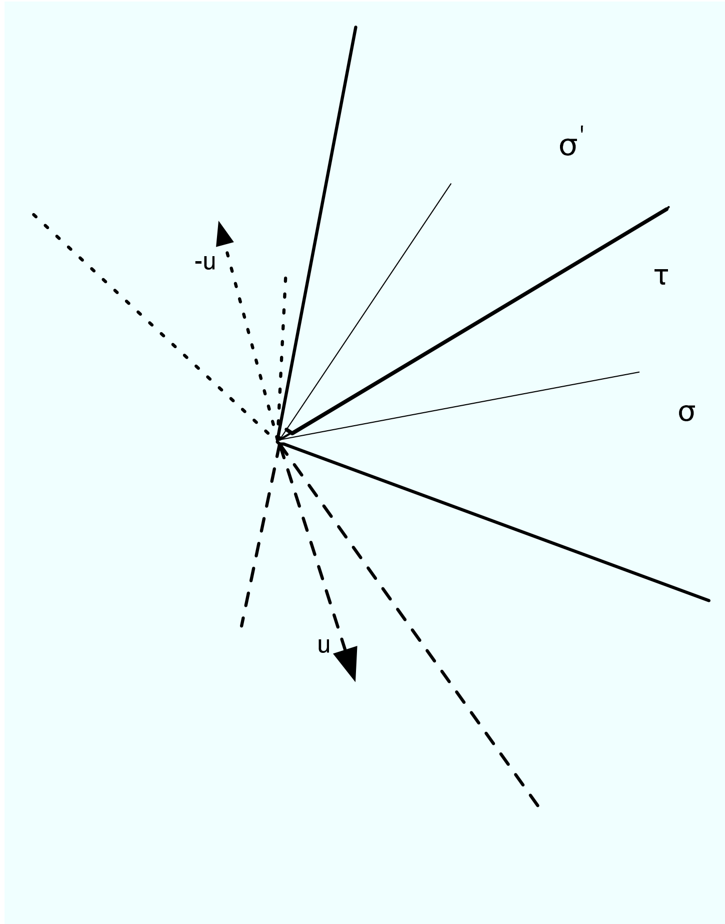
Remark. We note that under the hypotheses of the theorem one can show that there exists $u \in \sigma^\vee \cap (-\sigma')^\vee \cap M$ such that $\tau = H_u \cap \sigma$ and $\tau = H_u \cap \sigma'$. Then H_u is called a *separating hyperplane.*

At the end we say a few words about strongly convex polyhedral cones.

Definition. A convex polyedral cone σ is called *strongly convex* if $\sigma \cap (-\sigma) = \{0\}$.

Proposition 4.9. *Let σ be a convex polyhedral cone. The following assertions are equivalent.*

1. *σ is strongly convex.*
2. *σ contains no positive dimensional vector subspaces.*



3. $\{0\}$ *is a face of* σ.
4. $\dim \sigma^{\vee} = n$.

The proof is fairly obvious (see also Proposition 4.4). Strongly convex polyhedral cones have a natural set of generators, as the next proposition states. First, one-dimensional faces of a convex polyhedral cone are called *edges*.

Proposition 4.10. *Let* σ *be a strongly convex polyhedral cone with edges* $\rho_1, \ldots, \rho_s$. *Pick* $v_i \in \rho_i \setminus \{0\}$. *Then we have:*

a) $\sigma = \text{Cone}\{v_1, \ldots, v_s\}$,
b) $\{v_1, \ldots, v_s\}$ *is a* minimal *generating set for* σ *, in the sense that if* $\sigma = \text{Cone}(T)$, *then there exist numbers* $\lambda_i > 0$ *such that* $\{\lambda_1 v_1, \ldots, \lambda_m v_m\} \subset T$.

Sketch of the proof. (a) We look at the facets of $\sigma^{\vee}$, which by Proposition 4.4, are $\sigma^{\vee} \cap \rho_i^{\perp} = \sigma^{\vee} \cap H_{v_i}$, $1 \leq i \leq m$. By the proof of Theorem 4.3, we have that $\sigma^{\vee} = \bigcap_{i=1}^{m} H_{v_i}^{+}$ (note that $\langle \sigma^{\vee}, v_i \rangle \geq 0$), and

$$\sigma = \text{Cone}\{v_1, \ldots, v_s\}.$$

(b) Suppose that $\sigma = \text{Cone}(T)$. Then $\rho_i = \text{Cone}(\rho_i \cap T)$.. This implies that T contains an element of ρ_i, which is a positive multiple of v_i. $\square$

5. Affine toric varieties

Like in the previous section, we consider a lattice N, then we take the dual lattice $M = \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$ and the vector spaces $V = N \otimes_{\mathbb{Z}} (R)$, and also V^*. We fix a rational strongly convex polyhedral cone $\sigma \subset V$. We consider the semigroup $S_\sigma = \sigma^\vee \cap M$ and the group ring

$$A_\sigma := \mathbb{C}[S_\sigma].$$

As already mentioned in section 2, this consists of all linear combinations of formal powers of the form χ^u, where $u \in S_\sigma$ and χ is a formal variable.

Lemma 5.1. *The $\mathbb{C}$-algebra A_σ is finitely generated, it is a domain, and has no nonzero nilpotents.*

Proof. The first assertion follows from the fact that S_σ is a finitely generated semigroup (see Proposition 4.6). To prove the rest of the lemma, we note that $\{0\}$ is a face of σ. This implies that S_σ is a subsemigroup of $S_{\{0\}}$, hence A_σ is a subalgebra of $A_{\{0\}}$. But $S_{\{0\}}$ is the whole M. Let $\{e_1, \ldots, e_n\}$ be a basis of N and $\{e_1^*, \ldots, e_n^*\}$ the dual basis of M. As a semigroup, S_0 has the basis $\{e_1^*, -e_1^*, \ldots, e_n^*, -e_n^*\}$. Then $A_{\{0\}} = \mathbb{C}[S_0]$ is just the algebra $\mathbb{C}[X_1, X_1^{-1}, \ldots, X_n, X_n^{-1}]$ of Laurent polynomials, where $X_i := \chi^{e_i^*}$. We only have to note that $\mathbb{C}[\{X_i\}, \{X_i^{-1}\}]$ is a domain (i.e. it has no divisors of zero) and has no nonzero nilpotents. Consequently any of its subalgebras is a domain and has no nonzero nilpotents. $\square$

By Propositions 1.3 and 1.1, there exists a unique affine variety, call it U_σ, which is irreducible (see Proposition 3.1), and whose coordinate ring is A_σ. By Corollary 1.2, U_σ can be identified with $\mathrm{Hom}_{\mathbb{C}-\mathrm{alg}}(A_\sigma, \mathbb{C})$. But one can easily see that this is the same as[11] $\mathrm{Hom}_{\mathrm{sgp}}(S_\sigma, \mathbb{C})$, where $\mathrm{Hom}_{\mathrm{sgp}}$ stands for the space of all semigroup homomorphisms, and $\mathbb{C}$ is regarded as a semigroup with respect to the multiplication.

Examples. 1. Take the cone $\{0\} \subset V = \mathbb{R}^n$. Then $S_{\{0\}}$ is the whole M. Consequently, we have $U_{\{0\}} = \mathrm{Hom}_{\mathrm{sgp}}(M, \mathbb{C})$. This is the same as the space $\mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{C}^*)$ of all group homomorphisms $M \to \mathbb{C}^*$. This implies that

$$U_{\{0\}} = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{C}^*) = (\mathbb{C}^*)^n,$$

which is the complex torus (see section 2, example 1). Note that we can also write

$$U_{\{0\}} = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{C}^*) = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C}^* = N \otimes_{\mathbb{Z}} \mathbb{C}^* =: T_N.$$

2. To the cone $\sigma = \mathbb{R}_{\geq 0} e_1 + \ldots + \mathbb{R}_{\geq 0} e_n$ in $V = \mathbb{R}^n$ corresponds $\sigma^\vee = \mathbb{R}_{\geq 0} e_1^* + \ldots + \mathbb{R}_{\geq 0} e_n^* \subset V^*$. Hence S_σ has a basis consisting of $e_1^*, \ldots, e_n^*$. We have $U_\sigma = \mathrm{Hom}_{\mathrm{sgp}}(S_\sigma, \mathbb{C})$, which is obviously $\mathbb{C}^n$. Note that there is a natural embedding $(\mathbb{C}^*)^n = U_{\{0\}} \subset U_\sigma = (\mathbb{C})^n$.

One may also want to look again at the examples 2 and 3 in section 2.

Next we will discuss explicit descriptions of U_σ by polynomial equations. In other words, we need a presentation of A_σ in terms of generators and relations. Let $\{u_1, \ldots, u_k\} \subset S_\sigma$ be a set of generators. Then $Y_1 := \chi^{u_1}, \ldots, Y_k := \chi^{u_k}$ generate A_σ. Relations involving $Y_1, \ldots, Y_k$ are induced by equations of the form

$$\sum_{i=1}^{k} a_i u_i = \sum_{i=1}^{k} b_i u_i,$$

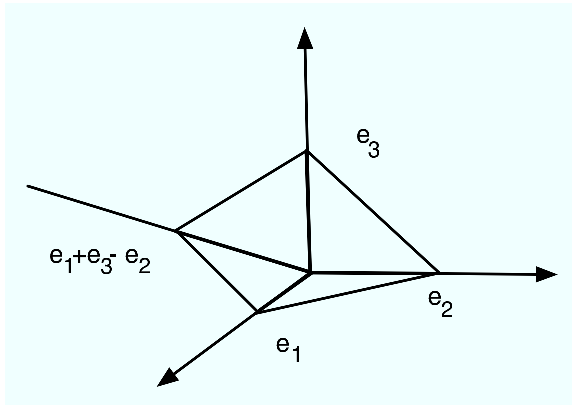
[11] More precisely, to $\alpha : S_\sigma \to \mathbb{C}$ corresponds the morphism $A_\sigma \to \mathbb{C}$, $\chi^u \mapsto \alpha(u)$.

where $a_i, b_i \in \mathbb{Z}_{\geq 0}$. The induced relation is obviously

$$\prod_{i=1}^{k} Y_i^{a_i} - \prod_{i=1}^{k} Y_i^{b_i} = 0. \tag{5}$$

These generate the ideal of relations of the affine variety $U_\sigma \subset \mathbb{C}^k$.

Example. Let $\sigma \subset \mathbb{R}^3$ be the cone generated by e_1, e_2, e_3, and $e_1 + e_3 - e_2$. Then $\sigma^\vee$ is



generated by $e_1^*, e_3^*, e_1^*+e_2^*$, and $e_2^*+e_3^*$ (note that e_1+e_2 is perpendicular to e_3 and $e_1+e_3-e_2$; also, e_2+e_3 is perpendicular to e_1 and $e_1+e_3-e_2$; in this way we obtain edges of $\sigma^\vee$). We deduce that the generators of the ring A_σ are $Y_1 := \chi^{e_1^*}, Y_2 = \chi^{e_3^*}, Y_3 = \chi^{e_1^*+e_2^*}, Y_4 = \chi^{e_2^*+e_3^*}$. In order to find the relations among those, we note that

$$e_1^* + (e_2^* + e_3^*) = e_3^* + (e_1^* + e_2^*)$$

which gives $Y_1Y_4 = Y_2Y_3$. Consequently, we have

$$U_\sigma = \mathcal{V}(Y_1Y_4 - Y_2Y_3) \subset \mathbb{C}^4.$$

Note that the point 0 on U_σ is a "singularity" (i.e. a singular point of the function $\mathbb{C}^4 \to \mathbb{C}$, $(y_1, y_2, y_3, y_4) \mapsto y_1y_4 - y_2y_3$).

The next result will relate the varieties U_σ and U_τ, where τ is a face of σ. In order to prove this relationship (see Proposition 5.3 below) we need to make some general considerations. Let $V \subset \mathbb{C}^n$ be an affine variety, and let $\mathbb{C}[V] = \mathbb{C}[X_1, \ldots, X_n]/\mathcal{I}(V)$ be its coordinate ring. This consists of cosets $[f]$, where $f \in \mathbb{C}[X_1, \ldots, X_n]$. We assume that V is irreducible, which implies that $\mathbb{C}[V]$ is a domain. If $[f] \in \mathbb{C}[V]$ is nonzero, we consider the *localization*[12] of $\mathbb{C}[V]$ at $[f]$, which is

$$\mathbb{C}[V]_{[f]} := \{\frac{[g]}{[f]^n} \ : \ n \geq 0 \text{ and } [g] \in \mathbb{C}[V]\}.$$

This has an obvious ring structure. It is (finitely) generated by the generators of $\mathbb{C}[V]$ together with $\frac{1}{[f]}$. It has obviously no nonzero nilpotents. According to Proposition 1.3, there exists an affine variety whose coordinate ring is $\mathbb{C}[V]_{[f]}$. The next lemma shows that this variety is just the principal open set V_f (see section 3 for the definition of this).

Proposition 5.2. *If V is an affine variety and $[f] \in \mathbb{C}[V]$ is different from 0, then the principal open set $V_f := \{x \in V \ : \ f(x) \neq 0\}$ is an affine variety whose coordinate ring is the localized ring $\mathbb{C}[V]_{[f]}$. If we identify $V = \mathrm{Hom}_{\mathbb{C}-\mathrm{alg}}(\mathbb{C}[V], \mathbb{C})$, and $V_f = \mathrm{Hom}_{\mathbb{C}-\mathrm{alg}}(\mathbb{C}[V]_{[f]}, \mathbb{C})$*

[12] This is actually a subring of the field of fractions of the domain $\mathbb{C}[V]$.

(see Corollary 1.2), then the inclusion $V_f \subset V$ is just the restriction map induced by the obvious inclusion $\mathbb{C}[V] \subset \mathbb{C}[V]_{[f]}$ (see Proposition 1.1).

Proof. We identify V_f with the set

$$\{(x_1, \ldots, x_n, x_{n+1}) \in \mathbb{C}^{n+1} : g(x_1, \ldots, x_n) = 0, \forall g \in \mathcal{I}(V) \text{ and } x_{n+1} f(x_1, \ldots, x_n) = 1\},$$

and it becomes an affine variety in $\mathbb{C}^{n+1}$. The coordinate ring of the latter variety is the quotient of $\mathbb{C}[X_1, \ldots, X_{n+1}]$ by the ideal generated by $g(X_1, \ldots, X_n)$, $g \in \mathcal{I}(V)$ and $X_{n+1} f(X_1, \ldots, X_n) - 1$. We identify the latter with $\mathbb{C}[V]_{[f]}$ via

$$\mathbb{C}[V_f] \ni [F(X_1, \ldots, X_n, X_{n+1})] \mapsto F(X_1, \ldots, X_n, \frac{1}{[f]}) \in \mathbb{C}[V]_{[f]}.$$

□

Example. Take $V = \mathbb{C}^n$ and $f = X_1 \ldots X_n$. We have

$$\mathbb{C}[V] = \mathbb{C}[X_1, \ldots, X_n], \quad \mathbb{C}[V]_{X_1 \ldots X_n} = \mathbb{C}[X_1, X_1^{-1}, \ldots, X_n, X_n^{-1}].$$

According to what we said above, $V_f = \mathcal{V}(X_1 \ldots X_n X_{n+1} - 1) \subset \mathbb{C}^{n+1}$, which is just the torus $(\mathbb{C}^*)^n$ (see Example 1, section 2).

Now let us consider a rational convex polyhedral cone σ and $\tau \subset \sigma$ a face of it. Because $S_\sigma \subset S_\tau$, we obtain the embedding

$$\mathrm{Hom}_{\mathrm{sgp}}(S_\tau, \mathbb{C}) = U_\tau \hookrightarrow U_\sigma = \mathrm{Hom}_{\mathrm{sgp}}(S_\sigma, \mathbb{C}), \tag{6}$$

given by the restriction map.

Proposition 5.3. *The inclusion (6) makes U_τ into a principal open subset of U_σ.*

Proof. By Proposition 4.7, we have

$$S_\tau = S_\sigma + \mathbb{Z}_{\geq 0}(-u_0),$$

for some $u_0 \in S_\sigma$. We deduce that the ring $\mathbb{C}[S_\tau] = \mathrm{Span}_{\mathbb{R}}\{\chi^u : u \in S_\tau\}$ coincides with the localized ring $\mathbb{C}[S_\sigma]_{\chi^{u_0}}$. In other words, A_τ is the localization of A_σ at χ^{u_0}. We use Proposition 5.2. □

Example. The prototypical situation is described by $\sigma = \mathbb{R}_{\geq 0} e_1 + \ldots + \mathbb{R}_{\geq 0} e_n$ like in example 2 from above, and $\tau = \{0\}$ like in example 1. Then $U_{\{0\}} = (\mathbb{C}^*)^n$ is the principal open subset of $U_\sigma = \mathbb{C}^n$ corresponding to $f = X_1 \ldots X_n$.

We note that in the previous example, $U_{\{0\}} = (\mathbb{C}^*)^n$ has an obvious group structure given by componentwise multiplication. This group acts on $U_\sigma = \mathbb{C}^n$ and the embedding $U_{\{0\}} \subset U_\sigma$ is equivariant. We will see next that this is a general situation:

Proposition 5.4. *If V is an affine toric variety then there exists a complex torus $(\mathbb{C}^*)^n$ which is contained in V as an open subset such that the action of $(\mathbb{C}^*)^n$ on itself by left multiplication can be extended to an algebraic action[13] on the whole V.*

Proof. Let U_σ be the toric variety associated to an arbitrary rational strongly convex polyhedral cone. We saw that $U_{\{0\}} = \mathrm{Hom}_{\mathrm{sgp}}(M, \mathbb{C}) = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{C}^*) = T_N$ is a complex torus which, by Proposition 5.3, is contained in U_σ as an open set. The action of T_N on itself by left multiplication is described as follows: if $t_1, t_2 \in \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{C}^*)$, then

$$t_1 \cdot t_2(u) := t_1(u) t_2(u).$$

[13] A group action on an affine variety is called *algebraic* if it is done by morphisms of varieties (in the sense defined in section 1).

The torus T_N acts on $U_\sigma = \mathrm{Hom}_{\mathrm{sgp}}(S_\sigma, \mathbb{C})$ as follows: if $t \in T_N$ and $x \in U_\sigma$, then $t{\cdot}x : S_\sigma \to \mathbb{C}$ is defined by

$$t \cdot x(u) := t(u)x(u), \quad u \in S_\sigma. \tag{7}$$

It is obvious that the inclusion $T_N \hookrightarrow U_\sigma$ is T_N-equivariant.

We show that the action of T_N on U_σ is induced by a $\mathbb{C}$-algebra endomorphism of $A_\sigma = \mathbb{C}[S_\sigma]$ — by using Proposition 1.1 we deduce that the action is algebraic. Regard $t \in T_N$ as a $\mathbb{C}$-algebra homomorphism $t : \mathbb{C}[M] \to \mathbb{C}$. It induces the endomorphism $A_\sigma \to A_\sigma$ given by $t.a := t^{-1}(a)a$, for all $a \in A_\sigma$ (note that $A_\sigma \subset \mathbb{C}[M]$). The endomorphism induced on $\mathrm{Hom}_{\mathbb{C}-\mathrm{alg}}(A_\sigma, \mathbb{C})$ is

$$(t.x)(a) := x(t^{-1}.a) = x(t(a)a) = t(a)x(a),$$

which is exactly the one given by (7). □

Finally, we will show that any affine toric variety is normal. In order to define the notion of normal affine variety, we recall that an integral domain R with field of fractions K is said *integrally closed* if

$$[k \in K \text{ and } f \in R[x] \setminus \{0\} \text{ is monic such that } f(k) = 0] \Rightarrow k \in R.$$

Example. The polynomial ring $\mathbb{C}[X_1, \ldots, X_n]$ is integrally closed (this is an easy exercise).

Definition. We say than an (irreducible) affine variety V is *normal* if $\mathbb{C}[V]$ is integrally closed.

Example. See [Su, Lecture 1, Exercise 1.10] for an example of a variety which is not normal.

We just mention the following result.

Proposition 5.5. *For any rational strongly convex polyhedral cone $\sigma \subset V$, the semigroup ring $\mathbb{C}[S_\sigma]$ is integrally closed. Consequently any affine toric variety U_σ is normal.*

For a proof one can see for instance [Su, Lecture 2, proof of Proposition 3.2].

6. General toric varieties

We have already presented in section 2 a few examples of varieties of the type mentioned in the title (see examples 4,5 and 6 in that section). They are constructed starting from what is called a fan. In order to define this notion, we place ourselves again in the context described in section 4. More precisely, we consider a lattice $N \simeq \mathbb{Z}^n$ of dimension n, $V = N \otimes_{\mathbb{Z}} \mathbb{R}$ the associated vector space and V^* the dual vector space, which is the same as $M \otimes_{\mathbb{Z}} \mathbb{R}$, for $M = \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$.

Definition. A *fan* in N is a finite non-empty collection Δ of rational strongly convex polyhedal cones with the following properties:

(a) each face of a cone in Δ belongs to Δ
(b) the intersection of any two cones in Δ is a face of both.

Example. An important example of a fan is any strongly rational convex polyhedral cone σ. To be more precise, the set of all faces of σ satisfies the conditions from above.

We note that a fan contains a certain number of *maximal* cones (in the sense of the inclusion). Their union is the same as the union of all cones of the fan.

If Δ is a fan, we associate to it the prevariety denoted $X(\Delta)$ as follows. For any $\sigma, \sigma' \in \Delta$, the intersection $\sigma \cap \sigma'$ is (not empty and) a face of both σ and σ'. Consequently, the variety

$U_{\sigma\cap\sigma'}$ is an open subspace of both U_σ and $U_{\sigma'}$ (see Proposition 5.3). Then $X(\Delta)$ is the prevariety obtained from the collection of all affine varieties U_σ, $\sigma \in \Delta$, by gluing them along $U_{\sigma\cap\sigma'}$ (see the definition of the gluing given in section 3).

Proposition 6.1. *The prevariety $X(\Delta)$ is actually a variety.*

Proof. We use Proposition 3.3. We need to show that for any $\sigma, \sigma' \in \Delta$, the image of the diagonal map $\delta : U_{\sigma\cap\sigma'} \to U_\sigma \times U_{\sigma'}$ is a closed subvariety of the product variety $U_\sigma \times U_{\sigma'}$. This is equivalent to the fact that the $\mathbb{C}$-algebra homomorphism

$$\delta^* : \mathbb{C}[U_\sigma \times U_{\sigma'}] \to \mathbb{C}[U_{\sigma\cap\sigma'}]$$

(see section 1 for the definition of this homomorphism) is surjective[14] . The coordinate rings are as follows[15]

$$\mathbb{C}[U_\sigma] = \mathbb{C}[S_\sigma], \mathbb{C}[U_\tau] = \mathbb{C}[S_\tau], \mathbb{C}[U_\sigma \times U_\tau] = \mathbb{C}[U_\sigma] \otimes \mathbb{C}[U_\tau] = \mathbb{C}[S_\sigma] \otimes \mathbb{C}[S_\tau], \mathbb{C}[U_{\sigma\cap\tau}] = \mathbb{C}[S_{\sigma\cap\tau}].$$

We need to prove that

$$\delta^* : \mathbb{C}[S_\sigma] \otimes \mathbb{C}[S_{\sigma'}] \to \mathbb{C}[S_{\sigma\cap\sigma'}]$$

is surjective. One can see that if we regard δ^* as a map $\mathbb{C}[U_\sigma] \otimes \mathbb{C}[U_{\sigma'}] \to \mathbb{C}[U_{\sigma\cap\sigma'}]$, then $\delta^*([f] \otimes [g]) = i_1^*([f])i_2^*([g])$, where $i_1 : U_{\sigma\cap\sigma'} \to U_\sigma$ and $i_2 : U_{\sigma\cap\sigma'} \to U_{\sigma'}$ are the inclusion maps. This implies that

$$\delta^*(\chi^u \otimes \chi^v) = \chi^{u+v},$$

for $u \in \mathbb{C}[S_\sigma], v \in \mathbb{C}[S_{\sigma'}]$. The decisive argument is Proposition 4.8, which says that $S_{\sigma\cap\sigma'} = S_\sigma + S_{\sigma'}$. $\square$

We discuss again the examples 4, 5 and 6 in section 2. The following general considerations will be useful. Assume that τ is a common face of σ and σ' and we want to understand the gluing of U_σ and $U_{\sigma'}$ along U_τ, which is contained in both of them as an open subspace. First, we describe U_τ by equations of the type (5), i.e. by finding generators $u_1, \ldots, u_m$ in S_τ and the relations among them. We stick to the variables $X_1 := \chi^{u_1}, \ldots, X_m = \chi^{u_m}$, which give coordinates $x_1, \ldots, x_m$ on U_τ. The embedding $U_\tau \subset U_\sigma$ has to be understood in terms of Proposition 5.3. That is, by Proposition 4.7, we can find $u \in S_\sigma$ such that

$$S_\tau = S_\sigma + \mathbb{Z}_{\geq 0}(-u). \tag{8}$$

If $v_1, \ldots, v_{m-1}$ is a generating set for S_σ, then S_τ is generated by $v_1, \ldots, v_{m-1}$, and $-u$. The embedding $U_\tau \subset U_\sigma$ is given by

$$(\chi^{u_1}, \ldots, \chi^{u_m}) = (\chi^{v_1}, \ldots, \chi^{v_{m-1}}, \chi^{-u}) \mapsto (\chi^{v_1}, \ldots, \chi^{v_{m-1}}).$$

To see it exactly, we need to express $v_1, \ldots, v_{m-1}$ in terms of the basis $u_1, \ldots, u_m$. The same has to be done for $S_{\sigma'}$, and then we can try to understand the gluing.

Example 4, section 2, revisited. The cones σ_0, σ_1, and σ_2 are a fan (according to the definition from above). Because $\sigma_0^\vee = \mathbb{R}$, we have $S_{\sigma_0} = \mathbb{Z}$. The generators are e and $-e$; consequently, if $X_1 = \chi^e$ and $X_2 = \chi^{-e}$, then $U_{\sigma_0} = \mathcal{V}(X_1X_2 - 1)$. In other words, the

[14] A general result says that that the image of an injective homomorphism of affine varieties $\varphi : V \to W$ is closed if and only if the map $\varphi^* : \mathbb{C}[W] \to \mathbb{C}[V]$ is surjective. See for instance Fantechi's notes [Fa], Proposition 2.11.

[15] A general result says that if V and W are two affine varieties and $V \times W$ the product variety, then $\mathbb{C}[V \times W] = \mathbb{C}[V] \otimes \mathbb{C}[W]$. To prove this, we note that $\mathcal{I}(V \times W) = \mathcal{I}(V) + \mathcal{I}(W)$ (where $\mathcal{I}(V)$ and $\mathcal{I}(W)$ are ideals in $\mathbb{C}[X_1, \ldots, X_n, Y_1, \ldots, Y_m]$). Then

$$\mathbb{C}[X_1, \ldots, X_n, Y_1, \ldots, Y_m]/(\mathcal{I}(V) + \mathcal{I}(W)) \simeq \mathbb{C}[X_1, \ldots, X_n]/\mathcal{I}(V) \otimes \mathbb{C}[Y_1, \ldots, Y_m]/\mathcal{I}(W).$$

coordinates on U_{σ_0} are x_1, x_2, and they satisfy $x_1 x_2 = 1$. We identify U_{σ_0} with $\mathbb{C}^*$. Now we discuss the embedding $U_{\sigma_0} \subset U_{\sigma_1}$. The semigroup S_{σ_1} is generated by e, hence $U_{\sigma_1} = \mathbb{C}$. We have

$$S_{\sigma_0} = S_{\sigma_1} \oplus \mathbb{Z}_{\geq 0}(-e).$$

According to what we said above, the embedding is $(\chi^e, \chi^{-e}) \mapsto \chi^e$, in other words,

$$U_{\sigma_0} = \mathbb{C}^* \ni x \mapsto x \in \mathbb{C} = U_{\sigma_1}.$$

The semigroup S_{σ_2} is generated by $-e$, hence $U_{\sigma_2} = \mathbb{C}$ again. We have

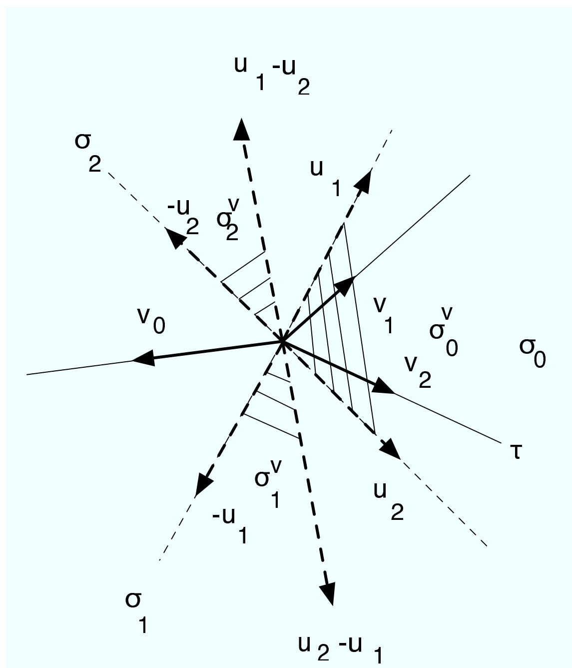
$$S_{\sigma_0} = S_{\sigma_2} \oplus \mathbb{Z}_{\geq 0}(e),$$

so the embedding is $(\chi^e, \chi^{-e}) \mapsto \chi^{-e}$, in other words,

$$U_{\sigma_0} = \mathbb{C}^* \ni x \mapsto x^{-1} \in \mathbb{C} = U_{\sigma_2}.$$

The way of identifying the toric variety $X(\Delta)$ with the projective space $\mathbb{P}^1$ has been described in Example 4, section 2.

Example 5, section 2, revisited.



In fact, this time we consider a slightly more general construction (which has the advantage of being symmetric). We start with a more general fan, but the corresponding variety is again the projective space $\mathbb{P}^2$. We consider the vectors v_0, v_1, v_2 in N which generate N and such that $v_0 + v_1 + v_2 = 0$. The cones of the fan are $\sigma_0 := \text{Cone}(v_1, v_2)$, $\sigma_1 := \text{Cone}(v_0, v_2)$, and $\sigma_2 := \text{Cone}(v_0, v_1)$ (see the figure). The shaded cones in the figure are the dual ones (as usually, we identify V^* with V by means of an inner product on V). For some vectors u_1, u_2 in M we have

$$\sigma_0^\vee = \text{Cone}(u_1, u_2), \quad \sigma_1^\vee = \text{Cone}(-u_1, u_2 - u_1), \quad \sigma_2^\vee = \text{Cone}(-u_2, u_1 - u_2).$$

Let us understand the gluing of U_{σ_0} and U_{σ_1} along U_τ. As we said above, we start by choosing (and fixing) coordinates on U_τ. Because $S_\tau = \tau^\vee \cap M$ is generated by $u_1, -u_1$, and u_2, we deduce that if we put $X_1 = \chi^{u_1}, X_2 = \chi^{u_2}, X_3 = \chi^{-u_1}$, then $U_\tau = \mathcal{V}(X_1 X_3 - 1)$ in $\mathbb{C}^3$. The induced coordinate functions on U_τ are x_1, x_2, x_3 and we have to bear in mind that $x_1 x_3 = 1$.

We can identify U_τ with $\mathbb{C}^* \times \mathbb{C}$. Now we look at the embedding $U_\tau \subset U_{\sigma_0}$. The semigroup S_{σ_0} is generated by u_1 and u_2, hence $U_{\sigma_0} = \mathbb{C}^2$. Note that

$$S_\tau = S_{\sigma_0} \oplus \mathbb{Z}_{\geq 0}(-u_1).$$

The embedding is $(\chi^{u_1}, \chi^{u_2}, \chi^{-u_1}) \mapsto (\chi^{u_1}, \chi^{u_2})$, hence

$$U_\tau = \mathbb{C}^* \times \mathbb{C} \ni (x_1, x_2) \mapsto (x_1, x_2) \in \mathbb{C}^2 = U_{\sigma_0}.$$

We look at $U_\tau \subset U_{\sigma_1}$. The semigroup S_{σ_1} is generated by $-u_1$ and $u_2 - u_1$, hence U_{σ_2} can be identified with $\mathbb{C}^2$ as well. This time we have

$$S_\tau = S_{\sigma_1} \oplus \mathbb{Z}_{\geq 0}(u_1).$$

The embedding is $(\chi^{u_1}, \chi^{u_2}, \chi^{-u_1}) \mapsto (\chi^{-u_1}, \chi^{u_2-u_1})$, hence

$$U_\tau = \mathbb{C}^* \times \mathbb{C} \ni (x_1, x_2) \mapsto (x_1^{-1}, x_1^{-1}x_2) \in \mathbb{C}^2 = U_{\sigma_1}.$$

The other gluings can be understood similarly. The way of identifying the toric variety $X(\Delta)$ with the projective space $\mathbb{P}^2$ has been described in Example 5, section 2. Note that the same construction can be done in a more symmetric way if instead of starting with $N = \mathbb{Z}^2$, we take $N = \mathbb{Z}^3/\mathbb{Z}(e_1 + e_2 + e_3)$. Then we take the fan consisting of the cones generated by the cosets of e_1, e_2, respectively e_1, e_3, respectively e_2, e_3 (just observe that $e_3 = -e_1 - e_2$ modulo $\mathbb{Z}(e_1 + e_2 + e_3)$.) The last two examples can be generalized, in the sense that any projective space $\mathbb{P}^n$ (i.e. space of all complex lines in $\mathbb{C}^{n+1}$) can be realized as $X(\Delta)$ for some fan Δ.

Any cone $\sigma \in \Delta$ has $\{0\}$ as a face, hence $T_N = U_{\{0\}}$ is a principal open subset of U_σ and it acts on U_σ in such a way that the embedding $T_N \subset U_\sigma$ is T_N-equivarient (see the end of the previous section). If $\tau \subset \sigma$ is a face, then the embedding $U_\tau \subset U_\sigma$ is T_N-equivariant: indeed, we only need to note that the map

$$U_\tau = \mathrm{Hom}_{\mathrm{sgp}}(S_\tau, \mathbb{C}) \hookrightarrow \mathrm{Hom}_{\mathrm{sgp}}(S_\sigma, \mathbb{C}) = U_\sigma$$

is given by restriction from S_τ to S_σ; then we take into account the definition of the action of T_N on U_σ (and U_τ) described in Proposition 5.4. Consequently, T_N is embedded in the glued space $X(\Delta)$ as an open subspace. Moreover, the embedding $T_N \hookrightarrow X(\Delta)$ is T_N-equivariant. We have proved "half" of the following theorem.

Theorem 6.2. *A variety X contains a torus T as an open subvariety in such a way that the embedding $T \subset X$ is T-equivariant if and only if $X = X(\Delta)$ for some fan Δ.*

The other implication is harder: a reference is given in [Fu] at the end of section 1.4.

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