CONNECTIVITY AND KIRWAN SURJECTIVITY FOR ISOPARAMETRIC SUBMANIFOLDS

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Abstract. Atiyah's formulation of what is nowadays called the convexity theorem of Atiyah-Guillemin-Sternberg has two parts: (a) the image of the moment map arising from a Hamiltonian action of a torus on a symplectic manifold is a convex polytope, and (b) all preimages of the moment map are connected. Part (a) was generalized by Terng to the wider context of isoparametric submanifolds in euclidean space. In this paper we prove a generalization of part (b) for a certain class of isoparametric submanifolds (more precisely, for those with all multiplicities strictly greater than 1). For generalized real flag manifolds, which are an important class of isoparametric submanifolds, we give a surjectivity criterium for a certain Kirwan map (involving equivariant cohomology with coefficients in $\mathbb{Q}$) which arises naturally in this context. Examples are also discussed.

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1. Introduction

A crucial result in symplectic geometry is the convexity theorem of Atiyah-Guillemin-Sternberg. It states that if (M, ω) is a symplectic manifold acted on by a torus T in a Hamiltonian way, then the image of the moment map $\mu : M \to \mathrm{Lie}(T)^*$ is a convex polytope. A closely related result says [1] that all pre-images of μ are connected (or empty). The convexity theorem has been generalized by Terng [12] to the case when M is an isoparametric submanifold in a euclidean space (see section 1 of our paper for the definition; we note that we always assume that M is compact). More precisely, she proved the following theorem.

Theorem 1.1. (Terng [12]) *Let $M \subset \mathbb{R}^{n+k}$ be an isoparametric submanifold, q a point of M, $\nu_q(M)$ the normal space at q to M, and $P : \mathbb{R}^{n+k} \to \nu_q(M)$ the orthogonal projection map. Then the image of the map*

$$\mu = P|_M : M \to \nu_q(M)$$

is a convex polytope.

For example, M can be a generalized real flag manifold, i.e. an orbit of the isotropy representation of a compact symmetric space (for more details,

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see section 5 of our paper). These can be realized as real loci (i.e. fixed point sets of antisymplectic involutions) of coadjoint orbits. Convexity results for such spaces had been obtained by Kostant [9] (see also Duistermaat [3]). But Terng's theorem does not fit entirely into the framework of the paper by Kostant. The reason is that there are several examples of isoparametric submanifolds which are not flag manifolds, i.e. are not homogeneous: such examples were found by Ferus, Karcher and Münzner in [5].

Now turning to the connectedness result of Atiyah mentioned above, the following question can be raised. In the context of Theorem 1.1, is any preimage of μ connected? One can easily see that the answer is in general negative. More specifically, the real flag manifold corresponding to the symmetric space $SU(2)/SO(2)$ is a circle and μ is just the height function on that circle; hence almost all preimages consist of two points. Our main result is (see section 2 for the definition of multiplicities):

Theorem 1.2. *Let $M \subset \mathbb{R}^{n+k}$ be an isoparametric submanifold with all multiplicities greater or equal to 2, and let $a \in \nu_q(M)$ be in the image of μ. The following statements are true.*

(i) The function $f : M \to \mathbb{R}$, $f(x) = \|\mu(x) - a\|^2$, is minimally degenerate in the sense defined by Kirwan [8] *(see section 3 of our paper).*

(ii) The level set $\mu^{-1}(a)$ is connected.

We note that in the case of the real flag manifold arising from $SU(2)/SO(2)$ mentioned above, the (only) multiplicity is equal to 1. So the hypothesis on multiplicities in the previous theorem is essential.

Remarks. (a) In fact, Terng [12] proved that if M_ξ is a manifold parallel to M (see section 2), then the image of $P|_{M_\xi} : M_\xi \to \nu_q(M)$ is a convex polytope. With the methods of our paper one can prove that any preimage of the latter map is connected as well.

(b) Here is a list of examples of isoparametric submanifolds with all multiplicities at least equal to 2, together with the corresponding parallel manifolds (see the previous remark):

- adjoint orbits of compact Lie groups
- isotropy orbits of the following symmetric spaces: $SU(2n)/Sp(n)$, E_6/F_4, $SU(m+n)/S(U(m)\times U(n))$, where m, n are positive integers, $m > n$, and $Sp(m+n)/Sp(m) \times Sp(n)$ (for the details, see Helgason [7])
- infinitely many of the Ferus-Karcher-Münzner [5] examples.

(c) The idea of the proof of Theorem 1.2 goes back to Kirwan [8]. We consider the Morse stratification of M induced by f. The strata are smooth submanifolds, and all but the minimum one have codimension at least 2, due to the hypothesis on multiplicities. Hence the minimum stratum is connected. Finally, we use the fact, also proved by Kirwan [8], that the latter stratum has the same Čech cohomology module $H^0(\cdot, \mathbb{Q})$ as the minimum set $f^{-1}(0) = \mu^{-1}(a)$.

(d) Terng [13] was able to extend Theorem 1.1 to the case when M is an *infinite dimensional* isoparametric submanifold in a Hilbert space, or any of its parallel manifolds, like in Remark (a). We conjecture that also our Theorem 1.2 can be extended to the infinite dimensional situation. The connectivity theorem proved recently by Harada, Holm, Jeffrey, and the author in [6] shows that the conjecture is true for the loop group $\Omega(G)$, which is a parallel manifold of a certain infinite dimensional isoparametric submanifold with all multiplicities equal to 2.

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2. Isoparametric submanifolds in euclidean space

We present some basic facts concerning the theory of isoparametric submanifolds. For more details, the reader can consult Palais and Terng's book [11] (especially chapter 6) and the references therein.

Let $M \subset \mathbb{R}^{n+k}$ be an n-dimensional embedded submanifold, which is closed, complete with respect to the induced metric, and full (i.e. not contained in any affine subspace). We say that M is *isoparametric* if any normal vector at a point of M can be extended to a parallel normal vector field ξ on M with the property that the eigenvalues of the shape operators $A_{\xi(p)}$ (i.e. the principal curvatures) are independent on $p \in M$, as values and multiplicities. It follows that for $p \in M$, the set $\{A_{\xi(p)} : \xi(p) \in \nu(M)_p\}$ is a commutative family of selfadjoint endomorphisms of $T_p(M)$, and so it determines a decomposition of $T_p(M)$ as a direct sum of common eigenspaces $E_1(p), E_2(p), ..., E_r(p)$. There exist normal vectors $\eta_1(p), \eta_2(p), ..., \eta_r(p)$ such that

$$A_{\xi(p)}|_{E_i(p)} = \langle \xi(p), \eta_i(p) \rangle \mathrm{id}_{E_i(p)},$$

for all $\xi(p) \in \nu_p(M)$, $1 \le i \le r$. By parallel extension in the normal bundle we obtain the vector fields $\eta_1, \dots, \eta_r$, which are the *principal curvature vectors*. The eigenspaces from above give rise to the distributions $E_1, \dots, E_r$ on M, which are called the *curvature distributions*. The numbers

$$m_i = \mathrm{rank} E_i,$$

$1 \le i \le r$, are the *multiplicities* of M.

To any parallel normal vector field ξ on M we assign the end-point map $\pi_\xi : M \to \mathbb{R}^{n+k}$,

$$\pi_\xi(p) = p + \xi(p),$$

$p \in M$. Its image is the "parallel" manifold M_ξ, which is also an embedded submanifold of $\mathbb{R}^{n+k}$. Now the differential map of π_ξ is

$$d(\pi_\xi)_p = I - A_{\xi(p)}.$$

So the focal points of M in the affine normal space $p+\nu_p(M)$ are those which are in one of the hyperplanes

$$\ell_i(p) := \{p+\xi(p) : \langle \eta_i(p), \xi(p)\rangle = 1\},$$

for some $i \in \{1, 2, \dots, r\}$. It turns out that $\ell_1(p), \dots, \ell_r(p)$ have a unique intersection point, call it c_0, independent on the point p. Moreover, M is contained in a sphere centered at c_0 (here we use the assumption that M is compact). We do not lose any generality if we assume that M is contained in the unit sphere S^{n+k-1}, hence c_0 is just the origin 0. One shows that the group of linear transformations of $\nu_p(M)$ generated by the reflections about $\ell_1(p), \dots, \ell_r(p)$ is a Coxeter group, whose isomorphism type is independent on p. We denote it by W and call it the *Weyl group* of M.

The map $\pi_\xi : M \to M_\xi$ is a submersion. If p is a point in M and $b := \pi_\xi(p) = p+\xi(p)$, we denote by $S_{p,b}$ the connected component of $\pi_\xi^{-1}(b)$ which contains p. This manifold is called the *slice* through p corresponding to ξ. Consider

$$I := \{i \in \{1, 2, \dots, r\} : p+\xi(p) \in \ell_i(p)\},$$

and also the subspace $(\bigcap_{i\in I} \ell_i(p))^\perp$ of $\nu_p(M)$, where the superscript $\perp$ indicates the orthogonal complement. An important result is the so-called slice theorem, which is stated as follows (for more details, see Theorem 6.5.9 of [11]).

Theorem 2.1. *The slice $S_{p,b}$ is a (full) isoparametric submanifold of the space $(\bigcap_{i\in I} \ell_i(p))^\perp + \sum_{i\in I} E_i(p)$.*

3. Minimally degenerate functions according to Kirwan

In this section we follow chapter 10 of Kirwan's book [8].

Definition. *A smooth function $f : X \to \mathbb{R}$ on a closed manifold X is called* minimally degenerate *if the following conditions hold.*

(a) *The set of critical points of f is a finite union of disjoint closed subsets $C_1, C_2, \dots, C_N$, on each of which f is constant. These subsets are called the* critical sets *of f.*

(b) *For every $j = 1, \dots, N$, there exists a submanifold Y_j of X, which contains C_j, with the property that the normal bundle*[1] *of Y_j in X is orientable, and such that*
   (i) *C_j is the subset of Y_j on which f takes its minimum value*
   (ii) *for any $x \in C_j$, the space T_xY_j is maximal among the subspaces of T_xX on which the Hessian $\mathrm{Hess}_x(f)$ is positive semi-definite.*

   A manifold Y_j with the properties (i) and (ii) is called a minimizing manifold *around C_j. The codimension of Y_j is called the* index *of f along C_j.*

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[1] More specifically, $(TX/TY_j)|_{Y_j}$

Even though minimally degeneracy is a condition weaker than nondegeneracy in the sense of Bott, it is still sufficient to induce a Morse stratification of M, as the following theorem shows.

Theorem 3.1. (Kirwan [8]) *Let $f : X \to \mathbb{R}$ be a minimally degenerate function like above and let g be a Riemannian metric on X. For any $j \in \{1, 2, \dots, N\}$ we denote by $\Sigma_j = \Sigma_j(g)$ the set of all points in X with the property that the ω-limit of the integral line through x of the vector field $-\nabla(f)$ is contained in C_j. Then we have as follows.*

(a) There exists a metric g with the property that Σ_j is a smooth submanifold of X of codimension equal to the index along C_j. We also have

$$X = \bigcup_{1 \le j \le N} \Sigma_j, \quad \Sigma_i \cap \Sigma_j = \phi, \text{ for } i \neq j.$$

The intersection of Y_j with a sufficiently small neighbourhood of C_j is contained in Σ_j.

(b) For any $j \in \{1, 2, \dots, N\}$, the inclusion map $C_j \hookrightarrow \Sigma_j$ induces an isomorphism in Čech cohomology.

4. Proof of Theorem 1.2

Let $M^n \subset \mathbb{R}^{n+k}$ be an isoparametric submanifold. We start with the following lemma.

Lemma 4.1. *Let q be a point of M, $b \in \nu_q(M)$, and let $S_{q,b}$ be the corresponding slice. Consider*

$$I := \{i \in \{1, 2, \dots, r\} : b \in \ell_i(q)\}.$$

If x is an arbitrary point in $S_{q,b}$, then we have:

(a) $S_{q,b}$ is a full isoparametric submanifold in $(\bigcap_{i\in I} \ell_i(q))^\perp + \sum_{i \in I} E_i(q)$, whose normal space at q is $(\bigcap_{i \in I} \ell_i(q))^\perp$

(b) $T_x S_{q,b} = \sum_{i\in I} E_i(x)$

(c) $\bigcap_{i\in I} \ell_i(x) + \sum_{i\in I} E_i(x) = \bigcap_{i\in I} \ell_i(q) + \sum_{i\in I} E_i(q)$

(d) $\bigcap_{i\in I} \ell_i(q) \subset \nu_x(M) \cap \nu_q(M)$

(e) $\sum_{j \notin I} E_j(x)$ is perpendicular to $\nu_q(M)$.

Proof. Points (a)-(d) have been proved for instance in [PT, Chapter 6]. We will prove (e). From (c) we deduce that $\sum_{j\notin I} E_j(x)$ is perpendicular to $\bigcap_{i\in I} \ell_i(q)$, and from (d), the same space is perpendicular to $\bigcap_{i\in I} \ell_i(q)$. □

We study the function $f : M \to \mathbb{R}$, $f(x) = \|\mu(x) - a\|^2$. First we determine its critical points. We note that this has been done already by Terng in section 3 of [13]. Let us consider an orthonormal basis $e_1, \dots, e_k$ in $\nu_q(M)$. We have

$$\mu(x) - a = \sum_{i=1}^{k} \langle x - a, e_i \rangle e_i,$$

hence

$$f(x) = \sum_{i=1}^{k} \langle x - a, e_i \rangle^2,$$

which implies

$$(1)\quad df_x(v) = 2\sum_{i=1}^{k} \langle x-a, e_i\rangle \langle v, e_i\rangle = 2\langle v, \sum_{i=1}^{k}\langle x-a, e_i\rangle e_i\rangle = 2\langle v, \mu(x) - a\rangle,$$

for any $v \in T_x(M)$. So x is a critical point of f if and only if the vector $b := \mu(x) - a \in \nu_q(M)$ is perpendicular to $T_x(M)$, in other words, when x is a critical point of the height function $h_b : M \to \mathbb{R}$, $h_b(x) = \langle b, x\rangle$. We can express this in a more concise way as

$$(2)\qquad Crit(f) = \mu^{-1}(a) \cup \bigcup_{b \in \nu_q(M)} \mu^{-1}(a+b) \cap Crit(h_b).$$

We prove that the intersection in the right hand side is nonempty only for finitely many $b \in \nu_q(M)$. We use the fact that

$$Crit(h_b) = \bigcup_{w\in W} S_{wq,b}$$

(see e.g. [13], subsection 2.9).

Lemma 4.2. *There exist finitely many $b \in \nu_q(M) \setminus \{0\}$ with the property that $\mu^{-1}(a+b) \cap S_{wq,b}$ is nonempty.*

Proof. For simplicity we assume that $w = 1$. Pick I an arbitrary subset of $\{1, 2, \ldots, r\}$. In $\nu_q(M)$ we consider the subspace

$$\ell_I := \bigcap_{i\in I} \ell_i(q).$$

We also consider the convex hull $\mathrm{cvx}[(W_I).q]$, where W_I denotes the stabilizer if ℓ_I. The affine span of this convex body is perpendicular to ℓ_I and its dimension is just the codimension of ℓ_I. To justify this, we note that this affine span is the affine normal space to a certain slice, which has codimension equal to $\dim(\ell_I)^\perp$, and which is contained in an affine space perpendicular to ℓ_I (see Lemma 4.1 (a)). Consequently the intersection $(a + \ell_I) \cap \mathrm{cvx}[(W_I).q]$ has at most one point. By letting I vary among all subsets of $\{1, 2, \ldots, r\}$, we obtain a finite set of points, call it F. Now, if $\mu^{-1}(a+b) \cap S_{wq,b} \neq \phi$, then $a + b$ must belong to $\mu(S_{wq,b})$, which is $\mathrm{cvx}[(W_b).q]$. Consequently, we have

$$a + b \in (a + \ell_I) \cap \mathrm{cvx}[(W_I).q],$$

where $I = \{i \in \{1, 2, \ldots, r\} : b \in \ell_i(q)\}$. This implies $a + b \in F$. □

We have proved that the set

$$B := \{b \in \nu_q(M) \setminus \{0\} : \mu^{-1}(a+b) \cap Crit(h_b) \neq \phi\}$$

is finite. The description (2) can be refined by taking into account that for any $b \in \nu_q(M)$ we have $h_b(x) = \langle x, b\rangle = \langle \mu(x), b\rangle$, $x \in M$. Consequently, if $x \in \mu^{-1}(a+b)$, then $h_b(x) = \langle a+b, b\rangle$. So

$$Crit(f) = \mu^{-1}(a) \cup \bigcup C_{b,w} \tag{3}$$

where $C_{b,w} := \mu^{-1}(a+b) \cap S_{wq,b}$ and the union runs over all $b \in B$ and $w \in W$ with the property that $h_b(S_{wq,b}) = \langle a+b, b\rangle$.

Fix b and w like above. Let $Y_{b,w}$ denote the stable manifold of the function h_b corresponding to the critical set $S_{wq,b}$. More specifically, this consists of all points $x \in M$ with the property that the limit at ∞ of the integral line through x of the vector field $-\nabla(h_b)$ is in $S_{wq,b}$. We will prove the following result.

Proposition 4.3. *(a) A minimizing manifold for f around $\mu^{-1}(a)$ is M itself.*

(b) For $b \in B$ and $w \in W$, the space $Y_{b,w}$ is a minimizing manifold for f around $C_{b,w}$.

(c) The only critical set of index 0 is $\mu^{-1}(a)$.

Proof. (b) First we show that the normal bundle to $Y_{b,w}$ is orientable. To this end we note that $Y_{b,w}$ is just a vector bundle over $S_{wq,b}$, hence it is homeomorphic to the latter space. But $S_{wq,b}$ is an isoparametric submanifold with all multiplicities at least 2, hence it is simply connected. Consequently, $Y_{b,w}$ is also simply connected. This implies the desired conclusion (we recall that any vector bundle over a simply connected space is orientable).

Next we note that if $x \in Y_{b,w}$, then $h_b(x) \geq h_b(S_{wq,b}) = \langle a+b, b\rangle$. So we have $\langle \mu(x), b\rangle \geq \langle a+b, b\rangle$, which implies

$$\langle \mu(x) - a, b\rangle \geq \langle b, b\rangle.$$

We deduce

$$\|\mu(x) - a\| \cdot \|b\| \geq \langle \mu(x) - a, b\rangle \geq \|b\|^2, \tag{4}$$

hence

$$f(x) \geq f(C_{b,w}).$$

Moreover, if $x \in Y_{b,w}$ has the property that $f(x) = f(C_{b,w})$, then we must have

- $h_b(x) = h_b(S_{wq,b})$ (from equation (4)), hence $x \in S_{wq,b}$
- $\mu(x) - a = \lambda b$, for a number λ (from equation (4)); we deduce that $\lambda = 1$, because $\langle \mu(x) - a, b\rangle = \langle b, b\rangle$.

Consequently, $x \in C_{b,w}$. We have proved that the condition (b) (i) from the definition of a minimally degenerate function is satisfied.

It remains to check condition (b) (ii). Let us consider a point x_0 in $C_{b,w}$. We construct a subspace $V \subset T_{x_0}(M)$ with the following properties.

1. $V \oplus T_{x_0}(Y_{b,w}) = T_{x_0}(M)$
2. $\mathrm{Hess}_{x_0}(f)|_V$ is negative definite.

First we determine the Hessians of h_b and f at the point x_0. To this end we consider the functions H_b and F given by

$$H_b(x) = \langle x, b\rangle, \quad F(x) = \|P(x) - a\|^2, \quad x \in \mathbb{R}^{n+k}$$

where P denotes the orthogonal projection $\mathbb{R}^{n+k} \to \nu_q(M)$. We know that for $v, w \in T_{x_0}(M)$ we have

$$\mathrm{Hess}(f)_{x_0}(v,w) := \langle \partial_v(\nabla f)(x_0), w\rangle = \langle \partial_v(\nabla F)(x_0), w\rangle + \langle A_{(\nabla F)^{\perp}_{x_0}} v, w\rangle,$$

where ∇ stands for gradient and the superscript $\perp$ indicates the orthogonal projection on $\nu_{x_0}(M)$. Because $(\nabla F)_x = 2(P(x) - a)$ (see equation (1)), and $P(x_0) - a = b$, we deduce that

$$\mathrm{Hess}(f)_{x_0} = 2(P + A_b). \tag{5}$$

Similarly, the Hessian of h_b is

$$\mathrm{Hess}(h_b)_{x_0} = A_b. \tag{6}$$

The tangent space $T_{x_0}(Y_{b,w})$ is the subspace of $T_{x_0}(M)$ where the hessian $\mathrm{Hess}(h_b)_{x_0}$ is negative semidefinite. From equation (6) we deduce that for $i \in \{1, 2, \dots, r\}$, the restriction of $\mathrm{Hess}(h_b)_{x_0}$ to $E_i(x_0)$ is given by scalar multiplication by $\langle b, \eta_i(x_0))\rangle$. Consequently we have

$$T_{x_0}(Y_{b,w}) = \sum E_i(x_0)$$

where the sum runs over all i with $\langle b, \eta_i(x_0)\rangle \geq 0$. From elementary Morse theoretical considerations, we know that

$$T_{x_0}(S_{wq,b}) \subset T_{x_0}(Y_{b,w}).$$

Set

$$V := \sum E_i(x_0), \tag{7}$$

where the sum runs over all i with $\langle b, \eta_i(x_0)\rangle < 0$. By Lemma 4.1 (e) (with q replaced by wq), P maps V to 0. The reason is that $b \in \ell_i(x_0)$ exactly when $\langle b, \eta_i(x_0)\rangle = 0$ (since $\ell_i(x_0)$ goes through the origin). By (5) and (6), the restrictions of $\mathrm{Hess}(f)_{x_0}$ and $\mathrm{Hess}(h_b)_{x_0}$ to V are the same, up to a factor of 2. Since the latter restriction is strictly negative definite, the former is like that as well.

(c) Suppose that for $b \in B$ and $w \in W$ the critical set $C_{wq,b}$ has index 0. The considerations from above show that the index of h_b at $S_{wq,b}$ is 0. This implies that the slice $S_{wq,b}$ is the minimum set of h_b on M. Consequently, for any $x \in M$ we must have $h_b(x) \geq h_b(S_{wq,b})$. Like in the proof of point (b), this implies $f(x) \geq f(C_{b,w}) = \|b\|^2$, for all $x \in M$. This is a contradiction (because f can reach the value 0), which concludes the proof. □

Remark. The arguments used above are similar to those used by Kirwan in the proof of Proposition 4.15 of [8] (in the context of Hamiltonian torus actions on symplectic manifolds).

Proof of Theorem 1.2 Only point (ii) has to be proved. By Theorem 3.1, there exists a metric g on M which induces the stratification

$$M = \Sigma_0 \cup \bigcup_{b\in B, w\in W} \Sigma_{b,w},$$

such that $\Sigma_0, \Sigma_{b,w}$ are smooth submanifolds. By Proposition 4.3 and equation (7), the manifolds $\Sigma_{b,w}$ have codimension at least 2 (recall that, by hypothesis, we have $\dim(E_i(x_0)) = m_i \geq 2$). Consequently, Σ_0 is connected. Since the inclusion map $\mu^{-1}(a) \hookrightarrow \Sigma_0$ induces the linear isomorphism $H^0(\mu^{-1}(a), \mathbb{Q}) \simeq H^0(\Sigma_0, \mathbb{Q})$ (see Theorem 3.1), we deduce that $\mu^{-1}(a)$ is connected as well. $\square$

5. Kirwan surjectivity for real flag manifolds

An important class of isoparametric submanifolds are the real flag manifolds. More specifically, we start with a non-compact symmetric space G/K, where G is a non-compact connected semisimple Lie group and $K \subset G$ a maximal compact subgroup. Then K is the fixed point set of an automorphism τ of G (see for instance chapter VI of [7]). The differential map $d(\tau)_e$ is an automorphism of $\mathfrak{g} = \mathrm{Lie}(G)$ and induces the Cartan decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p},$$

where $\mathfrak{k} = \mathrm{Lie}(K)$ and $\mathfrak{p}$ are the $(+1)$-, respectively (-1)-eigenspaces of $(d\tau)_e$.

Now let us consider $\mathfrak{a} \subset \mathfrak{p}$ a maximal abelian subspace. The number $k := \dim(\mathfrak{a})$ is the rank of the symmetric space G/K. The roots of the symmetric space are linear functions $\alpha : \mathfrak{a} \to \mathbb{R}$ with the property that the space

$$\mathfrak{g}_\alpha := \{z \in \mathfrak{g} : [x, z] = \alpha(x)z \text{ for all } x \in \mathfrak{a}\}$$

is non-zero. The set Π of all roots is a root system in $(\mathfrak{a}^*, \langle\ ,\ \rangle)$. Let $\Pi^+ \subset \Pi$ be the set of positive roots with respect to a simple root system. For any $\alpha \in \Pi^+$ we have

$$\mathfrak{g}_\alpha + \mathfrak{g}_{-\alpha} = \mathfrak{k}_\alpha + \mathfrak{p}_\alpha,$$

where $\mathfrak{k}_\alpha = (\mathfrak{g}_\alpha + \mathfrak{g}_{-\alpha}) \cap \mathfrak{k}$ and $\mathfrak{p}_\alpha = (\mathfrak{g}_\alpha + \mathfrak{g}_{-\alpha}) \cap \mathfrak{p}$. We have the direct decompositions

$$\mathfrak{p} = \mathfrak{a} + \sum_{\alpha\in\Pi^+} \mathfrak{p}_\alpha, \quad \mathfrak{k} = \mathfrak{k}_0 + \sum_{\alpha\in\Pi^+} \mathfrak{k}_\alpha,$$

where $\mathfrak{k}_0$ denotes the commutator of $\mathfrak{a}$ in $\mathfrak{k}$.

Since $[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}$, the space $\mathfrak{p}$ is $Ad_G(K) := Ad(K)$-invariant. The orbits of the action of $Ad(K)$ on $\mathfrak{p}$ are called *generalized real flag manifolds.* The restriction of the Killing form of $\mathfrak{g}$ to $\mathfrak{p}$ is an $Ad(K)$-invariant inner product on $\mathfrak{p}$, which we denote by $\langle\ ,\ \rangle$.

Proposition 5.1. (see e.g. [11], Example 6.5.6) *Let $M = Ad(K).q$ be the orbit of $q \in \mathfrak{a}$.*

(i) If q is regular (i.e. not contained in any of the hyperplanes $\ker(\alpha)$, $\alpha \in \Pi$), then M is an isoparametric submanifold of $(\mathfrak{p}, \langle\ ,\ \rangle)$. The curvature distributions at q are

$$E_\alpha(q) = [q, \mathfrak{k}_\alpha + \mathfrak{k}_{2\alpha}],$$

where α is a positive indivisible root. Hence the multiplicities of M are

$$m_\alpha = \dim(\mathfrak{k}_\alpha) + \dim(\mathfrak{k}_{2\alpha}).$$

The normal space to M at the point q is

$$\nu_q(M) = \mathfrak{a}.$$

(ii) If q is not regular, then M is a manifold parallel to an isoparametric submanifold.

In [10] we have investigated the action of

$$K_0 := Z_K(\mathfrak{a}) = \{k \in K : Ad(k)(x) = x, \text{ for all } x \in \mathfrak{a}\}$$

on M. It turns out that if all m_α are strictly greater than 1, then K_0 is connected and the action of K_0 on M is equivariantly formal. In this paper we will address the following question.

Problem. *If $\mu : M \to \mathfrak{a}$ is the restriction to M of the orthogonal projection map $P : \mathfrak{p} \to \mathfrak{a}$ and a is an arbitrary point in $\mathfrak{a}$, is it true that the Kirwan type map*

$$\kappa : H^*_{K_0}(M, \mathbb{Q}) \to H^*_{K_0}(\mu^{-1}(a), \mathbb{Q}) \tag{8}$$

is surjective?

We will prove that the answer to this question is affirmative under certain restrictions.

Proposition 5.2. (Surjectivity criterium) *Assume that all multiplicities m_α are strictly greater than 1 and for any $b \in \mathfrak{a}$, the set*

$$Z_{\mathfrak{p}}(b) := \{x \in \mathfrak{p} : [x, b] = 0\}$$

is the fixed point set of a certain torus $T_b \subset K_0$. Then the map κ described by equation (8) is surjective, for any $a \in \mathfrak{a}$.

Proof. We consider the function $f : M \to \mathbb{R}$, $f(x) = \|\mu(x) - a\|^2$. By Theorem 1.2 (i), this is a minimally degenerate function. Moreover, f is K_0-invariant. Let C be a critical set of f (by equation (3), C can be $\mu^{-1}(a)$ or $C_{b,w}$). Denote $M^\pm = f^{-1}((-\infty, f(C) \pm \epsilon))$, for $\epsilon > 0$ sufficiently small. By [8], chapter 10 (see also [4], section 9) we have the commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & H^*_{K_0}(M_+, M_-) \arrow[r] \arrow[d, "\simeq"] & H^*_{K_0}(M_+) \arrow[r] \arrow[d] & H^*_{K_0}(M_-) \arrow[r] & \cdots \\
& H^{*-\mathrm{index}(C)}_{K_0}(C) \arrow[r, "\cup e_C"] & H^*_{K_0}(C) & &
\end{tikzcd} \tag{9}$$

where $e_C \in H^*_{K_0}(C)$ denotes the equivariant Euler class of the normal bundle $\nu(\Sigma_C)|_C$.

We will prove that e_C is not a divisor of zero. If $C = \mu^{-1}(a)$, then $e_C = 1$ and the claim is obvious. Let us consider the case when $C = C_{b,w} = \mu^{-1}(a+b) \cap S_{wq,b}$ (see equation (3)). According to a criterium of Atiyah and Bott (see [2], Proposition 13.4), it is sufficient to prove that there exists a torus $T \subset K_0$ with the property that the only points in $\nu(\Sigma_C)|_C$ which are fixed by T are those from C. But $\nu(\Sigma_C)|_C$ is contained in $\nu(S_{wq,b})|_C$ (because Σ_C contains $Y_{b,w}$ on a neighbourhood of C, and $\dim \Sigma_C = \dim Y_{b,w}$, see Theorem 3.1). We will show that the fixed points of the torus T_b (see the statement of the proposition) on $\nu(S_{wq,b})$ are exactly those from $S_{wq,b}$. Indeed, the fixed point set of T_b on M is

$$M \cap Z_{\mathfrak{p}}(b) = Crit(h_b),$$

which is the disjoint union of all slices $S_{wq,b}$, where $w \in W$ (see section 4). □

We will give an example where the criterium applies.

Example 1. We consider the symmetric space $SU(2n)/Sp(n)$. In order to give a precise description of this space and the corresponding isotropy representation, we write quaternions as $a+ib+jc+kd = a+ib+j(c-id)$, where $a,b,c,d \in \mathbb{R}$. In this way we obtain the $\mathbb{R}$-linear isomorphism

$$\mathbb{H}^n \simeq \mathbb{C}^{2n}, u + jv = (u,v), \tag{10}$$

where $u, v \in \mathbb{C}^n$. The multiplication from the right endows $\mathbb{H}^n$ with a structure of a $\mathbb{H}$-vector space. The symplectic group $Sp(n)$ consists of all elements of $SU(2n)$ which are $\mathbb{H}$-linear via the isomorphism (10). One can easily see that the transformation of $\mathbb{C}^{2n}$ given by the multiplication from the right with j is given by the composition of

$$J_n := \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix},$$

with the complex conjugation. Consequently, $Sp(n)$ is the fixed point set of the involution σ of $SU(2n)$ given by

$$\sigma(A) = J_n \bar{A} J_n^{-1},$$

$A \in SU(2n)$. More precisely, $Sp(n)$ is the set of all $A \in SU(2n)$ of the form

$$A = \begin{pmatrix} b & -\bar{c} \\ c & \bar{b} \end{pmatrix},$$

where $b, c \in \mathrm{Mat}(n \times n, \mathbb{C})$. Via the map

$$A = \begin{pmatrix} b & -\bar{c} \\ c & \bar{b} \end{pmatrix} \mapsto a := b + jc,$$

we obtain the description of the symplectic group as

$$Sp(n) = \{a \in \mathrm{Mat}(n \times n, \mathbb{H}) \ : \ a \cdot a^* = I_n\}.$$

In the Cartan decomposition $\mathfrak{su}(2n)=\mathfrak{sp}(n)\oplus\mathfrak{p}$, the space $\mathfrak{p}$ consists of all $X\in\mathfrak{su}(2n)$ with $\sigma(X)=-X$. This gives

$$\mathfrak{p}=\{X=\left(\begin{array}{cc} y & \bar{z} \\ z & -\bar{y}\end{array}\right) \ : \ y,z\in \mathrm{Mat}(n\times n,\mathbb{C}), y=-y^*, Tr(y)=0, z=-z^T\}.$$

On the other hand, $Sp(n)$ acts by conjugation on the space

$$\tilde{\mathfrak{p}}:=\{x\in \mathrm{Mat}(n\times n,\mathbb{H}) \ : \ x=x^*, Tr(x)=0\}$$

of all traceless hermitian matrices with coefficients in $\mathbb{H}$.

Claim 1. The map

$$\mathfrak{p}\to\tilde{\mathfrak{p}}, \quad X=\left(\begin{array}{cc} y & \bar{z} \\ z & -\bar{y}\end{array}\right)\mapsto iy+j(iz)$$

is an $Sp(n)$-equivariant linear isomorphism.

The proof of the claim is a direct consequence of the fact that the map

$$\mathrm{Mat}(n\times n,\mathbb{H})\to \mathrm{Mat}(2n\times 2n,\mathbb{C}), \quad b+jc\mapsto \left(\begin{array}{cc} b & -\bar{c} \\ c & \bar{b}\end{array}\right)$$

is a ring homomorphism.

Consequently, the isotropy representation of $SU(2n)/Sp(n)$ is given by the conjugation action of $Sp(n)$ on $\tilde{\mathfrak{p}}$. Note that $\mathfrak{a}$ is the space of all real $n\times n$ diagonal matrices with trace 0 and

$$K_0=\{Diag(q_1,\ldots,q_n)\mid q_l\in\mathbb{H}, |q_l|=1, l=1,2,\ldots n\}.$$

We will show that the hypothesis of Propositon 5.2 is satisfied. Take $b\in\mathfrak{a}$ of the form $b=Diag(b_1,\ldots,b_n)$, where $b_1,\ldots,b_n\in\mathbb{R}$. An easy calculation shows that

$$Z_{\tilde{\mathfrak{p}}}(b)=\{x=(x_{ij})_{1\le i,j\le n}\in\tilde{\mathfrak{p}} \ : \ x_{ij}=0 \text{ if } b_i\neq b_j\}.$$

We show that this is the fixed point set of the subtorus of K_0 given by

$$T_b:=\{Diag(z_1,\ldots,z_n)\in\mathbb{C}^n \ : \ |z_1|=\ldots=|z_n|=1, z_i=z_j \text{ exactly when } b_i=b_j\}$$

where $\mathbb{C}$ is regarded as the set of all quaternions of type $a+ib$, with $a,b\in\mathbb{R}$. Indeed, this is a straightforward consequence of the following elementary fact.

Claim 2. If $q\in\mathbb{H}$ such that $z_1qz_2=\gamma$ for any $z_1,z_2\in\mathbb{H}$ of the form $a+ib$, where $a,b\in\mathbb{R}$, $a^2+b^2=1$, then $q=0$.

Now we give another example, where the criterium does not apply.

Example 2. Consider the symmetric space $\mathbb{C}P^2=SU(3)/U(2)$, where $U(2)$ is embedded in $SU(3)$ via

$$A\mapsto\left(\begin{array}{cc} \frac{1}{\det(A)} & 0 \\ 0 & A\end{array}\right),$$

$A\in U(2)$. It turns out that $\mathfrak{p}$ consists of all matrices of the type

$$\left(\begin{array}{ccc} 0 & -\bar{z}_1 & -\bar{z}_2 \\ z_1 & 0 & 0 \\ z_2 & 0 & 0\end{array}\right),$$

where $z_1, z_2 \in \mathbb{C}$. We choose $\mathfrak{a}$ the space of all matrices

$$a := \begin{pmatrix} 0 & -x & 0 \\ x & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

with $x \in \mathbb{R}$. The positive roots are α and 2α, where $\alpha(a) = -x$ and the corresponding root spaces in $\mathfrak{p}$ are

$$\mathfrak{p}_\alpha = \{ \begin{pmatrix} 0 & 0 & -\bar{z} \\ 0 & 0 & 0 \\ z & 0 & 0 \end{pmatrix} : z \in \mathbb{C}\}, \quad \mathfrak{p}_{2\alpha} = \{ \begin{pmatrix} 0 & iy & 0 \\ iy & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} : y \in \mathbb{R}\}.$$

We deduce that $m_\alpha = 2 + 1 = 3$. An easy calculation shows that $K_0 := Z_K(\mathfrak{a})$ is the subgroup of $U(2)$ which consists of

$$\begin{pmatrix} z & 0 & 0 \\ 0 & z & 0 \\ 0 & 0 & \frac{1}{z^2} \end{pmatrix}$$

for $z \in \mathbb{C} \setminus \{0\}$. One can see that K_0 acts trivially not only on $\mathfrak{a}$, but also on $\mathfrak{p}_{2\alpha}$. We deduce that it acts trivially on $\mathfrak{k}_{2\alpha}$ as well. This implies that there is no subgroup of K_0 whose fixed point set in $\mathfrak{k}$ is just $\mathfrak{k}_0$. So the hypothesis of Proposition 5.2 is not satisfied.

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