PRODUCTS OF CONJUGACY CLASSES IN $SU(2)$

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ABSTRACT. We obtain a complete description of the conjugacy classes $C_1, \dots, C_n$ in $SU(2)$ with the property that $C_1 \dots C_n = SU(2)$. The basic instrument is a characterization of the conjugacy classes $C_1, \dots, C_{n+1}$ in $SU(2)$ with $C_1 \dots C_{n+1} \ni I$, which generalizes a result of [Je-We].

1. INTRODUCTION

The following problem was posed by D. Burago:

Problem: Let G be a group. For which conjugacy classes $C_1, \dots, C_n$ of G is it true that the multiplication map

$$C_1 \times \cdots \times C_n \to G$$

is surjective?

We give a solution to this problem in the case $G = SU(2)$. In this case the conjugacy classes are parametrized by their eigenvalues

$$\mathrm{diag}(e^{i\lambda}, e^{-i\lambda})$$

so they are determined by one number $\lambda \in [0, \pi]$.

2. EIGENVALUES OF A MULTIPLE PRODUCT

For any $\lambda \in [0, \pi]$ we denote by $C(\lambda)$ the conjugacy class of the matrix

$$\mathrm{Diag}(e^{i\lambda}, e^{-i\lambda})$$

in $SU(2)$. Note that any conjugacy class in $SU(2)$ is of the form $C(\lambda)$ for a unique $\lambda \in [0, \pi]$. The following result was proved in [Je-We]:

Proposition 2.1. *For $\lambda_1, \lambda_2, \lambda_3 \in [0, \pi]$ we have*

$$C(\lambda_1)C(\lambda_2)C(\lambda_3) \ni I$$

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iff

$$|\lambda_1 - \lambda_2| \leq \lambda_3 \leq \min\{\lambda_1 + \lambda_2, 2\pi - (\lambda_1 + \lambda_2)\} \tag{1}$$

Note that (1) is equivalent to

$$\begin{aligned} \lambda_1 + \lambda_2 + \lambda_3 &\leq 2\pi \\ -\lambda_1 - \lambda_2 + \lambda_3 &\leq 0 \\ -\lambda_1 + \lambda_2 - \lambda_3 &\leq 0 \\ \lambda_1 - \lambda_2 - \lambda_3 &\leq 0 \end{aligned}$$

The goal of this section is to prove the more general result:

Theorem 2.2. *For $n \geq 2$ an integer number and $\lambda_1, \dots, \lambda_{n+1} \in [0, \pi]$ we have*

$$C(\lambda_1) \dots C(\lambda_{n+1}) \ni I$$

iff the following system of inequalities holds:

a) If $n + 1 = 2k$ is an even number:

$$S^1_{n+1}(\{\lambda_i\}) \leq (n-1)\pi, \quad S^3_{n+1}(\{\lambda_i\}) \leq (n-3)\pi, \quad \dots \quad S^{2k-1}_{n+1}(\{\lambda_i\}) \leq 0 \tag{2}$$

where $S^j_{n+1}(\{\lambda_i\})$ is any sum of the type $\sum_{i=1}^{n+1} \pm\lambda_i$ which contains exactly j minus signs.

b) If $n + 1 = 2k + 1$ is an odd number:

$$S^0_{n+1}(\{\lambda_i\}) \leq n\pi, \quad S^2_{n+1}(\{\lambda_i\}) \leq (n-2)\pi, \quad \dots \quad S^{2k}_{n+1}(\{\lambda_i\}) \leq 0 \tag{3}$$

where $S^j_{n+1}(\{\lambda_i\})$ has the same meaning as before.

Remarks. 1. A more concise way to express both (2) and (3) is

$$S^{n-2j}_{n+1}(\{\lambda_i\}) \leq 2j\pi$$

for any $0 \leq j \leq n/2$ and any sum of the type S^{n-2j}_{n+1}.

2. An elementary computation involving the binomial formula shows that the number of inequalities in both (2) and (3) is

$$\binom{n+1}{0} + \binom{n+1}{2} + \dots = \binom{n+1}{1} + \binom{n+1}{3} + \dots = 2^n.$$

We will use induction on n to prove this theorem. In order to make the induction step we will need the following result:

Lemma 2.3. *We have*

$$C(\lambda_1)\dots C(\lambda_{n+1}) \ni I$$

iff there exists $\lambda \in [0,\pi]$ *such that*

$$C(\lambda_1)\dots C(\lambda_{n-1})C(\lambda) \ni I \tag{4}$$

and

$$C(\lambda)C(\lambda_n)C(\lambda_{n+1}) \ni I. \tag{5}$$

Proof. The fundamental group of the $(n+1)$-punctured sphere Σ_{n+1} in two dimensions is the free group on n generators, or the group

$$\Pi_n = \langle x_1,\dots,x_{n+1} \mid x_1\dots x_{n+1} = 1\rangle$$

with $n{+}1$ generators and one relation. We can form the $(n{+}1)$-punctured sphere by gluing an $(n-1)$-punctured sphere and a 3-punctured sphere along one of the boundary components of each. Call S the common boundary resulting from this construction and consider the fundamental groups of the two components as follows:

$$\Pi_{n-2} = \langle x_1,\dots,x_{n-2},x \mid x_1\dots x_{n-2}x = 1\rangle$$

and

$$\Pi_2 = \langle x',x_n,x_{n+1} \mid x'x_nx_{n+1} = 1\rangle,$$

where x and x' represent the loop S in each of the two components. From the theorem of Seifert-van Kampen, we have that

$$\Pi_{n+1} = (\Pi_{n-1}\times\Pi_2)/\langle xx' = 1\rangle \tag{6}$$

Now we consider representations of these groups into $G = SU(2)$. The condition (7)is equivalent to the existence of a representation ρ of Π_{n+1} such that

$$\rho(x_i) \in C(\lambda_i)$$

for any $1 \le i \le n+1$. From (6), this is equivalent to the existence of two representations ρ_{n-2} of Π_{n-2} and ρ_2 of Π_2 which coincide with ρ on $x_1,\dots,x_{n-2}$, respectively x_n, x_{n+1} and satisfy

$$\rho_{n-2}(x)\rho_2(x') = I.$$

The latter equality implies that the conjugacy classes of $\rho_{n-2}(x)$ and $\rho_2(x')$ are equal, call them $C(\lambda)$ (note that in $SU(2)$ every element is conjugate to its inverse). The conditions 4 and 5 correspond respectively to the representations ρ_{n-2} and ρ_2.

□

Proof of Theorem 2.2 Just the induction step has to be performed. We want to prove that

$$C(\lambda_1)\dots C(\lambda_{n+1}) \ni I \tag{7}$$

iff equation (2) or (3) holds. Suppose that $n = 2k$ is an even number. Condition (4) of Lemma 2.3 is equivalent to

$$S_n^1(\lambda_1,\dots,\lambda_{n-1},\lambda) \le (n-2)\pi, S_n^3(\lambda_1,\dots,\lambda_{n-1},\lambda) \le (n-4)\pi,\dots, S_n^{2k-1}(\lambda_1,\dots,\lambda_{n-1},\lambda) \le 0 \tag{8}$$

where we have used the induction hypothesis, and condition 5 is equivalent to

$$|\lambda_n - \lambda_{n+1}| \le \lambda \le \min\{\lambda_n + \lambda_{n+1}, 2\pi - (\lambda_n + \lambda_{n+1})\} \tag{9}$$

where we have used Proposition 2.1 By Lemma 2.3, condition (7) is equivalent to the system of inequalities obtained by considering each of the 2^{n-1} inequalities from (8) and deriving from it two inequalities, as follows:

(i) if λ occurs with a *plus* sign in that sum, replace it by $\lambda_n - \lambda_{n+1}$ and $-\lambda_n + \lambda_{n+1}$
(ii) if λ occurs with a *minus* sign in that sum, replace it by $\lambda_n + \lambda_{n+1}$ and $-\lambda_n - \lambda_{n+1}$, but in the latter situation add 2π to the right hand side of the original inequality.

One sees that in the case (i) we replace an inequality of the type

$$S_n^j \le (n-j-1)\pi \tag{10}$$

by two different inequalities, both of the type

$$S_{n+1}^{j+1} \le (n-j-1)\pi. \tag{11}$$

In the case (ii) one again replaces an inequality of the type (10) by an inequality of the type (11) and an inequality of the type

$$S_{n+1}^{j-1} \le (n-j+1)\pi.$$

One obtains 2^n distinct inequalities of type (3), which means that (7) is really equivalent to (3).

A similar argument can be made when $n = 2k-1$ is an odd number. □

Remark. The result stated in Theorem 1.2 can also be obtained from [Ag-Wo, Theorem 3.1] by using the structure of the quantum cohomology ring of $\mathbb{C}P^1$. More precisely, let us consider the two Schubert classes in $H^*(\mathbb{C}P^1)$:

$$[\sigma_1] \in H^2(\mathbb{C}P^1) \text{ and } [\sigma_2] = 1 \in H^0(\mathbb{C}P^1).$$

The quantum cohomology ring of $\mathbb{C}P^1$ is

$$(QH^*(\mathbb{C}P^1) = H^*(\mathbb{C}P^1) \otimes \mathbb{R}[q], \star),$$

where q is a formal variable of degree 4 and $\star$ is an $\mathbb{R}[q]$-linear, commutative and associative product which satisfies

$$[\sigma_1] \star [\sigma_1] = q. \tag{12}$$

Each of the 2^n inequalities indicated in Theorem 2.2 can be obtained by choosing $i_1, \ldots, i_n \in \{1,2\}$ and evaluating the product

$$[\sigma_{i_1}] \star \ldots \star [\sigma_{i_n}]$$

in $QH^*(\mathbb{C}P^1)$. By the equation (12), this product is of the form $q^d \sigma_k$, where d is a positive integer and $k \in \{1,2\}$. The inequality of the type (2) or (3) which corresponds to $i_1, \ldots, i_n$ is

$$\sum_{j=1}^{n} (-1)^{i_j - 1} \lambda_j + (-1)^k \lambda_{n+1} \leq 2d\pi.$$

3. Surjectivity of a multiple product

Our main result is

Theorem 3.1. *We have*

$$C(\lambda_1) \ldots C(\lambda_n) = SU(2) \tag{13}$$

iff for any integer j with $0 \leq j < n/2$ and for any sum of the type $S_n^j = S_n^j(\{\lambda_i\})$ (see Thm. 1.2) we have

$$-(j-1)\pi \leq S_n^j \leq (n-j-1)\pi. \tag{14}$$

Proof. The idea of the proof is that (13) holds iff (7) holds for any $\lambda_{n+1} \in [0, \pi]$. In turn, (7) is equivalent to (2), respectively (3). We just have to take each inequality from (2) (respectively (3)) and make the following formal replacements in its left hand side:

(i) λ_{n+1} by π
(ii) $-\lambda_{n+1}$ by 0.

Let us consider the case $n = 2k - 1$. We have to show that if we perform (i) and (ii) for each inequality contained in (2), we obtain exactly one of the following inequalities:

$$\pi \leq S_n^0 \leq (n-1)\pi \tag{15}$$

$$0 \leq S_n^1 \leq (n-2)\pi \tag{16}$$

$$-\pi \leq S_n^2 \leq (n-3)\pi \tag{17}$$

$$-2\pi \leq S_n^3 \leq (n-4)\pi \tag{18}$$

....

We claim that if we label the inequalities given by (2) as [1], [3],..., $[2k-3]$, $[2k-1]$ then each of [1] and $[2k-1]$ gives exactly one of (15) and (16), each of [3] and $[2k-3]$ gives exactly one of (17) and (18), ... and finally

- if $k = 2p$ is even, then each of $[2p-1]$ and $[2p+1]$ gives exactly one of

$$-(k-1)\pi \le S_n^{k-2} \le (k-2)\pi$$
$$-(k-2)\pi \le S_n^{k-1} \le (k-1)\pi$$

- if $k = 2p+1$ is odd, then each of $[2p+1]$ gives exactly one of

$$-(k-2)\pi \le S_n^{k-1} \le (k-1)\pi$$

Consider first [1] together with $[2k-1]$: the only S_{n+1}^1 which contains $-\lambda_{n+1}$ leads to

$$\lambda_1 + \ldots + \lambda_n \le (n-1)\pi$$

whereas the only S_{n+1}^{2k-1} which contains λ_{n+1} leads to

$$\lambda_1 + \ldots + \lambda_n \ge \pi.$$

The remaining inequalities of type $S_{n+1}^1 \le (n-1)\pi$ lead to all possible inequalities of the type

$$S_n^1 \le (n-2)\pi$$

and the remaining inequalities of the type $S_{n+1}^{2k-1} \le 0$ lead to all possible inequalities of the type

$$S_n^1 \ge 0.$$

The same idea applies[1] to each pair $[2j+1]$, $[2(k-j)-1]$, $0 \le j < k/2$ (if $k = 2p+1$ is an odd number, then for $j = p$ we have $2j+1 = 2(k-j)-1$ and the corresponding pair reduces to just one type of inequalities).

Similar ideas can be used in the case when $n = 2k$ is an even number. $\square$

Remark. The system of inequations (14) admit solutions for any $n \ge 2$. For $n = 2$ the *unique* solution is

$$\lambda_1 = \lambda_2 = \frac{\pi}{2}. \tag{19}$$

For $n \ge 3$ there are several solutions, one of them consisting of λ_1, λ_2 given by (19) and

$$\lambda_3 = \ldots \lambda_n = 0.$$

[1]If we compare the total number of inequalities we start with to the number of inequalities obtained via (i) and (ii) we "deduce" that $\binom{n+1}{2j+1} + \binom{n+1}{n+1-(2j+1)} = 2(\binom{n}{2j+1} + \binom{n}{2j})$. The latter equation is obviously true, by properties of the Pascal triangle.

References

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