THE KERNEL OF THE EQUIVARIANT KIRWAN MAP AND THE RESIDUE FORMULA

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Abstract. Using the notion of equivariant Kirwan map, as defined by Goldin [3], we prove that — in the case of Hamiltonian torus actions with isolated fixed points — Tolman and Weitsman's description of the kernel of the Kirwan map can be deduced directly from the residue theorem of [6] and [7]. A characterization of the kernel of the Kirwan map in terms of residues of one variable (i.e. associated to Hamiltonian *circle* actions) is obtained.

§1 Introduction

Let H be a torus acting in a Hamiltonian fashion on a compact symplectic manifold N and $S \subset H$ a circular subgroup. Assume that $0 \in \mathfrak{s}^*$ is a regular value of the moment map $\mu_S : M \to \mathfrak{s}^*$ which corresponds to the S action on N. Define the symplectic reduction $N_{\rm red} = \mu_S^{-1}(0)/S$ and consider

$$\kappa_S : H^*_H(N) \to H^*_H(\mu_S^{-1}(0)) = H^*_{H/S}(N_{\rm red}).$$

R. Goldin [3] called this map the *equivariant Kirwan map* and then she proved:

Theorem 1.1. (see [3, Theorem 1.2]) *The map κ_S is surjective.*

Assume that the fixed point set $\mathcal{F} = N^S$ consists of isolated points. Consider

$$\mathcal{F}_+ = \{F \in \mathcal{F} | \mu_S(F) > 0\}$$

and

$$\mathcal{F}_- = \{F \in \mathcal{F} | \mu_S(F) < 0\}.$$

Let $X, Y_1, \dots, Y_m$ be variables corresponding to an integral basis of $\mathfrak{h}^*$ such that

$$X|_{\mathfrak{s}} \text{ corresponds to an integral basis of } \mathfrak{s}^* \text{ and } Y_j|_{\mathfrak{s}} = 0,\ j \geq 1.$$

We will identify

$$H^*_H(\mathrm{pt}) = S(\mathfrak{h}^*) = \mathbb{C}[X, Y_1, \dots, Y_m].$$

The following residue formula has been proved in [6] and [7] (see also [8], [4]):

Theorem 1.2. *For any $\eta \in H_H^*(N)$ we have*[1]

$$\kappa_S(\eta)[N_{\mathrm{red}}] = c \sum_{F\in\mathcal{F}_+} \mathrm{Res}_X^+ \frac{\eta|_F}{e_F}. \tag{1.1}$$

Here $e_F \in H_H^*(F)$ denotes the H-equivariant Euler class of the normal bundle $\nu(F)$, c is a non-zero constant[2]*, and Res_X^+ is defined below in (1.3).*

According to [7, Definition 3.3 and Proposition 3.4], the value of Res_X^+ on a rational function of the type

$$h(X, Y_1, \dots, Y_m) = \frac{p(X, Y_1, \dots, Y_m)}{q(X, Y_1, \dots, Y_m)} = \frac{p(X, Y_1, \dots, Y_m)}{\prod_k (m_k X + \sum_i \beta_{ki} Y_i)} \tag{1.2}$$

where $p \in \mathbb{C}[X, \{Y_i\}]$, $m_i \in \mathbb{Z} \setminus \{0\}$ and $\beta_{ki} \in \mathbb{C}$, can be obtained as follows: regard $Y_1, \dots, Y_m$ as constants (i.e. as complex numbers) and set

$$\mathrm{Res}_X^+(h) = \sum_{b\in\mathbb{C}} \mathrm{Res}_{X=b} \frac{p}{q} \tag{1.3}$$

where on the right hand side $\frac{p}{q}$ is interpreted as a meromorphic function in the variable X on $\mathbb{C}$. It remains to note that only residues of expressions of the type (1.2) are involved in (1.1). For we may assume that each normal bundle $\nu(F)$, $F \in \mathcal{F}_+$, is a direct sum of H-invariant line bundles and in this way the Euler class e_F is the product of H-equivariant first Chern classes of those line bundles.

Remark 1.3. For future reference, we mention that

$$\mathrm{Res}_X^+ \frac{1}{X + \sum_i \beta_i Y_i} = 1.$$

Remark 1.4. Guillemin and Kalkman gave another definition of the residue of h given by (1.2) (see [4, section 3]). Denote $\beta_k(Y) = \sum_i \beta_{ki} Y_i$ and write

$$\frac{1}{\prod_k (m_k X + \beta_k(Y))} = \frac{1}{\prod_k m_k X (1 + \frac{\beta_k(Y)}{m_k X})} = \prod_k (m_k X)^{-1} \prod_k (1 - \frac{\beta_k(Y)}{m_k X} + (\frac{\beta_k(Y)}{m_k X})^2 - \dots).$$

Multiply the right hand side by the polynomial p, and add together all coefficients of X^{-1}: the result is the Guillemin-Kalkman residue. It is a simple exercise to show that the latter coincides with the residue Res_X^+ of [7] (one uses the fact that the sum of the residues of the 1-form $h(X)dX$ on the Riemann sphere equals zero).

Since κ_S is a ring homomorphism, from Theorem 1.2 we deduce:

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[1] Note that both sides of the equation are in $\mathbb{C}[Y_1, \dots, Y_m]$.

[2] The exact value of c is not needed in our paper, but the interested reader can find it in [7, §3].

Corollary 1.5.

$$\kappa_S(\eta)\kappa_S(\zeta)[N_{\mathrm{red}}] = \sum_{F\in\mathcal{F}_+} \mathrm{Res}_X^+ \frac{(\eta\zeta)|_F}{e_F}.$$

Since $0 \in \mathfrak{s}^*$ is a regular value of μ_S, S acts on $\mu_S^{-1}(0)$ with finite stabilizers. Hence $\mu_S^{-1}(0)/S$ has at worst orbifold singularities and in particular it satisfies Poincaré duality. From Corollary 1.5 it follows that $\eta \in H_H^*(N)$ is in $\ker \kappa_S$ iff

$$\sum_{F\in\mathcal{F}_+} \mathrm{Res}_X^+ \frac{(\eta\zeta)|_F}{e_F} = 0, \tag{1.4}$$

for all $\zeta \in H_H^*(N)$. We refer to the set of all such η as the "residue kernel".

The main goal of our paper is to compare the latter description of $\ker \kappa_S$ to the one given by Goldin in [3]. We will assume that the fixed point sets N^S and N^H are equal, which holds for a generic choice of the subgroup S of H (see Remark 1.9 below).

Theorem 1.6. (see [3, Theorem 1.5]) *We have*

$$\ker \kappa_S = K_- \oplus K_+$$

where K_- is the set of all elements of $H_H^*(N)$ vanishing on all components $F \in \mathcal{F}_-$, and similarly for K_+.

We refer to $K_- \oplus K_+$ as the "Tolman-Weitsman kernel", since the importance of this object was first identified in Tolman and Weitsman's paper [9].

The goal of our paper is to give a direct proof of the following result:

Theorem 1.7. *Assume that the fixed point set $N^H = N^S$ is finite. A class $\eta \in H_H^*(N)$ satisfies (1.4) for all $\zeta \in H_H^*(N)$ if and only if $\eta \in K_- \oplus K_+$.*

In other words, we shall give a direct proof that the residue kernel is equal to the Tolman-Weitsman kernel. In [3] Goldin worked entirely with the Tolman-Weitsman kernel; our objective in this article is to give a direct proof that the residue kernel encodes the same information.

It is obvious that any $\eta \in K_+$ satisfies (1.4). On the other hand, by the Atiyah-Bott-Berline-Vergne localization formula (see [1], [2]) the sum

$$\sum_{F\in\mathcal{F}} \frac{(\eta\zeta)|_F}{e_F}$$

is a polynomial in $X, \{Y_i\}$, so that its residue is zero. It follows that any η in K_- satisfies (1.4) as well. The hard part will be to prove that if η satisfies (1.4) for any $\zeta \in H^*_H(N)$, then η is in $K_- \oplus K_+$.

Remark 1.8. Theorem 1.7 is a generalization of the main result of [5].

Remark 1.9. Let M be a symplectic manifold acted on by a torus G in a Hamiltonian fashion such that the fixed point set M^G is finite and let $\mu_G : M \to \mathfrak{g}^*$ be the corresponding moment map. Suppose that $0 \in \mathfrak{g}^*$ is a regular value of μ_G and consider the corresponding symplectic reduction

$$M//G = \mu_G^{-1}(0)/G$$

as well as the Kirwan surjection

$$\kappa : H^*_G(M) \to H^*(M//G).$$

Using reduction in stages, results like Theorem 1.6 or Theorem 1.7 can be used in order to obtain descriptions of $\ker \kappa$. The following construction is needed: There exists a "generic" (in the sense of [3, section 1]) circle $S_1 \subset G$ with Lie algebra $\mathfrak{s}_1$ such that

- $M^{S_1} = M^G$
- 0 is a regular value of the moment map $\mu_{S_1} : M \to \mathfrak{s}_1^*$.

Consider the action of G/S_1 on the reduced space $M//S_1 = \mu_{S_1}^{-1}(0)/S_1$. As before, there exists a generic circle $S_2 \subset G/S_1$ such that

- $(M//S_1)^{S_2} = (M//S_1)^{G/S_1}$
- 0 is a regular value of the moment map $\mu_{S_2} : M//S_1 \to \mathfrak{s}_2^*$.

The equivariant Kirwan map corresponding to the subtorus $S_1 \times S_2$ of G is the following composition of maps:

$$H^*_G(M) \xrightarrow{\kappa_1} H^*_{G/S_1}(M//S_1) \xrightarrow{\kappa_2} H^*_{G/(S_1\times S_2)}(M//(S_1 \times S_2)),$$

where we have used "reduction in stages". In fact the procedure can be continued giving rise to a sequence of tori

$$\{1\} = T_0 \subset T_1 \subset T_2 \subset \ldots \subset T_m = G$$

where $T_1 = S_1, T_2 = S_1 \times S_2, \ldots$ with the following properties:

- $(M//T_{j-1})^{T_j/T_{j-1}} = (M//T_{j-1})^{G/T_{j-1}}$
- 0 is a regular value of the moment map $\mu_{T_j/T_{j-1}}$ on $M//T_{j-1}$.

By reduction in stages, the Kirwan map κ decomposes as

$$\kappa = \kappa_m \circ \ldots \circ \kappa_1 \tag{1.5}$$

where

$$\kappa_j = \kappa_{T_j/T_{j-1}} : H^*_{G/T_{j-1}}(M//T_{j-1}) \to H^*_{G/T_j}(M//T_j)$$

is the T_j/T_{j-1}-equivariant Kirwan map. Goldin [3] used this decomposition in order to deduce from Theorem 1.6 the Tolman-Weitsman description of $\ker \kappa$ (see [9]). Theorem 1.7 of our paper — with $N = M//T_{j-1}$, $H = G/T_{j-1}$ and $S = T_j/T_{j-1}$ — shows that the Tolman-Weitsman description of $\ker \kappa$ is equivalent to the obvious characterization in terms of residues arising from the components κ_j of κ.

Remark 1.10 Our methods are somewhat similar to those used by Goldin in [3]. Our objectives however differ substantially from the aims of [3], since the residue plays no role in [3].

§2 Expressing $\ker \kappa_S$ in terms of residues

We will now give a proof of Theorem 1.7. Take $\xi \in \mathfrak{s}$ a non-zero vector and $f = \mu_S^\xi = \langle \mu_S, \xi \rangle : N \to \mathbb{R}$ the function induced on N by the moment map $\mu_S : N \to \mathfrak{s}^*$. This is an H-equivariant Morse function, whose critical set is $N^S = N^H$. For any $F \in \mathcal{F}$, we have the H-equivariant splitting of the tangent space

$$T_F N = \nu_f^-(F) \oplus \nu_f^+(F),$$

determined by the sign of the Hessian on the two summands. Let $e(\nu_f^-(F)), e(\nu_f^+(F)) \in H^*_H(F)$ be the corresponding H-equivariant Euler classes. The following two results can be proved by using Morse theory (see for example [3]):

Proposition 2.1. *Suppose $\eta \in H^*_H(N)$ restricts to zero on all $G \in \mathcal{F}$ for which $f(G) < f(F)$. Then $\eta|_F$ is some $H^*_H(\mathrm{pt})$-multiple of $e(\nu_f^- F)$.*

Proposition 2.2. *For any $F \in \mathcal{F}$ there exists a class $\alpha^-(F) \in H^*_H(N)$ with the following properties:*

1. *$\alpha^-(F)|_G = 0$, for any $G \in \mathcal{F}$ which cannot be joined to F along a sequence of integral lines of the negative gradient field $-\nabla f$ (in particular, for any $G \in \mathcal{F}$ with $f(G) < f(F)$)*
2. *$\alpha^-(F)|_F = e(\nu_f^- F)$.*

In the same way there exists $\alpha^+(F) \in H^*_H(N)$ such that:

1. $\alpha^+(F)|_G = 0$, *for any* $G \in \mathcal{F}$ *which cannot be joined to* F *along a sequence of integral lines of the gradient field* ∇f *(in particular, for any* $G \in \mathcal{F}$ *with* $f(G) > f(F)$*)*
2. $\alpha^+(F)|_F = e(\nu_f^+ F)$.

From the injectivity theorem of Kirwan — which says that $\alpha \in H^*_H(N)$ is uniquely determined by its restrictions to N^H — we deduce:

Corollary 2.3. *The set* $\{\alpha^-(F)|F \in \mathcal{F}\}$ *is a basis of* $H^*_H(N)$ *as a* $H^*_H(pt)$*-module.*

Consider the space $\hat{H}^*_H(N)$ consisting of all expressions of the type

$$\sum_{F\in\mathcal{F}} r_F \alpha^-(F),$$

where r_F is in the ring $\mathbb{C}(X, \{Y_i\})$ of rational expressions in $X, \{Y_i\}$ (i.e. quotients p/q, with $p, q \in \mathbb{C}[X, \{Y_i\}]$, $q \neq 0$). The space $\hat{H}^*_H(N)$ is obviously a $\mathbb{C}(X, \{Y_i\})$-algebra.

Lemma 2.4. *Take* $\eta \in H^*_H(N)$ *of degree* d. *Then we can decompose*

$$\eta = \eta_+ + \eta_-$$

where $\eta_+, \eta_- \in \hat{H}^*_H(N)$, *such that*

(i) $\eta_+|_{\mathcal{F}_-} = 0$, $\eta_-|_{\mathcal{F}_+} = 0$;

(ii) η_+ *and* η_- *are linear combinations of* $\alpha^-(F)$, $F \in \mathcal{F}$, *where the coefficients* r_F *are rational functions whose denominators can be decomposed as products of linear factors of the type*

$$X + \sum_i \beta_i Y_i, \quad \beta_i \in \mathbb{C}.$$

Proof. By Corollary 2.3, we have

$$\eta = \sum_{F\in\mathcal{F}} p_F \alpha^-(F),$$

where $p_F \in \mathbb{R}[X, \{Y_i\}]$ are homogeneous polynomials. Take

$$\tilde{\eta}_- = \sum_{F\in\mathcal{F}_-} p_F \alpha^-(F), \quad \tilde{\eta}_+ = \sum_{F\in\mathcal{F}_+} p_F \alpha^-(F).$$

By Proposition 2.2, $\tilde{\eta}_+$ restricts to 0 on all $G \in \mathcal{F}_-$.

Consider the ordering $F_1, F_2, \ldots$ of the elements of $\mathcal{F}_+$ such that

$$0 < f(F_1) < f(F_2) < \ldots .$$

There exists a rational function $r_1(X, \{Y_i\}) \in \mathbb{C}(X, \{Y_i\})$ such that

$$\tilde{\eta}_-|_{F_1} = r_1(X, \{Y_i\})e(\nu_f^- F_1),$$

which means that the form

$$\tilde{\eta}_1 := \tilde{\eta}_- - r_1(X, \{Y_i\})\alpha^-(F_1)$$

vanishes at F_1. There exists another rational function $r_2(X, \{Y_i\}) \in \mathbb{C}(X, \{Y_i\})$, with the property that

$$\tilde{\eta}_1|_{F_2} = r_2(X, \{Y_i\})e(\nu_f^- F_2),$$

which implies that the form

$$\tilde{\eta}_1 - r_2(X, \{Y_i\})\alpha^-(F_2)$$

vanishes at both F_1 and F_2. We continue this process and we get the decomposition claimed in the proposition as follows:

$$\eta_- = \tilde{\eta}_- - r_1\alpha^-(F_1) - r_2\alpha^-(F_2) - \ldots , \quad \eta_+ = \tilde{\eta}_+ + r_1\alpha^-(F_1) + r_2\alpha^-(F_2) + \ldots .$$

Property (ii) follows from the fact that for each $F \in \mathcal{F}_+$, the weights of the representation of S on $\nu_f^-(F)$ are all non-zero. ■

We are now ready to prove the main result of the paper.

Proof of Theorem 1.7. Take $\eta \in H_H^d(N)$ satisfying

$$\mathrm{Res}_X^+ \sum_{F \in \mathcal{F}_+} \frac{(\eta\zeta)|_F}{e_F} = 0 \tag{2.1}$$

for all $\zeta \in H_H^*(N)$. We consider the decomposition

$$\eta = \eta_- + \eta_+$$

with $\eta_-, \eta_+ \in \hat{H}^*_H(N)$ given by Lemma 2.4. We show that η_- and η_+ are actually in $H^*_H(N)$. More precisely, if η_+ is of the form

$$\eta_+ = \sum_{G\in\mathcal{F}_+} \frac{p_G}{q_G}\alpha^-(G)$$

with $p_G, q_G \in \mathbb{C}[X, Y_i]$ we show that

$$q_G \text{ divides } p_G$$

for any $G \in \mathcal{F}_+$.

From (2.1) and the fact that $\eta_-|_{\mathcal{F}_+} = 0$, we deduce that

$$\mathrm{Res}^+_X \sum_{F\in\mathcal{F}_+} \frac{(\eta_+\zeta)|_F}{e_F} = 0$$

which is equivalent to

$$\mathrm{Res}^+_X \sum_{F,G\in\mathcal{F}_+} \frac{p_G}{q_G} \cdot \frac{\alpha^-(G)|_F\zeta|_F}{e_F} = 0 \tag{2.2}$$

for all $\zeta \in H^*_H(N)$. Consider again the ordering $F_1, F_2, \ldots$ of the elements of $\mathcal{F}_+$ such that

$$f(F_1) < f(F_2) < \ldots .$$

We prove by induction on $k \geq 1$ that q_{F_k} divides p_{F_k}.

In (2.2) we put $\zeta = p\alpha^+(F_1)$, where $p \in \mathbb{C}[X, Y_i]$ is an arbitrary polynomial. Since

$$e(\nu^-_f F_1)e(\nu^+_f F_1) = e_{F_1},$$

we deduce that

$$\mathrm{Res}^+_X p\frac{p_{F_1}}{q_{F_1}} = 0$$

for any $p \in \mathbb{C}[X, Y_1, \ldots, Y_m]$ (we are using the fact that $\alpha^+(F_1)|_F = 0$ if $\mu(F) > \mu(F_1)$ while $\alpha^-(G)|_F = 0$ if $\mu(F) < \mu(G)$: hence the only nonzero contribution comes from $F = G = F_1$). From Lemma 2.5 (see below) we deduce that

$$q_{F_1} \text{ divides } p_{F_1}.$$

Now we fix $k \geq 2$, assume that q_{F_i} divides p_{F_i} for any $i < k$ and show that q_{F_k} divides p_{F_k}. In (2.2) put $\zeta = p\alpha^+(F_k)$, with $p \in \mathbb{C}[X, Y_1, \ldots, Y_m]$. We obtain

$$\mathrm{Res}^+_X \sum_{i\leq j\leq k} p\frac{p_{F_i}}{q_{F_i}} \cdot \frac{\alpha^-(F_i)|_{F_j}\alpha^+(F_k)|_{F_j}}{e_{F_j}} = 0.$$

The sum in the left hand side is over i and j (k being fixed). We divide into sums of the type

$$\Sigma^i := \mathrm{Res}^+_X \sum_{i\le j\le k} p\frac{p_{F_i}}{q_{F_i}} \cdot \frac{\alpha^-(F_i)|_{F_j}\alpha^+(F_k)|_{F_j}}{e_{F_j}}, \quad i \le k.$$

If $i<k$, then $\Sigma^i = 0$, by the hypothesis that q_{F_i} divides p_{F_i} and the Atiyah-Bott-Berline-Vergne localization formula for $p\frac{p_{F_i}}{q_{F_i}}\cdot\alpha^-(F_i)\alpha^+(F_k)$ (note that $\alpha^-(F_i)|_F\alpha^+(F_k)|_F = 0$ for any F which is not of the form F_j with $i\le j\le k$).

Finally we show that $\Sigma^k = 0$, which is equivalent to

$$\mathrm{Res}^+_X p\frac{p_{F_k}}{q_{F_k}} = 0$$

for any $p \in \mathbb{C}[X, Y_i]$. By Lemma 2.5,

$$q_{F_k} \text{ divides } p_{F_k},$$

which concludes the proof. ■

We have used the following result:

Lemma 2.5. *Let f, g be in $\mathbb{C}[X, \{Y_i\}]$, where*

$$g = \prod_k (X + \sum_i \beta_{ik} Y_i),$$

$\beta_{ik} \in \mathbb{C}$. If

$$\mathrm{Res}^+_X(p \cdot \frac{f}{g}) = 0, \tag{2.3}$$

for any $p \in \mathbb{C}[X, \{Y_i\}]$, then g divides f.

Proof. Suppose that g does not divide f. We can assume that g and f are relatively prime. Then there exist $p_1, p_2 \in \mathbb{C}[X, \{Y_i\}]$ such that

$$p_1 f + p_2 g = 1.$$

From (2.3) it follows that

$$\mathrm{Res}^+_X \frac{p}{g} = \mathrm{Res}^+_X(pp_1 \cdot \frac{f}{g} + p_2) = 0$$

for any $p \in \mathbb{C}[X, \{Y_i\}]$. Now fix k_0 and set

$$p = \prod_{k\ne k_0}(X + \sum_i \beta_{ik} Y_i).$$

We deduce that

$$\mathrm{Res}^+_X \frac{1}{X + \sum_i \beta_{ik_0} Y_i} = 0.$$

But the left hand side is actually 1 (see Remark 1.3), which is a contradiction. ■

§3 Residues and the kernel of the Kirwan map

In this section we shall give a direct characterization of the kernel of the Kirwan map in terms of residues of one variable.

Let M be a compact symplectic manifold equipped with a Hamiltonian G action, where G is a torus and let $\kappa : H_G^*(M) \to H^*(M//G)$ be the Kirwan map. Consider the decomposition of κ described in Remark 1.9 (see equation (1.5)). For each j between 1 and m we consider the commutative diagram

$$\begin{array}{ccc} H^*_{G/T_{j-1}}(M//T_{j-1}) & \xrightarrow{\kappa_j} & H^*_{G/T_j}(M//T_j) \\ \downarrow \pi_j & & \downarrow \pi \\ H^*_{T_j/T_{j-1}}(M//T_{j-1}) & \xrightarrow{\tilde{\kappa}_j} & H^*(M//T_j) \end{array} \tag{3.1}$$

where $\tilde{\kappa}_j$ is the Kirwan map associated to the action of the circle T_j/T_{j-1} on $M//T_{j-1}$. For the same action we consider the moment map μ_j, the fixed point set $\mathcal{F}^j = (M//T_{j-1})^{T_j/T_{j-1}}$ and the partition of the latter into $\mathcal{F}^j_-$ and $\mathcal{F}^j_+$ which consist of fixed points where μ_j is negative, respectively positive.

Definition 3.1. *An element $\eta \in H^*_{T_j/T_{j-1}}(M//T_{j-1})$ is in* $\ker_{\mathrm{res},j}$ *if*

$$Res_{X_j=0} \sum_{F \in \mathcal{F}^j_+} \frac{(\eta\zeta)|_F}{e_F(X_j)} = 0$$

for any $\eta \in H^*_{T_j/T_{j-1}}(M//T_{j-1})$, where X_j is a variable corresponding to a basis of the dual of the Lie algebra of T_j/T_{j-1} and $Res_{X_j=0}$ means the coefficient of X_j^{-1}.

For any $\alpha \in H_G^*(M)$ we define

$$\alpha_j = (\kappa_j \circ \cdots \circ \kappa_1)(\alpha)$$

$0 \le j \le m$. The goal of this section is to prove

Theorem 3.2. *Let $\alpha \in H_G^*(M)$. Then $\kappa(\alpha) = 0$ if and only if $\pi_j(\alpha_{j-1}) \in \ker_{\mathrm{res,j}}$ for some $j \ge 1$.*

Proof. If $\kappa(\alpha) = 0$, by equation (1.5) there exists j such that $\alpha_j = 0$. Since $\alpha_j = \kappa_j(\alpha_{j-1})$, we can use first the commutativity of the diagram (3.1) to deduce that $\tilde{\kappa}_j(\pi_j\alpha_{j-1}) = 0$ and then the residue formula of [6] and [7] to deduce that $\pi_j(\alpha_{j-1}) \in \ker_{\mathrm{res,j}}$.

The opposite direction is less obvious: Take $\alpha \in H^*_G(M)$ of degree greater than zero such that $\pi_j\alpha_{j-1} \in \ker_{\mathrm{res,j}}$. If $j = m$ then π_j and π are the identity maps, $\kappa_j = \tilde{\kappa}_j$ and we just have to apply the residue formula of [6] and [7] to deduce that $\kappa_m(\alpha_{m-1}) = \kappa(\alpha) = 0$. If $j < m$, by the same residue formula, we have $\tilde{\kappa}_j(\pi_j\alpha_{j-1}) = 0$, which implies $\pi(\kappa_j\alpha_{j-1}) = 0$. Note that $\ker(\pi) = H^*_{G/T_j}(\mathrm{pt})$ so $\pi(\kappa_j\alpha_{j-1}) = 0$ is equivalent to $\kappa_j\alpha_{j-1} \in H^*_{G/T_j}(\mathrm{pt})$. The latter implies that $(\kappa_m \circ \cdots \circ \kappa_j)\alpha_{j-1} = 0$ (since the map $\kappa_m \circ \cdots \circ \kappa_{j+1}$ sends all elements of $H^*_{G/T_j}(\mathrm{pt})$ of degree larger than zero to 0, because the image of this map is $H^*(M//G)$ and the image of the equivariant cohomology of a point under this map is the ordinary cohomology of a point). But this means $\kappa(\alpha) = 0$ and the proof is now complete. $\blacksquare$

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