The Kirwan map, equivariant Kirwan maps, and their kernels

Lisa C. Jeffrey
Department of Mathematics, University of Toronto, Toronto ON, M5S 3G3, Canada
E-mail address: `jeffrey@math.toronto.edu`

Augustin-Liviu Mare
Department of Mathematics and Statistics, University of Regina, Regina SK, S4S 0A2, Canada, E-mail address: `mareal@math.uregina.ca`

Jonathan M. Woolf
Christ's College, Cambridge, CB2 3BU, UK, E-mail address: `jw301@cam.ac.uk`

Abstract. Consider a Hamiltonian action of a compact Lie group K on a compact symplectic manifold. We find descriptions of the kernel of the Kirwan map corresponding to a regular value of the moment map κ_K. We start with the case when K is a torus T: we determine the kernel of the equivariant Kirwan map (defined by Goldin in [Go]) corresponding to a generic circle $S \subset T$, and show how to recover from this the kernel of κ_T, as described by Tolman and Weitsman in [To-We]. (In the situation when the fixed point set of the torus action is finite, similar results have been obtained in our previous papers [Je], [Je-Ma]). For a compact nonabelian Lie group K we will use the "non-abelian localization formula" of [Je-Ki1] and [Je-Ki2] to establish relationships — some of them obtained by Tolman and Weitsman in [To-We] — between $\mathrm{Ker}(\kappa_K)$ and $\mathrm{Ker}(\kappa_T)$, where $T \subset K$ is a maximal torus. In the appendix we prove that the same relationships remain true in the case when 0 is no longer a regular value of μ_T.

1. Introduction

Let M be a compact symplectic manifold acted[1] on by a torus T in a Hamiltonian fashion. Assume that 0 is a regular value of the moment map $\mu : M \to \mathfrak{t}^* = (\text{Lie } T)^*$ and consider the *symplectic reduction*

$$M//T = \mu^{-1}(0)/T.$$

By a well-known result (see [Ki]), the Kirwan map $\kappa : H^*_T(M) \to H^*(M//T)$ induced by the inclusion $\mu^{-1}(0) \hookrightarrow M$ via the identification $H^*_T(M) = H^*(M//T)$ is surjective[2]. Determining the kernel of κ is an important problem: combined with a presentation of the equivariant cohomology ring $H^*_T(M)$, this gives a description of the cohomology ring of $M//T$.

Now consider a 1-dimensional torus $S \subset T$ with Lie algebra $\mathfrak{s}$ and let $\mu_S : M \to \mathfrak{s}^*$ be the moment map corresponding to the action of S on M. We will call S *generic* if

[1]The fixed point set of the action is not assumed to consist of isolated points, unlike the hypothesis in [Je] and [Je-Ma]).

[2]In this paper all cohomology groups are with complex coefficients.

(i) the two fixed point sets M^S and M^T are equal

(ii) 0 is a regular value of the moment map $\mu_S : M \to \mathfrak{s}^*$.

Denote by $M//S = \mu_S^{-1}(0)/S$ the symplectic reduction corresponding to the S action on M. In the same way as before, we can consider the map induced by the inclusion $\mu_S^{-1}(0) \hookrightarrow M$ at the level of the T-equivariant cohomology and identify at the same time

$$H_T^*(\mu_S^{-1}(0)) = H_{T/S}^*(\mu_S^{-1}(0)/S).$$

Goldin [Go] called the resulting map

$$\kappa_S : H_T^*(M) \to H_{T/S}^*(M//S)$$

the *equivariant Kirwan map* and she proved:

Theorem 1.1. (see [Go]) *If 0 is a regular value of μ_S, then the map κ_S is surjective.*

Our first theorem (Theorem 1.2) shows that the knowledge of $\mathrm{Ker}(\kappa_S)$, for a generic circle $S \subset T$, is crucial for determining the kernel of the Kirwan map

$$\kappa : H_T^*(M) \to H^*(M//T).$$

Theorem 1.2.

$$\mathrm{Ker}(\kappa) = \sum_S \mathrm{Ker}(\kappa_S),$$

where the sum runs over all generic circles $S \subset T$.

One aim of our paper is to describe the kernel of κ_S. Let us denote first by $\mathcal{F}$ the set of all connected components of the fixed point set M^T: note that the moment map μ is constant on each $F \in \mathcal{F}$. To any generic circle $S \subset T$ we assign the partition of $\mathcal{F}$ given by $\mathcal{F} = \mathcal{F}_- \cup \mathcal{F}_+$, where

$$\mathcal{F}_+ = \{F \in \mathcal{F} : \mu_S(F) > 0\}$$

and

$$\mathcal{F}_- = \{F \in \mathcal{F} : \mu_S(F) < 0\}.$$

Also, let $X, Y_1, \ldots, Y_m$ be variables corresponding to an integral basis of $\mathfrak{t}^*$ such that

(1) $\qquad X|_{\mathfrak{s}}$ corresponds to an integral element of $\mathfrak{s}^*$ and $Y_j|_{\mathfrak{s}} = 0, j \geq 1$.

In this way, we have

$$H_T^*(\mathrm{pt}) = S(\mathfrak{t}^*) = \mathbb{C}[X, Y_1, \ldots, Y_m].$$

The following definition (Definition 1.3 for $\mathrm{Ker}_{\mathrm{res}}$)[3] is suggested by the residue formula of [Je-Ki1] and [Je-Ki2]:

$$\mathrm{Ker}(\kappa_S) = \mathrm{Ker}_{\mathrm{res}}(\kappa_S). \tag{2}$$

[3] For more details, we refer the reader to section 2.

Definition 1.3. *(i) The* residue kernel $\mathrm{Ker}_{\mathrm{res}}(\kappa_S)$ *is the set of all classes* $\eta \in H_T^*(M)$ *with the property that*

$$\mathrm{Res}_X^+ \sum_{F \in \mathcal{F}_+} \int_F \frac{i_F^*(\eta\zeta)}{e_F} = 0$$

for all $\zeta \in H_T^*(M)$*. Here* $e_F = e(\nu F) \in H_T^*(F)$ *is the equivariant Euler class of the normal bundle of* F *and* Res_X^+ *is defined by*

$$\mathrm{Res}_X^+(h) = \sum_{b \in \mathbb{C}} \mathrm{Res}_{X=b} h \tag{3}$$

for every rational function h *in the variables* $X, Y_1, \ldots, Y_m$*.*

(ii) We define the residue kernel of κ *as*

$$\mathrm{Ker}_{\mathrm{res}}(\kappa) = \sum_S \mathrm{Ker}_{\mathrm{res}}(\kappa_S)$$

We shall prove the following result:

Theorem 1.4. *If* $S \subset T$ *is a generic circle, then we have that*

$$\mathrm{Ker}_{\mathrm{res}}(\kappa_S) = K_-^S \oplus K_+^S, \tag{4}$$

where K_-^S *(resp.* K_+^S*) denote the set of all equivariant cohomology classes* η *whose restriction to* $\mathcal{F}_-^S$ *(resp.* $\mathcal{F}_+^S$*) is zero.*

We remark that we prove Theorem 1.4 directly, without using the fact that the right and left hand sides of (4) are known to be the kernel of κ_S (by results of [Je-Ki1] and Goldin [Go]).

As a corollary of Theorem 1.4, we deduce the following description of $\mathrm{Ker}_{\mathrm{res}}(\kappa)$:

Corollary 1.5. *The residue kernel of the Kirwan map* $\kappa : H_T^*(M) \to H^*(M//T)$ *is given by*

$$\mathrm{Ker}_{\mathrm{res}}(\kappa) = \sum_S (K_-^S \oplus K_+^S),$$

where S *is as in Theorem 1.2. Here* $K_\pm^S$ *consist of all equivariant cohomology classes* α *which restrict to zero on all components* $F \in \mathcal{F}$ *with the property that*

$$\pm\mu(F)(\xi) > 0$$

where ξ *is a fixed non-zero vector in the Lie algebra of* S*.*

Proof of Corollary: The corollary follows immediately from Theorem 1.4 because Theorem 1.4 tells us that

$$\mathrm{Ker}_{\mathrm{res}}(\kappa_S) = K_-^S \oplus K_+^S$$

so

$$\sum_S \mathrm{Ker}_{\mathrm{res}}(\kappa_S) = \sum_S (K_-^S \oplus K_+^S)$$

which is $\mathrm{Ker}_{\mathrm{res}}(\kappa)$ by definition (Definition 1.3). □

Tolman-Weitsman's theorem [To-We] is as follows:

Theorem 1.6. *We have*

$$\mathrm{Ker}(\kappa) = \sum_S (K_-^S \oplus K_+^S). \tag{5}$$

Proof of Tolman-Weitsman's theorem: We obtain Tolman-Weitsman's theorem by the following steps.

Step 1: Theorem 1.2 tells us that $\mathrm{Ker}(\kappa) = \sum_S \mathrm{Ker}(\kappa_S)$
Step 2: Results of [Je-Ki1] and [Je-Ki2] tell us that $\mathrm{Ker}(\kappa_S) = \mathrm{Ker}_{\mathrm{res}}(\kappa_S)$
Step 3: By Theorem 1.4 $\mathrm{Ker}_{\mathrm{res}}(\kappa_S) = K_-^S \oplus K_+^S$. □

The key step is Step 3, which uses residues which enable us to easily exhibit the structure of the residue kernel as a sum of classes vanishing on one side of certain hyperplanes. We must emphasize that it is unexpected that one can deduce Tolman-Weitsman's result (Theorem 1.6) by this means; obtaining a new proof of this result was not one of our original goals, but rather an unanticipated byproduct of our analysis.

Our goal was to prove directly that the residue kernel $\mathrm{Ker}_{\mathrm{res}}$ (see Definition 1.3) equals the Tolman-Weitsman kernel (i.e. the right hand side of (5)). We obtain this result (Corollary 1.5) by combining Theorem 1.2 and Theorem 1.4. This directly generalizes the main result of [Je]. The two descriptions of the kernel of the Kirwan map are apparently quite different, so it is illuminating to see directly that they coincide.

The final goal of the paper is to describe the kernel of the Kirwan map in the case of the Hamiltonian action of a non-abelian Lie group. Let K be an arbitrary compact connected Lie group with maximal torus $T \subset K$ and let $W = N_K(T)/T$ be the corresponding Weyl group. If M is a K manifold, then there exists a natural action of W on $H_T^*(M)$. If $H_T^*(M)^W$ denotes the space of W-invariant cohomology classes, then we have the isomorphism

$$H_K^*(M) = H_T^*(M)^W,$$

(see [At-Bo1]) which allows us to identify the two spaces. Let us also denote by $\mathcal{D}$ the element of $H_T^*(\mathrm{pt}) = S(\mathfrak{t}^*)$ given by the product of all positive roots. Suppose now that M is a compact symplectic manifold and the action of K is Hamiltonian. Assume that 0 is a regular value of the moment map

$$\mu_K : M \to \mathfrak{k}^*$$

and consider the symplectic quotient

$$M//K = \mu_K^{-1}(0)/K.$$

The Kirwan surjection (see [Ki])

$$\kappa_K : H_K^*(M) \to H^*(M//K)$$

is, exactly as in the abelian case, induced by restriction from M to $\mu_K^{-1}(0)$ followed by the isomorphism

$$H_K^*(\mu_K^{-1}(0)) \simeq H^*(\mu_K^{-1}(0)/K).$$

The following result was proved in [Je-Ki1]:

Theorem 1.7. (non-abelian residue formula [Je-Ki1], [Je-Ki2]). *For every cohomology class $\eta \in H_K^*(M) = H_T^*(M)^W$ we have that*

$$\kappa_K(\eta)[M//K] = c_1 \mathrm{Res}(\sum_{F \in \mathcal{F}} \int_F \frac{\mathcal{D}^2 \eta|_F}{e_F}) \tag{6}$$

where c_1 is a non-zero constant, $\mathcal{F}$ is the set of connected components of the fixed point set M^T and Res *is a certain complex valued operator*[4] *on the space of functions of the form p/q, with $p, q \in S(\mathfrak{t}^*)$, $q \neq 0$.*

In the case $K = T$, the class $\mathcal{D}$ is 1, and we obtain immediately the *abelian residue formula*: for every $\eta \in H_T^*(M)$ we have that

$$\kappa_T(\eta)[M//T] = c_2 \mathrm{Res}(\sum_{F \in \mathcal{F}} \int_F \frac{\eta|_F}{e_F}) \tag{7}$$

where Res is the same operator as the one from equation (6) and c_2 is a non-zero constant. Also note that the residue formula (6) gives a complete description of $\mathrm{Ker}(\kappa_K)$, since for $\eta \in H_K^*(M)$ we have that

$$\kappa_K(\eta) = 0 \quad \text{iff} \quad \kappa_K(\eta\zeta)[M//K] = 0 \text{ for every } \zeta \in H_K^*(M). \tag{8}$$

We would like to make this description more explicit, by giving a direct relationship between $\mathrm{Ker}(\kappa_K)$ and $\mathrm{Ker}(\kappa_T)$ (the latter being given by Corollary 1.5). We will prove that:

Theorem 1.8. *Take $\eta \in H_K^*(M) = H_T^*(M)^W$. The following assertions are equivalent:*

(i) $\kappa_K(\eta) = 0$,
(ii) $\kappa_T(\mathcal{D}\eta) = 0$,
(iii) $\kappa_T(\mathcal{D}^2\eta) = 0$.

The proof will be given in section 5. Some explicit relationships between $\mathrm{Ker}(\kappa_K)$ and $\mathrm{Ker}(\kappa_T)$ will also be discussed there.

The layout of our paper is as follows. In Section 2 we recall the characterizations of residues from [Gu-K], [Je-Ki1], [Je-Ki2], and summarize some results from Morse theory which will be used later. Section 3 is devoted to a proof of Theorem 1.4, and Section 4 to the proof of Theorem 1.2. Finally Section 5 gives a characterization of the kernel of κ for nonabelian group actions.

[4]For the exact "residue formula", the reader is referred to [Je-Ki1] and [Je-Ki2]; but for the purposes of the present paper, equation (6) is sufficient. A more detailed description is given in Section 2.2, where it is used.

Remarks.

1: The chief advances described in this article are as follows.

(a): We prove Theorem 1.2 (which relates the kernel of the Kirwan map to the kernels of the equivariant Kirwan maps κ_S).

(b): We prove Theorem 1.4 (which characterizes the residue kernel $\mathrm{Ker}_{\mathrm{res}}(\kappa_S)$ as linear combinations of elements of equivariant cohomology vanishing on the preimage of one side of a hyperplane).

(c): Our construction does not assume isolated fixed points. Theorem 1.4 is a generalization of the main results of [Je] and [Je-Ma], where actions with isolated fixed points were considered.

(d): We give an independent characterization (Theorem 1.8) of the kernel of the Kirwan map for nonabelian group actions. According to Tolman and Weitsman [To-We], the equivalence (i) $\iff$ (ii) from Theorem 1.8 can be deduced from S. Martin's integral formula (see [Ma, Theorem B]). Our proof of this equivalence relies on the residue formulas (6) and (7).

2: The characterization of $\mathrm{Ker}(\kappa_S)$ given in Theorem 1.4 (see equation (2)) was proved by Goldin in [Go], by using the theorem of Tolman and Weitsman [Theorem 3, To-We]. The strategy of our paper is different: we first prove the formula stated in Theorem 1.4 and then deduce the theorem of Tolman and Weitsman starting from Corollary 1.5 using Theorem 1.2 and the known fact that $\mathrm{Ker}(\kappa_S) = \mathrm{Ker}_{\mathrm{res}}(\kappa_S)$.

2. Residue formulas and Morse-Kirwan theory

The goal of this section is to provide some fundamental definitions and results — which will be needed later — concerning the notions mentioned in the title.

2.1. **The Kirwan map in terms of residues.** Let us consider the compact symplectic manifold M equipped with the Hamiltonian action of the torus T. Let $S \subset T$ be a generic circle and consider the variables $X, Y_1, \ldots, Y_m$ determined by (1) (see section 1). The following result was proved in [Je-Ki1] and [Je-Ki2]:

Theorem 2.1.1. *For all $\eta \in H_T^*(M)$ we have*[5]

$$\kappa_S(\eta)[M//S] = c \sum_{F \in \mathcal{F}_+} \mathrm{Res}_X^+ \int_F \frac{i_F^*(\eta)}{e_F}. \tag{9}$$

Here $e_F \in H_T^*(F)$ denotes the T-equivariant Euler class of the normal bundle $\nu(F)$, c is a non-zero constant[6]*, and the meaning of $\mathrm{Res}_X^+ \int_F$ is given in (11)*

[5]Note that both sides of the equation are in $\mathbb{C}[Y_1, \ldots, Y_m]$.

[6]The precise value of c is not needed in our paper, but the interested reader can find it in [Je-Ki2, §3].

For all $\alpha \in H_T^*(F)$, we obtain

$$\mathrm{Res}_X^+ \int_F \frac{\alpha}{e_F}$$

as follows: We may assume that the normal bundle $\nu(F)$ splits as

$$\nu(F) = L_1 \oplus \ldots \oplus L_k$$

where L_i, $1 \leq i \leq k$ are T-equivariant complex line bundles. We express the reciprocal of e_F as

$$\frac{1}{e_F} = \prod_{i=1}^{k} \frac{1}{(m_i X + \beta_i(Y) + c_1(L_i))} = \prod_{i=1}^{k} \sum_{r_j \geq 0} \frac{(-c_1(L_i))^{r_j}}{(m_i X + \beta_i(Y))^{r_j+1}}. \tag{10}$$

Here m_i is the weight of the representation of S on L_i, $\beta_i(Y)$ is a certain linear combination of $Y_1, \ldots, Y_m$ with integer coefficients, and $c_1(L_i)$ is the first Chern class of L_i. Since $c_1(L_i)$ is in $H^2(F)$, it is nilpotent, hence all sums involved in (10) are finite. The class α is in $H_T^*(F) = H^*(F) \otimes \mathbb{C}[X, \{Y_i\}]$, hence by (10),

$$\int_F \frac{\alpha}{e_F}$$

appears as a sum of rational functions of the type

$$h(X, Y_1, \ldots, Y_m) = \frac{p(X, Y_1, \ldots, Y_m)}{q(X, Y_1, \ldots, Y_m)}$$

where $p, q \in \mathbb{C}[X, \{Y_i\}]$, with $q = \prod_j (n_j X + \sum_l n_{jl} Y_l)$, $n_j \in \mathbb{Z} \setminus \{0\}$ and $n_{jl} \in \mathbb{Z}$. In order to define $\mathrm{Res}_X^+(h)$, regard $Y_1, \ldots, Y_m$ as complex constants and set

$$\mathrm{Res}_X^+(h) = \sum_{b \in \mathbb{C}} \mathrm{Res}_{X=b} \frac{p}{q} \tag{11}$$

where on the right hand side $\frac{p}{q}$ is interpreted as a meromorphic function in the variable X on $\mathbb{C}$.

Remark 2.1.2 Guillemin and Kalkman gave another definition of the residue involved in Theorem 2.1.1. (see [Gu-Ka, section 3]). Write

$$\begin{aligned}
\frac{1}{e_F} &= \prod_{i=1}^{k} \frac{1}{(m_i X + \beta_i(Y) + c_1(L_i))} \\
&= \prod_{i=1}^{k} \frac{1}{m_i X (1 + \frac{\beta_i(Y) + c_1(L_i)}{m_i X})} \\
&= \prod_{i=1}^{k} \left(\frac{1}{m_i X} - \frac{\beta_i(Y) + c_1(L_i)}{(m_i X)^2} + \frac{(\beta_i(Y) + c_1(L_i))^2}{(m_i X)^3} - \ldots\right).
\end{aligned}$$

If we multiply the k power series in X in the last expression and then multiply the result by the polynomial $\alpha \in H^*(F) \otimes \mathbb{C}[X, \{Y_i\}]$, we obtain a series

$$\sum_{r=r_0}^{-\infty} \gamma_r X^r,$$

where γ_r are in $H^*(F) \otimes \mathbb{C}[\{Y_i\}]$. The Guillemin-Kalkman residue is $\int_F \gamma_{-1}$. It is a simple exercise to show that the latter coincides with the expression

$$\mathrm{Res}_X^+ \int_F \frac{\alpha}{e_F}$$

of [Je-Ki2] which we have defined above.

Definition 2.1. *(see Section 3 of* [Je-Ki2]*) If h is a meromorphic function of $m+1$ variables $X_1, \dots, X_{m+1}$ then*

$$\mathrm{Res}(h(X)) = \triangle \mathrm{Res}_{X_1}^+ \circ \dots \circ \mathrm{Res}_{X_{m+1}}^+ (h(X)) \tag{12}$$

where the variables $X_1, \dots, X_m$ are held constant while calculating $\mathrm{Res}_{X_{m+1}}^+$, and $\triangle$ is the determinant of some $(m+1) \times (m+1)$ matrix whose columns are the coordinates of an orthonormal basis of $\mathbf{t}$ defining the same orientation as the chosen coordinate system. Here the symbols $\mathrm{Res}_{X_j}^+$ were defined in (11).

Definition 2.2. *If $\alpha, \beta \in H_T^*(M)$ then we define the complex number*

$$\mathrm{Res}(\alpha\beta) = \mathrm{Res}(\sum_F \int_F \frac{\alpha\beta}{e_F}).$$

Here F are the components of the fixed point set of the T action, and e_F is the equivariant Euler class of the normal bundle.

By (2) and Theorem 1.2 (which will be proved in Section 4) we have

$$\mathrm{Ker}(\kappa) = \mathrm{Ker}_{\mathrm{res}}(\kappa). \tag{13}$$

By Corollary 1.5 we have also that

$$\mathrm{Ker}(\kappa) = \sum_S (K_-^S \oplus K_+^S).$$

By [Je-Ki1] we have that

$$\mathrm{Ker}(\kappa) = \{\alpha \in H_T^*(M) | \mathrm{Res}(\alpha\beta) = 0 \ \forall \ \beta \in H_T^*(M)\}. \tag{14}$$

Combining these observations we obtain

Theorem 2.3. *A class $\alpha \in H_T^*(M)$ satisfies $\mathrm{Res}(\alpha\beta) = 0$ for all $\beta \in H_T^*(M)$ if and only if $\alpha \in \sum_S K_S^- \oplus K_S^+$.*

$\square$

2.2. **Kirwan-Morse theory.** Let $S \subset T$ be a generic circular subgroup of a torus T which acts in a Hamiltonian manner on the compact symplectic manifold M. One knows that the critical points of the moment map

$$\mu_S = f : M \to \mathfrak{s}^* = \mathbb{R}$$

are the fixed points of the S action: as usual, we will denote by $\mathcal{F}$ the set of connected components of

$$M^T = M^S = \mathrm{Crit}(f).$$

Moreover, for every $F \in \mathcal{F}$, the restriction of the Hessian of f to the normal space to F is nondegenerate, which means that f is a Morse-Bott function. We assume for simplicity that the values of f on any two elements of $\mathcal{F}$ are different.

Fix $F \in \mathcal{F}$ and let νF be the normal bundle. Consider the T-invariant splitting,

$$\nu F = \nu^- F \oplus \nu^+ F$$

into the negative and positive subspaces of the Hessian. The rank of the bundle $\nu^- F$ is just the index $\mathrm{Ind}(F)$ of the critical level F. The following result follows from [At-Bo2] (for more details, see [Go, section 2]):

Theorem 2.2.1. *Suppose that ϵ is sufficiently small that $f(F)$ is the only critical value in the interval $(f(F)-\epsilon, f(F)+\epsilon)$ and consider*

$$M_- = f^{-1}(-\infty, f(F)-\epsilon), \quad M_+ = f^{-1}(-\infty, f(F)+\epsilon).$$

a) There exists a $H_T^*(\mathrm{pt})$-linear map[7]

$$r : H_T^{*-\mathrm{Ind}F}(F) \to H_T^*(M_+)$$

such that

- *the sequence*

$$0 \to H_T^{*-\mathrm{Ind}F}(F) \xrightarrow{r} H_T^*(M_+) \to H_T^*(M_-) \to 0 \tag{15}$$

is exact,

- *the composition of the restriction map*

$$H_T^*(M_+) \to H_T^*(F)$$

with r is the multiplication by the equivariant Euler class $e_T(\nu^-(F))$; moreover, the image of r consists of all classes in $H_T^*(M_+)$ whose restriction to F is a multiple of $e_T(\nu_f^- F)$.

b) The maps $H_T^*(F) \to H_T^*(F)$ given by the multiplication by $e_T(\nu^-(F))$ and $e_T(\nu^+(F))$ are injective (in other words $e_T(\nu^\pm(F))$ are not zero divisors).

[7] The map r is in fact the restriction map $H_T^*(M_+, M_-) \to H_T^*(M_+)$ composed with the inverse of the excision isomorphism $H_T^*(M_+, M_-) \to H_T^*(D^{\mathrm{Ind}F}, S^{\mathrm{Ind}F-1})$ and then with the inverse of the Thom isomorphism $H_T^*(D^{\mathrm{Ind}F}, S^{\mathrm{Ind}F-1}) \to H_T^{*-\mathrm{Ind}F}(F)$, where $D^{\mathrm{Ind}F}$ is the cell bundle over F attached to M_- after passing the critical level F, and $S^{\mathrm{Ind}F-1}$ is the corresponding sphere bundle over F.

c) The restriction map

$$H_T^*(M) \to H_T^*(M_+)$$

is surjective.

d) There exists a class $\alpha^+(F) \in H_T^*(M)$ *with the following properties:*

- $\alpha^+(F)|_G = 0$ *for all* $G \in \mathcal{F}$ *with* $f(G) > f(F)$*).*
- $\alpha^+(F)|_F = e_T(\nu^+(F))$

Remark 2.2.2 A similar result holds if we take instead of f (a generic component of the moment map $\mu : M \to \mathfrak{t}^*$) the function

$$g(x) = \|\mu(x)\|^2, \quad x \in M.$$

Although g is no longer Morse-Bott, like f, it is *minimally degenerate* in the sense defined by Kirwan [Ki]. In particular the critical set $\mathrm{Crit}(g)$ is a disjoint union of T-invariant closed subspaces C, call them the *critical sets of* g, with the following properties:

- μ is constant on C;
- the subbundle $\nu_g^-(C)$ of $TM|_C$ given by the negative space of the Hessian of g has constant dimension along C, call it $\mathrm{Ind}C$;
- the equivariant Euler class $e_T(\nu_g^-(C))$ is not a zero-divisor in $H_T^*(C)$.

This is sufficient to deduce that for any critical set C, the result stated at point a) of the previous theorem remains true if we replace f by g and F by C.

3. The residue kernel of the equivariant Kirwan map

The goal of this section is to give a proof of Theorem 1.4. This theorem is similar to the main result of [Je]: it is a generalization of [Je] in that the proof does not require the hypothesis of isolated fixed points and the argument is generalized to equivariant Kirwan maps. Like the argument in [Je], our argument makes use of residues.

From the definition of $\mathrm{Ker}_{\mathrm{res}}(\kappa_S)$ (see Definition 1.4), it is obvious that the latter contains K_+. In order to prove that it contains K_- as well, we only have to notice that, by the Atiyah-Bott-Berline-Vergne localization formula (see [At-Bo1] and [Be-Ve]), we have that

$$\mathrm{Res}_X^+ \sum_{F \in \mathcal{F}_+} \int_F \frac{i_F^*(\eta\zeta)}{e_F} = -\mathrm{Res}_X^+ \sum_{F \in \mathcal{F}_-} \int_F \frac{i_F^*(\eta\zeta)}{e_F}.$$

The difficult part of the proof will be to show that

$$\mathrm{Ker}_{\mathrm{res}}(\kappa_S) \subset K_- \oplus K_+.$$

Let us start with the following two lemmas.

Lemma 3.1. *If F is in $\mathcal{F}$ and $\alpha \in H_T^*(F)$ satisfies*

$$\int_F i_F^*(\zeta)\alpha = 0 \tag{16}$$

for all $\zeta \in H^*(M)$, then

$$\alpha = 0.$$

Proof. We have

$$\int_F i_F^*(\zeta)\alpha = \int_M \zeta i_{F*}(\alpha),$$

where $i_{F*} : H_T^*(F) \to H_T^{*+\mathrm{codim} F}(M)$ is the push-forward of the inclusion map $i_F : F \to M$. So equation (16) leads to

$$i_{F*}(\alpha) = 0.$$

We apply i_F^* on both sides of this equation and use the fact that the map $i_F^* \circ i_{F*}$ is just the multiplication by the Euler class e_F in order to deduce that

$$\alpha e_F = 0.$$

But e_F is the same as the product $e(\nu^- F)e(\nu^+ F)$, where neither factor is a zero divisor (see Theorem 2.2.1 (b)). So we must have $\alpha = 0$. □

Lemma 3.2. *Fix $F \in \mathcal{F}$ and suppose that $\eta \in H_T^*(M)$ satisfies*

$$\mathrm{Res}_X^+ \int_F \frac{i_F^*(\eta\beta)}{e(\nu^- F)} = 0 \tag{17}$$

for all $\beta \in H_T^*(M)$. Then there exists $\gamma \in H_T^*(M)$ such that

$$\eta|_F = \gamma|_F$$

and

$$\gamma|_G = 0 \quad \textit{for all } G \in \mathcal{F} \text{ with } f(G) < f(F).$$

Proof. Consider a T-invariant splitting of the normal bundle into a sum of complex line bundles. As explained in Remark 2.1.2, this decomposition leads to the series

$$\frac{i_F^*(\eta)}{e(\nu^- F)} = \sum_{r \le r_0} \gamma_r X^r,$$

with $\gamma_r \in H^*(F) \otimes \mathbb{C}[\{Y_i\}]$. For $\zeta \in H^*(M)$ we have that

$$\frac{i_F^*(\eta\zeta)}{e(\nu^- F)} = \sum_{r \le r_0} i_F^*(\zeta)\gamma_r X^r.$$

The hypothesis of the lemma implies that

$$\int_F i_F^*(\zeta)\gamma_{-1} = 0,$$

for all $\zeta \in H^*(M)$. From Lemma 3.1 it follows that

$$\gamma_{-1} = 0.$$

Now take an arbitrary class ζ in $H^*(M)$ and put $\beta = X\zeta$ in (17). We deduce that $\gamma_{-2} = 0$. Then we take $\beta = X^2\zeta$ and deduce that $\gamma_{-3} = 0$ etc. Consequently we have that

$$\eta|_F = i_F^*(\eta) = \alpha e(\nu^- F),$$

where

$$\alpha = \sum_{r=0}^{r_0} X^r \gamma_r \in H_T^*(F).$$

Theorem 2.2.1 (a) says that the multiplication by $e(\nu^- F)$ is the composition of the restriction map $H_T^*(M_+) \to H_T^*(F)$ with r; so we have

$$\eta|_F = r(\gamma')|_F$$

where $\gamma' \in H_T^*(M_+)$. Note that by the exact sequence

$$0 \to H_T^{*-\mathrm{Ind} F}(F) \xrightarrow{r} H_T^*(M_+) \to H_T^*(M_-) \to 0$$

(see Theorem 2.2.1 (a)) we have that

$$r(\gamma')|_G = 0$$

for all $G \in \mathcal{F}$ with $f(G) < f(F)$. Finally we obtain $\gamma \in H_T^*(M)$ with the properties required in the lemma by extending $r(\gamma') \in H_T^*(M_+)$ to the whole M (see Theorem 2.2.1 (c)). $\square$

We are now ready to prove Theorem 1.4.

Proof of Theorem 1.4 Denote

$$f = \mu_S : M \to \mathfrak{s}^* = \mathbb{R}$$

the moment map of the S action on M. Take $\eta \in H_T^*(M)$ satisfying

$$\mathrm{Res}_X^+ \sum_{F \in \mathcal{F}_+} \int_F \frac{i_F^*(\eta\zeta)}{e_F} = 0 \tag{18}$$

for all $\zeta \in H_T^*(M)$. Consider the ordering $F_1, F_2, \ldots, F_N$ of the elements of $\mathcal{F}_+$ such that

$$0 < f(F_1) < f(F_2) < \ldots < f(F_N).$$

We shall inductively construct classes $\gamma_1, \gamma_2, \ldots, \gamma_N \in H_T^*(M)$ which vanish when restricted to $\mathcal{F}_-$ and such that for all $1 \le k \le N$, the form

$$\eta_k = \eta - \gamma_1 - \ldots - \gamma_k$$

vanishes when restricted to $F_1, F_2, \ldots, F_k$. If we set

$$\eta_- = \gamma_1 + \ldots + \gamma_N, \quad \eta_+ = \eta - \eta_-,$$

then the decomposition

$$\eta = \eta_- + \eta_+$$

has the desired properties.

First, in (18) we put

$$\zeta = \beta\alpha^+(F_1),$$

where $\beta \in H_T^*(M)$ is an arbitrary cohomology class and the notation $\alpha^+(F_1)$ was introduced in Theorem 2.2.1 (d). Since

$$e(\nu_f^-(F_1))e(\nu_f^+(F_1)) = e_{F_1},$$

we deduce that

$$\mathrm{Res}_X^+ \int_{F_1} \frac{i_{F_1}^*(\beta\eta)}{e(\nu^-(F_1))} = 0 \text{ for all } \beta \in H_T^*(M),$$

where we are using the fact that $\alpha^+(F_1)|_F = 0$ if $f(F) > f(F_1)$. From Lemma 3.2 it follows that there exists $\gamma_1 \in H_T^*(M)$ such that

$$\eta|_{F_1} = \gamma_1|_{F_1},$$

and

$$\gamma_1|_G = 0, \text{ for all } G \in \mathcal{F} \text{ with } f(G) < f(F_1).$$

Next suppose that we have constructed $\gamma_1, \ldots, \gamma_k$ which vanish when restricted to $\mathcal{F}_-$ and such that

$$\eta_k = \eta - \gamma_1 - \ldots - \gamma_k$$

vanishes when restricted to $F_1, \ldots, F_k$. We claim that η_k satisfies

$$\mathrm{Res}_X^+ \sum_{F \in \mathcal{F}_+} \int_F \frac{i_F^*(\eta_k\zeta)}{e_F} = 0.$$

This follows from equation (18) and the Atiyah-Bott-Berline-Vergne localization formula for $(\gamma_1 + \ldots + \gamma_k)\zeta$ (note that the restriction of the form $(\gamma_1 + \ldots + \gamma_k)\zeta$ to $\mathcal{F}_-$ is zero). Since $\eta_k|_G = 0$ for all $G \in \mathcal{F}_+$ with $f(G) \leq f(F_k)$, we have that

$$\mathrm{Res}_X^+ \sum_{F \in \mathcal{F}_+, f(F) f(F_k)} \int_F \frac{i_F^*(\eta_k\zeta)}{e_F} = 0. \tag{19}$$

In (19) we put $\zeta = \beta\alpha^+(F_{k+1})$, where β is an arbitrary element in $H_T^*(M)$. Since

$$e(\nu_f^-(F_{k+1}))e(\nu_f^+(F_{k+1})) = e_{F_{k+1}},$$

we deduce that

$$\mathrm{Res}_X^+ \int_{F_{k+1}} \frac{i_{F_{k+1}}^*(\beta\eta)}{e(\nu^-(F_{k+1}))} = 0$$

where we are using the fact that $\alpha^+(F_{k+1})|_F = 0$ if $f(F) > f(F_{k+1})$. From Lemma 3.2 it follows that there exists $\gamma_{k+1} \in H_T^*(M)$ such that

$$\eta_k|_{F_{k+1}} = \gamma_{k+1}|_{F_{k+1}},$$

and

$$\gamma_{k+1}|_G = 0$$

for all $G \in \mathcal{F}$ with $f(G) < f(F_{k+1})$. This means that the form

$$\eta_{k+1} = \eta_k - \gamma_{k+1} = \eta - \gamma_1 - \ldots - \gamma_{k+1}$$

vanishes when restricted to F_{k+1} and for all $1 \leq j \leq k$, we have that

$$\eta_{k+1}|_{F_j} = \eta_k|_{F_j} = 0.$$

$\square$

4. The kernel of $\kappa : H_T^*(M) \to H^*(M//T)$ via equivariant Kirwan maps

We will provide a proof of Theorem 1.2. A similar result is stated in [Go] (where it is proved by expressing κ as a composition of κ_S). The general strategy we will use is the same as in the proof of Theorem 3 of [To-We].

Proof of Theorem 1.2 First, it is simple to prove that for every generic circle $S \subset T$, we have that

$$\mathrm{Ker}\kappa_S \subset \mathrm{Ker}\kappa.$$

We only have to take into account the residue formulas (9) and (7), where in the latter formula res is an iterated residue which starts with Res_X^+.

Let us prove that

$$\mathrm{Ker}\kappa \subset \sum_S \mathrm{Ker}\kappa_S.$$

To this end, we consider the ordering $\mu^{-1}(0) = C_0, C_1, C_2, \ldots, C_p$ of the critical sets of g such that

$$i < j \Rightarrow g(C_i) < g(C_j).$$

Take η in $\mathrm{Ker}\kappa$, i.e. $\eta|_{C_0} = 0$. We will show by induction on $0 \leq k \leq p$ that there exists $\eta_k \in \sum_S \mathrm{Ker}\kappa_S$ such that

$$\eta_k|_{C_i} = \eta|_{C_i}, \quad \text{for all } 0 \leq i \leq k.$$

We start with $\eta_0 = 0$, and at the end of the process we will obtain the form $\eta_p \in \sum_S \mathrm{Ker}\kappa_S$ which, by the Kirwan injectivity theorem[8], is the same as η.

Suppose that we have η_k and want to construct η_{k+1}. The form $\eta - \eta_k$ vanishes on all critical sets C with

$$g(C) < g(C_{k+1}).$$

By Remark 2.2.2, $(\eta - \eta_k)|_{C_{k+1}}$ is a multiple of $e_T(\nu_g^-(C_{k+1}))$. Consider the function

$$h(x) = \|\mu(x) + \lambda\mu(C_{k+1})\|^2, \quad x \in M,$$

where λ is a sufficiently large positive real number (see below). One can easily check that C_{k+1} is a critical set of g and the negative spaces of the Hessians of f and g at any point

[8]Note that $M^T \subset \mathrm{Crit}(g)$.

in C_{k+1} are the same. We use Remark 2.2.2 for the function h, which is the norm squared of a moment map: Because $(\eta - \eta_k)_{C_{k+1}}$ is a multiple of $e_T(\nu_g^-(C_{k+1})) = e_T(\nu_h^-(C_{k+1}))$, we deduce that there exists a form $\beta_k \in H_T^*(M)$ which vanishes on $h^{-1}(-\infty, h(C_{k+1}) - \epsilon)$ and such that

$$\beta_k|_{C_{k+1}} = (\eta - \eta_k)|_{C_{k+1}}.$$

In particular we have that

$$\beta_k|_C = 0, \quad \text{for all critical sets } C \text{ of } g \text{ with } h(C) < h(C_{k+1}). \tag{20}$$

So if we set $\eta_{k+1} := \eta_k + \beta_k$ then obviously

$$(\eta - \eta_{k+1})|_{C_j} = 0, \quad \text{for all } 0 \le j \le k+1.$$

Now we claim that there exists $\lambda \in \mathbb{R}$ such that the form β_k is in $\sum_S \mathrm{Ker}\kappa_S$. In fact we only need to choose λ sufficiently large that $h(F) < h(C_{k+1})$ for any $F \in \mathcal{F}$ with $\langle \mu(F), \mu(C_{k+1})\rangle < 0$. Note that any such F is contained in a critical set C, because $M^T \subset \mathrm{Crit}(g)$. Consequently $h(C) < h(C_{k+1})$, thus, by (20), we have that

$$\beta_k|_F = 0 \quad \text{for all } F \in \mathcal{F} \text{ with } \langle \mu(F), \mu(C_{k+1})\rangle < 0.$$

We deduce that there exists a generic circle S such that $\beta_k \in \mathrm{Ker}\kappa_S$. Indeed, the generic circle $S \subset T$ can be chosen such that the two components μ_S and $\langle \mu(\cdot), \mu(C_{k+1})\rangle$ of μ are sufficiently close such that

$$\mu_S(F) < 0 \Leftrightarrow \langle \mu(F), \mu(C_{k+1})\rangle < 0,$$

where $F \in \mathcal{F}$. We also use the characterization of $\mathrm{Ker}\kappa_S$ given by Theorem 1.4 (note that, in the notation of Theorem 1.4, we have $\beta_k \in K_-$). This finishes the proof. □

5. Hamiltonian actions of non-abelian Lie groups

The goal of this section is to prove Theorem 1.8. We will use the notations from the introduction. The action of W on $H_T^*(M)$ plays an important role. We say that an element η of $H_T^*(M)$ is *anti-invariant* if

$$w.\eta = \epsilon(w)\eta$$

for all $w \in W$, where $\epsilon(w)$ denotes the signature[9] of w. Note that

$$\epsilon(vw) = \epsilon(v)\epsilon(w) \tag{21}$$

for all $v, w \in W$. One can easily see that the element $\mathcal{D}$ of $H_T^*(\mathrm{pt}) = S(\mathfrak{t}^*)$ which is obtained by multiplying all positive roots (see also (6) and (7) from the introduction) is anti-invariant. The following result is proved in [Br]:

Lemma 5.1. (see [Br]) *The set of anti-invariant elements in $H_T^*(M)$ is $\mathcal{D} \cdot H_T^*(M)^W$.*

[9] By definition, $\epsilon(w)$ is $(-1)^{l(w)}$, where $l(w)$ is the length of w with respect to the generating set of W consisting of the reflections into the walls of a fixed Weyl chamber in $\mathfrak{t}$.

Now we are ready to prove Theorem 1.8.

Proof of Theorem 1.8. Let us consider the pairing $\langle\ ,\ \rangle$ on $H^*_T(M)$ given by

$$\langle \eta, \zeta \rangle = \kappa_T(\eta\zeta)[M//T],$$

$\eta, \zeta \in H^*_T(M)$. The kernel of $\langle\ ,\ \rangle$ is just $\mathrm{Ker}(\kappa_T)$. Since the map

$$\kappa_T : H^*_T(M) \to H^*(M//T)$$

is W-equivariant, the pairing $\langle\ ,\ \rangle$ is W-invariant.

First we show that (i) is equivalent to (iii). Take $\eta \in H^*_T(M)^W$. By equations (6), (7) and (8), the condition

$$\kappa_K(\eta) = 0$$

is equivalent to

$$\langle \mathcal{D}^2\eta, \zeta \rangle = 0, \text{ for all } \zeta \in H^*_T(M)^W \tag{22}$$

Now we show that (22) is equivalent to $\mathcal{D}^2\eta \in \mathrm{Ker}\langle\ ,\ \rangle$. Indeed, if ζ is an arbitrary element in $H^*_T(M)$, we can consider

$$\zeta' = \sum_{w\in W} w\zeta$$

which is W-invariant, hence

$$\langle \mathcal{D}^2\eta, \zeta' \rangle = |W| \langle \mathcal{D}^2\eta, \zeta \rangle \tag{23}$$

where we have used the W-invariance of both the pairing $\langle\ ,\ \rangle$ and the class $\mathcal{D}^2\eta$. Equation (23) finishes the proof of the equivalence between (i) and (iii).

Now we prove that (ii) is equivalent to (iii). In fact only the implication (iii) $\Rightarrow$ (ii) is non-trivial. So let us consider $\eta \in H^*_T(M)^W$ with the property that $\kappa_T(\mathcal{D}^2\eta) = 0$, which is equivalent to

$$\langle \mathcal{D}^2\eta, \zeta \rangle = 0$$

for every $\zeta \in H^*_T(M)$.
Claim 1. $\langle \mathcal{D}\eta, \zeta \rangle = 0$, for all $\zeta \in H^*_T(M)$ which is anti-invariant.
Indeed, if ζ is anti-invariant, by Lemma 5.1, it is of the form $\zeta = \mathcal{D}\xi$, with $\xi \in H^*_T(M)$, thus we have that

$$\langle \mathcal{D}\eta, \zeta \rangle = \langle \mathcal{D}\eta, \mathcal{D}\xi \rangle = \langle \mathcal{D}^2\eta, \xi \rangle = 0.$$

Claim 2. $\langle \mathcal{D}\eta, \zeta \rangle = 0$, for all $\zeta \in H^*_T(M)$.
This is because we can use Claim 1 for

$$\zeta' = \sum_{w\in W} \epsilon(w) w.\zeta$$

which is an anti-invariant equivariant cohomology class. More precisely, we notice that

$$\langle \mathcal{D}\eta, \zeta' \rangle = |W| \langle \mathcal{D}\eta, \zeta \rangle, \tag{24}$$

where we have used the W-invariance of $\langle \ , \ \rangle$ and the anti-invariance of $\mathcal{D}\eta$. The left hand side of (24) is zero by Claim 1, hence so must be the right hand side. $\square$

Remark. Explicit descriptions of $\mathrm{Ker}(\kappa_K)$ in terms of $\mathrm{Ker}(\kappa_T)$ can be easily deduced from Theorem 1.8, as follows:

$$\mathrm{Ker}(\kappa_K) = \{\frac{1}{\mathcal{D}} \sum_{w\in W} \epsilon(w) w.\eta \mid \eta \in \mathrm{Ker}(\kappa_T)\} \tag{25}$$

Alternatively, we obtain all elements of the kernel of κ_K if we consider all cohomology classes of the type $\sum_{w\in W} w.\eta$ with $\eta \in \mathrm{Ker}(\kappa_T)$ which are multiples of $\mathcal{D}^2$, and divide those by $\mathcal{D}^2$ (of course the description (25) is simpler, since by Lemma 5.1, for all $\eta \in H^*_T(M)$, the cohomology class $\sum_{w\in W} \epsilon(w) w.\eta$ is a multiple of $\mathcal{D}$). The presentation (25) is also given in [To-We, Proposition 6.1] (see also [Br, Corollaire 1]).

Appendix A. (By Jonathan Woolf)

This appendix explains how to prove Theorem 1.8 under weaker assumptions. In particular the assumption that 0 is a regular value for the T moment map can be relaxed in a wide class of cases. When 0 is not a regular value for the T moment map $M//T$ is, in general, singular. It has a natural stratification by symplectic orbifolds [Sj-Le] and, in particular, is a pseudomanifold with even dimensional strata. Thus we can define its intersection cohomology $IH^*(M//T)$ (with coefficients in $\mathbb{C}$) and there is a natural non-degenerate intersection pairing

$$\langle \ , \ \rangle_{M//T} : IH^*(M//T) \times IH^*(M//T) \to \mathbb{C}$$

generalising the intersection pairing on the cohomology of a manifold. We can extend the definition of the Kirwan map to obtain a map

$$\kappa_T : H^*_T(M) \to IH^*(M//T).$$

This construction was first carried out for the closely related case of geometric invariant theory quotients in algebraic geometry in [Ki3]. A version for symplectic quotients can be found in [Kie-Wo1]. It should be noted that when 0 is not a regular value of μ_T there are choices involved in this construction, and so κ_T is not canonical. We briefly sketch the construction.

Meinrenken and Sjamaar describe a partial desingularisation (orbifold singularities may remain) procedure for singular symplectic quotients in [Me-Sj]. (This is modelled on the algebro-geometric version in [Ki2].) Singularities in the quotient arise from Lie subgroups of T with fixed point subsets contained within $\mu_T^{-1}(0)$. There is a finite set $\mathcal{S}_M$ of such subgroups with dimension ≥ 1, and 0 is a regular value of μ_T if, and only if, this set is empty. If $S \in \mathcal{S}_M$ is of maximal dimension then the components of its fixed point subset contained within $\mu_T^{-1}(0)$ form a closed symplectic submanifold of M. Performing a T-equivariant symplectic blowup along this submanifold produces a symplectic manifold M_1 with a Hamiltonian T action such that $\mathcal{S}_{M_1} = \mathcal{S}_M - \{S\}$. Proceeding inductively we obtain a symplectic manifold $\widetilde{M} = M_r$ with a Hamiltonian T action and moment map $\tilde{\mu}_T$ such that $\mathcal{S}_{\widetilde{M}} = \emptyset$, or equivalently, 0 is a regular value of $\tilde{\mu}_T$. The blowdown $M_i \to M_{i-1}$ induces a continuous surjection $\pi_i : M_i//T \to M_{i-1}//T$. Composing these we obtain a continuous surjection from the symplectic orbifold $\widetilde{M}//T$ onto $M//T$ i.e. a partial desingularisation.

For each $i > 0$ we can choose, non-canonically, a surjection

$$IH^*(M_i//T) \to IH^*(M_{i-1}//T),$$

see [Kie-Wo1, Theorem 5]. We define a Kirwan map κ_T by composing the equivariant pullbacks with the Kirwan map $\tilde{\kappa}_T$ and these surjections:

$$\begin{tikzcd}
H_T^*(M) \arrow[r] \arrow[d, dashed, "\kappa_T"'] & H_T^*(M_1) \arrow[r] & \cdots \arrow[rr] & & H_T^*(\widetilde{M}) \arrow[d, "\tilde{\kappa}_T"] \\
IH^*(M//T) & IH^*(M_1//T) \arrow[l] & \cdots \arrow[l] & IH^*(\widetilde{M}//T) \arrow[l] & \cong H^*(\widetilde{M}//T)
\end{tikzcd}$$

What makes the Kirwan map for nonsingular quotients (when 0 is a regular value of μ_T) useful is its surjectivity. Unfortunately, it is not known whether the map defined above is always surjective when 0 is not a regular value of μ_T. However, we do have

Theorem A.1. *Suppose* 0 *is not a regular value of* μ_T. *Then, if the action of* T *on* M *is almost-balanced in the sense of* [Kie, §5] or *if* M *is a complex projective variety with symplectic structure given by the Fubini-Study form and the action of* T *is the restriction of an algebraic action of* $(\mathbb{C}^*)^n$, *the Kirwan map* κ_T *is surjective.*

Proof. See [Kie-Wo2], [Ki3] and [Wo]. □

We can use the Kirwan map to define a pairing on $H_T^*(M)$ by

$$\langle \eta, \zeta \rangle = \langle \kappa_T(\eta), \kappa_T(\zeta) \rangle_{M//T}$$

where the RHS is the intersection pairing on $IH^*(M//T)$. This extends the previous definition since $IH^*(M//T) \cong H^*(M//T)$ and $\langle \kappa_T(\eta), \kappa_T(\zeta) \rangle_{M//T} = \kappa_T(\eta\zeta)[M//T]$ when 0 is a regular value. If the Kirwan map is surjective we have

$$\kappa_T(\eta) = 0 \iff \langle \eta, \zeta \rangle = 0 \qquad \forall \zeta \in H_T^*(M). \tag{26}$$

Suppose the Hamiltonian T action on M arises as the restriction of a Hamiltonian K action. Then the normaliser $N_K(T)$ acts and preserves $\mu_T^{-1}(0)$, thereby inducing an action of the Weyl group $W = N_K(T)/T$ on $M//T$.

Lemma A.2. *The Kirwan map* $\kappa_T : H_T^*(M) \to IH^*(M//T)$ *can be chosen to be* W*-equivariant.*

Proof: The symplectic blowups in the partial desingularisation procedure can be done $N_K(T)$-equivariantly (so that $N_K(T)$ acts on each M_i and W on each $M_i//T$). The Kirwan map $\tilde{\kappa}_T$ is W-equivariant. It remains to see that the surjections $IH^*(M_i//T) \to$

$IH^*(M_{i-1}//T)$ can be chosen W-equivariantly. These surjections are constructed in [Kie-Wo1, §2] by decomposing appropriate complexes of sheaves in the constructible derived category of $M_{i-1}//T$. In the presence of a finite group action we can carry out a formally identical argument in the equivariant constructible derived category to obtain the required equivariant surjections. □

Corollary A.3. *The pairing $\langle\ ,\ \rangle$ on $H_T^*(M)$ is W-invariant.*

Proof. This follows immediately from the W-equivariance of κ_T and the W-invariance of the pairing $\langle\ ,\ \rangle_{M//T}$ on $IH^*(M//T)$. □

Now suppose further that 0 is a regular value for the K moment map μ_K on M. Then $\mu_K^{-1}(0)$ is a submanifold of M and the inclusion $\mu_K^{-1}(0) \hookrightarrow \mu_T^{-1}(0)$ has trivial normal bundle of dimension $d = \dim \mathfrak{k}^*/\mathfrak{t}^*$. However it is not *equivariantly* trivial — the equivariant Thom class is the product $\mathcal{D}$ of the positive roots. Provided that no finite subgroups of T fix points of $\mu_K^{-1}(0)$ then the inclusion $\mu_K^{-1}(0)/T \hookrightarrow \mu_T^{-1}(0)/T = M//T$ is also normally nonsingular.

Recall from [Go-Ma, §5.4] that, if $\imath : X \hookrightarrow Y$ is a normally nonsingular inclusion of pseudomanifolds with even dimensional stratifications, then there are maps

$$\imath^* : IH^*(Y) \to IH^*(X) \text{ and } \imath_* : IH^*(X) \to IH^{*+d}(Y).$$

These are adjoint to one another with respect to the intersection pairings i.e.

$$\langle \alpha, \imath_* \beta \rangle_Y = \langle \imath^* \alpha, \beta \rangle_X.$$

Lemma A.4. *Suppose that no finite subgroups of T fix points of $\mu_K^{-1}(0)$. Let $\imath : \mu_K^{-1}(0)/T \hookrightarrow M//T$ be the normally nonsingular inclusion. Then $\kappa_T(\mathcal{D}\eta) = \imath_* \imath^* \kappa_T(\eta)$.*

Proof. For the case when 0 is regular for μ_T this reduces to the statement that the identification $H_T^*(\mu_T^{-1}(0)) \cong H^*(\mu_T^{-1}(0)/T)$ takes the equivariant Thom class $\mathcal{D}$ of the submanifold $\mu_K^{-1}(0)$ to the Thom class of the quotient $\mu_K^{-1}(0)/T$ in $\mu_T^{-1}(0)/T$. Moreover this statement is local to $\mu_K^{-1}(0)$. Hence the identity holds more generally because, when we partially desingularise, we only blow up along centres which do not meet $\mu_K^{-1}(0)$ i.e. we do not alter anything in a neighbourhood of $\mu_K^{-1}(0)$. □

Remark. The restriction that no finite subgroups of T fix points of $\mu_K^{-1}(0)$ is not necessary for the proof of Lemma A.4 (and hence for what follows). If we relax it then the inclusion $\mu_K^{-1}(0)/T \hookrightarrow M//T$ will not necessarily be normally nonsingular, the normal fibres may be orbifolds. However, the extensions to Goresky and MacPherson's results on normally nonsingular inclusions can be obtained by working in an appropriate equivariant derived category cf. [Kie-Wo1, Remark 5 and §3].

Corollary A.5. *For $\eta, \zeta \in H_T^*(M)$ we have $\langle \mathcal{D}\eta, \mathcal{D}\zeta \rangle = \langle \mathcal{D}^2\eta, \zeta \rangle$. If both η and ζ are invariant under the Weyl group action i.e. $\eta, \zeta \in H_T^*(M)^W \cong H_K^*(M)$ then*

$$\langle \mathcal{D}\eta, \mathcal{D}\zeta \rangle = \frac{a}{b}|W| \cdot \kappa_K(\eta\zeta)[M//K]$$

where a is the order of the largest subgroup of K which fixes $\mu_K^{-1}(0)$ point-wise and b the order of the largest subgroup of T which fixes $\mu_T^{-1}(0)$ point-wise.

Proof: Using Lemma A.4 and the fact that $\imath_*$ is adjoint to $\imath^*$ we obtain

$$\langle \mathcal{D}\eta, \mathcal{D}\zeta \rangle = \langle \imath^* \kappa_T(\mathcal{D}\eta), \imath^* \kappa_T(\zeta) \rangle_{\mu_K^{-1}(0)/T}$$

A second use of Lemma A.4 and adjointness yields

$$\langle \imath^* \kappa_T(\mathcal{D}\eta), \imath^* \kappa_T(\zeta) \rangle_{\mu_K^{-1}(0)/T} = \langle \mathcal{D}^2\eta, \zeta \rangle.$$

If $\eta, \zeta \in H_T^*(M)^W$ then, since 0 is a regular value of μ_K, Theorem B$''$ of [Ma] tells us that

$$\langle \imath^* \kappa_T(\mathcal{D}\eta), \imath^* \kappa_T(\zeta) \rangle_{\mu_K^{-1}(0)/T} = \frac{a}{b} |W| \cdot \kappa_K(\eta\zeta)[M//K].$$

□

Theorem A.6. *Suppose 0 is a regular value of μ_K. Choose a W-equivariant Kirwan map $\kappa_T : H_T^*(M) \to IH^*(M//T)$ and suppose that it is surjective. For $\eta \in H_T^*(M)^W \cong H_K^*(M)$ the following assertions are equivalent:*

(i) $\kappa_K(\eta) = 0$,
(ii) $\kappa_T(\mathcal{D}\eta) = 0$,
(iii) $\kappa_T(\mathcal{D}^2\eta) = 0$.

Proof. Note that $\kappa_K(\eta) = 0$ if, and only if, $\kappa_K(\eta\zeta)[M//K] = 0$ for all $\zeta \in H_T^*(M)^W$. By Corollary A.5 this occurs if, and only if, $\langle \mathcal{D}\eta, \mathcal{D}\zeta \rangle = 0$ for all $\zeta \in H_T^*(M)^W$. Recall from Lemma 5.1 (which does not require the hypothesis that 0 is a regular value of μ_T) that

$$\{\mathcal{D}\zeta : \zeta \in H_T^*(M)^W\} = \{\text{Anti-invariant } \xi \in H_T^*(M)\}.$$

So $\langle \mathcal{D}\eta, \xi \rangle = 0$ for all anti-invariant $\xi \in H_T^*(M)$ and hence, by the argument in the proof of Theorem 1.8, for all $\xi \in H_T^*(M)$. By (26) we see that $\kappa_T(\mathcal{D}\eta) = 0$ i.e. (i) implies (ii). The converse is easy.

On the other hand, using Corollary A.5 we see that $\kappa_K(\eta) = 0$ if, and only if, $\langle \mathcal{D}^2\eta, \zeta \rangle = 0$ for all $\zeta \in H_T^*(M)^W$. Since $\mathcal{D}^2\eta$ is W-invariant this is equivalent to $\langle \mathcal{D}^2\eta, \xi \rangle = 0$ for all $\xi \in H_T^*(M)$. By (26) this is equivalent to $\kappa_T(\mathcal{D}^2\eta) = 0$. So (i) and (iii) are equivalent. □

In particular this theorem applies under the conditions in the statement of Theorem A.1. It can be read either as allowing us to determine the kernel of κ_K in terms of the kernel of *any* surjective W-equivariant Kirwan map κ_T for the T action, or, alternatively, as telling us that the subspace $\{\mathcal{D}\eta : \eta \in \mathrm{Ker}(\kappa_K)\}$ is contained in the kernel of *every* surjective W-equivariant Kirwan map κ_T.

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