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Cohomology of Isoparametric Hypersurfaces in Hilbert Space

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Abstract. One obtains descriptions of the cohomology ring of the manifolds mentioned in the title in terms of their multiplicities and the Euler, respectively Stiefel–Whitney classes of the curvature distributions. Lifts of equifocal hypersurfaces in symmetric spaces are also discussed.

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1. Introduction

Let M be an immersed submanifold of the Hilbert space V. Denote by $\nu(M) = \{(x, v): x \in M, v \in \nu(M)_x\}$ the normal bundle and consider the end-point map $\eta: \nu(M) \to V$, $\eta(x, v) = x + v$. We say that M is *proper Fredholm* if it has a finite codimension and

(i) the map η is Fredholm, i.e. its derivative at any point is a Fredholm linear map;
(ii) the restriction of η to any normal disk bundle of finite radius r is proper.

The notion was considered for the first time by C.-L. Terng in [13]. One of the main ideas of her paper is that under the two hypotheses above the major problems which occur when passing from the geometry of submanifolds in Euclidean spaces to that of submanifolds in infinite-dimensional Hilbert space are dropped: we can manage the spectrum of the shape operators and we can apply the classical Morse theory of submanifolds. More precisely, if M is proper Fredholm, then the shape operator A_ξ associated to any normal vector ξ is compact and the squared distance function $f_a: M \to \mathbb{R}$, $f_a(x) = \|x - a\|^2$, satisfies the condition (C) of Palais and Smale.

A proper Fredholm submanifold M of V is called *isoparametric* if

(a) the normal bundle $\nu(M)$ is flat with respect to the normal connection,
(b) for any parallel normal vector field ξ and any $p, q \in M$, the shape operators $A_{\xi(p)}$ and $A_{\xi(q)}$ are orthogonally conjugate.

The isoparametric submanifolds here will be always supposed as being connected and complete.

So far there exists only one class of examples of isoparametric submanifolds in infinite-dimensional Hilbert space, namely the lifts of so-called equifocal submanifolds of symmetric spaces to certain path spaces (we shall return to this in more details in Section 5); for sake of simplicity we call them 'lifts of equifocal hypersurfaces'. Since the isoparametric hypersurfaces in a sphere are equifocal, we can lift them and get an important subclass.

The main goal of our paper is to obtain descriptions of the cohomology ring of an arbitrary isoparametric hypersurface in Hilbert space in terms of multiplicities and characteristic classes (Euler, respectively Stiefel–Whitney) of the curvature distributions. More precisely, fix M as an isoparametric hypersurface in the infinite dimensional Hilbert space V and choose E_- and E_+ as the curvature distributions whose corresponding curvature spheres S_-, respectively S_+, have the smallest radii among all curvature spheres, m_- and m_+ their ranks. The antipodal maps of the leaves S_- and S_+ give rise to the involutive automorphisms φ_- and φ_+ of M. The group W generated by them is what we call the Weyl group of M (it is really an affine Weyl group of type $\tilde{A}_1$). The homology module of M can be easily obtained by Morse theory applied to a nondegenerate distance function. Its critical set is a W-orbit $W.p, p \in M$ and passing through the critical level corresponding to an arbitrary critical point $x = w.p$ brings in homology a new one-dimensional direct summand, generated by a cycle of the Bott–Samelson type. We modified the cycle originally constructed by Hsiang, Palais and Terng in [5]; the advantage of our cycle is that it makes use only of the spheres S_- and S_+. More exactly, if

$$w = \varphi_{i_r}\varphi_{i_{r-1}} \dots \varphi_{i_1}, \quad i_j = (-1)^{j+1},$$

then our Bott–Samelson cycle at $x = w.p$ is

$$F_r = \{(z_1, \dots, z_r) \in M^{\times r}: z_1 \in S_{i_1}(p), z_2 \in S_{i_2}(z_1), \dots, z_r \in S_{i_r}(z_{r-1})\},$$

together with $u_r: F_r \to M$, $u_r(z_1, \dots, z_r) = z_r$ (it is the homology cycle $u_{r*}([F_r])$ which has to be added as a generator in $H_*(M)$ after passing the critical level of x). The map

$$\pi_r: F_r \to F_{r-1}, \quad \pi_r(z_1, \dots, z_r) = (z_1, \dots, z_{r-1})$$

is a sphere bundle map and

$$s_r: F_{r-1} \to F_r, \quad s_r(z_1, \dots, z_{r-1}) = (z_1, \dots, z_{r-1}, z_{r-1})$$

is a section of it. The space F_r being an iterated sphere bundle, its homology module and even cohomology ring are easily manageable. Through reasons related to the orientability of F_r, we shall take the coefficients ring $\mathcal{R} = \mathbb{Z}$ if both m_- and m_+ are greater than 1 and $\mathcal{R} = \mathbb{Z}_2$ if the contrary (of course, a simple modulo 2 reduction will furnish the $\mathbb{Z}_2$- cohomology ring of M also in the first situation). In the first case, we choose arbitrary orientations on E_- and E_+ and fix them. Because $E_\pm$ restricted to $S_\pm$ is the tangent bundle, in this way we get orientations on $S_\pm$.

The following two numbers associated to M play an important role:

$$(d_1, d_2) = \begin{cases} (e(E_+|_{S_-}), e(E_-|_{S_+})), & \text{if } \mathcal{R}=\mathbb{Z}, \\ (w_1(E_+|_{S_-}), w_1(E_-|_{S_+})), & \text{if } \mathcal{R}=\mathbb{Z}_2. \end{cases}$$

THEOREM 1.1. *Put* $i_k = (-1)^{k+1}$. *We have*

(a) $H_*(F_r) \simeq H_*(S_{i_1}(p)) \otimes \ldots \otimes H_*(S_{i_r}(p))$.

(b) *If* $\xi_j \in H^{m_{i_j}}(F_r)$ *is determined by* $\xi_j([S_{i_k}]) = \delta_{jk}$ *(Kronecker delta),* $1 \leqslant j, k \leqslant r$, *then* $H^*(F_r)$ *is the graded commutative ring generated by* $\xi_1, \ldots, \xi_r$, *subject to the relations* $(1 - d_{i_k i_k})\xi_k^2 - \sum_{j=1}^{k-1} d_{i_j i_k} \xi_j \xi_k = 0$, *where*

$$d_{i_j i_k} = \begin{cases} e(E_{i_k}|_{S_{i_j}}), & \text{if } \mathcal{R} = \mathbb{Z} \\ w_1(E_{i_k}|_{S_{i_j}}), & \text{if } \mathcal{R} = \mathbb{Z}_2 \end{cases} = \begin{cases} d_1, & \text{if } k \text{ is odd}, \ j \text{ even}, \\ d_2, & \text{if } k \text{ is even}, \ j \text{ odd}, \\ 1 + (-1)^{m_{i_k}}, & \text{if } k \equiv j \bmod 2. \end{cases}$$

The inclusions of the chain $F_1 \overset{s_2}{\hookrightarrow} F_2 \overset{s_3}{\hookrightarrow} F_3 \overset{s_4}{\hookrightarrow} \cdots \overset{s_r}{\hookrightarrow} F_r$ satisfy $\pi_k \circ s_k = \pi_k$, $1 \leqslant k \leqslant r$, hence they make possible the definition of $F = \bigcup_{k \geqslant 1} F_k$ and

$$u = \lim_{\overrightarrow{k}} u_k.$$

Through the reasons sketched above, $u_*: H_*(F) \to H_*(M)$ is surjective, hence $u^*: H^*(M) \to H^*(F)$ is injective.

THEOREM 1.2. (a) *Regarded as a module,* $H^*(F)$ *is the set of all formal homogeneous series* $\sum_{J \subset \mathbb{N}\setminus\{0\} finite} n_J \xi_J$, *where* $n_J \in \mathcal{R}$ *and* $\xi_J := \xi_{j_1} \ldots \xi_{j_k}$, *for any* $J = \{j_1 < \ldots < j_k\} \subset \mathbb{N} \setminus \{0\}$. *The structure of* $H^*(F)$ *as graded commutative ring is completely determined by the expression of* ξ_k^2 *given by Theorem 1.1* (b).

(b) *Associate to any* $J = \{j_1 < \ldots < j_k\} \subset \mathbb{N} \setminus \{0\}$ *the element* $g(J) = \varphi_{i_{j_k}} \ldots \varphi_{i_{j_1}}$ *of* W, *whose length* $l(g(J))$ *is at most* k. *Then the ring* $H^*(M, \mathcal{R})$ *is the subring of* $H^*(F, \mathcal{R})$ *which is freely generated as* $\mathcal{R}$*-module by all formal series of the form*

$$\sum_{J \subset \mathbb{N}\setminus\{0\}, g(J)=w, Card(J)=l(w)} \xi_J,$$

with $w \in W$. *Consequently,* $H^*(M, \mathcal{R})$ *is uniquely determined by* m_-, m_+, d_1 *and* d_2.

In order to obtain more concrete expressions of $H^*(M, \mathcal{R})$ we look, for given m_-, m_+, d_1 and d_2, for prototypes which are easy to manage. These will be always lifts of equifocal hypersurfaces. In this way, we get

THEOREM 1.3. (a) *If* $m_- \neq m_+$ *or* $m_- = m_+ > 1$ *is an odd number, then*

$$H^*(M, \mathcal{R}) \simeq H^*(S^{m_-} \times S^{m_+} \times \Omega S^{m_- + m_+ + 1}, \mathcal{R}).$$

(b) *If* $m_- = m_+ = 1$, *then*

$$H^*(M, \mathbb{Z}_2) \simeq \begin{cases} H^*(S^1 \times \Omega S^2, \mathbb{Z}_2), & \text{if } (d_1, d_2) = (0, 0), \\ H^*(B_2/T \times \Omega S^9, \mathbb{Z}_2)_{\frac{1}{2}}, & \text{if } (d_1, d_2) = (1, 0), \\ H^*(A_2/T \times \Omega S^7, \mathbb{Z}_2)_{\frac{1}{2}}, & \text{if } (d_1, d_2) = (1, 1), \end{cases}$$

where A_2, B_2 *denote the simply connected simple Lie groups of these types,* T *the corresponding maximal torus and the subscript* $\frac{1}{2}$ *means that we divide by 2 the dimensions of all generators of the corresponding ring.*

Regarding the remaining case, when $m_- = m_+$ is an even integer, it seems difficult to obtain restrictions for the integers d_1 and d_2 and then concrete formulae for the cohomology rings. But if we restrict ourselves to lifts of equifocal hypersurfaces, this task can be almost completely achieved:

THEOREM 1.4. *Let* $\tilde{M}$ *be the lift of an equifocal hypersurface, having multiplicities* $m_- = m_+$ *even and let* d_1 *and* d_2 *be the integers given by Theorem 1.1. Then*

(a) *It holds that* $d_1 d_2 \in \{0, 1, 2, 3, 4\}$.
(b) *If* $d_1 = d_2 = 0$, *then* $H^*(\tilde{M}, \mathbb{Z}) \simeq H^*(S^m \times S^m \times \Omega S^{2m+1}, \mathbb{Z})$.
(c) *If* $d_1 = d_2 = \pm 2$, *then* $H^*(\tilde{M}, \mathbb{Z}) \simeq H^*(S^m \times \Omega S^{m+1}, \mathbb{Z})$.
(b) *If* $d_1 d_2 \in \{0, 4\}$ *and* $(d_1, d_2) \neq (0, 0)$, *then*

$$H^*(\tilde{M}, \mathbf{Q}) \simeq H^*(S^m \times \Omega S^{m+1}, \mathbf{Q}).$$

(c) *We have*

$$H^*(\tilde{M}, \mathbb{Z}) \simeq \begin{cases} H^*(A_2/T \times \Omega S^7, \mathbb{Z})_{\frac{m}{2}}, & \text{if } d_1 d_2 = 1, \\ H^*(B_2/T \times \Omega S^9, \mathbb{Z})_{\frac{m}{2}}, & \text{if } d_1 d_2 = 2, \\ H^*(G_2/T \times \Omega S^{13}, \mathbb{Z})_{\frac{m}{2}}, & \text{if } d_1 d_2 = 3, \end{cases}$$

where A_2, B_2, G_2 *denote the simply connected simple Lie groups of these types,* T *the corresponding maximal tori and the subscript* $m/2$ *means that we multiply the dimensions of all generators of the corresponding ring by* $m/2$.

We are not able to elucidate the situations $d_1 d_2 = 0$ and $d_1 d_2 = 4$. Does the first one imply $d_1 = d_2 = 0$? Does the second one imply $d_1 = d_2 = \pm 2$? We do not know the answer.

Notice that such evaluations can be relevant for the following fundamental questions:

(1) Does there exist isoparametric hypersurfaces in Hilbert space which are not lifts of equifocal hypersurfaces?

(2) Does there exist a lift of an equifocal hypersurface in a certain symmetric space which is not in the same time lift of an isoparametric hypersurface in a sphere?

A positive answer to the first question could be given (Theorem 1.3) by an isoparametric hypersurface in Hilbert space for which $m_- = m_+$ even and $d_1 d_2 > 4$. As to the second question, we can easily see that the lift of an isoparametric hypersurface in a sphere cannot lead to $(d_1, d_2) = (1, 4)$ or to $d_1 d_2 = 0$ and $(d_1, d_2) \neq (0, 0)$ (see Proposition 5.4 (e)). Hence it would be enough to find an equifocal hypersurface in a symmetric space which furnishes these values of d_1 and d_2. We couldn't find examples able to illustrate either of these two situations and the questions still remain open.

As a direct consequence of Theorem 1.3, we would like to notice the following result, closely related to question 2:

COROLLARY 1.5. *Let $\tilde{M}$ be the lift of an equifocal hypersurface in a symmetric space, having multiplicities $m_- = m_+$ even. Then there exists M' an isoparametric hypersurface in a sphere, whose lift $\tilde{M}'$ satisfies:*

$$H^*(\tilde{M}, \mathbf{Q}) \simeq H^*(\tilde{M}', \mathbf{Q}).$$

Our calculations use the ideas initiated by Hsiang, Palais and Terng in [5] and carried over by us in [7] which had as the main goal the cohomology of isoparametric submanifolds in euclidean spaces. The proof of Theorem 1.4 involves estimations obtained by Grove and Halperin in [4] about the rational homotopy fiber associated to a double mapping cylinder. There are two basic facts which allow us to do that: if M is an equifocal hypersurface in the simply connected Lie group G and M_1, M_2 are its focal manifolds, then, by a result of [14], G is the double mapping cylinder DM associated to the mappings $M_1 \leftarrow M \rightarrow M_2$. Moreover, the lift of M to the Hilbert space is homeomorphic to the homotopy fiber F of the inclusion $M \hookrightarrow DM$ (for more details, see Section 5). The basic tool of [4] is the classification of F up to the rational homotopy type.

2. Homology of Isoparametric Hypersurfaces

Take V to be an (always separable) infinite-dimensional Hilbert space and $M \subset V$ an isoparametric hypersurface. One of the main ingredients we need in studying the topology of M is the so-called F-cycle associated to a critical point of a nondegenerate distance function. The goal of this section is to describe this notion.

We begin with some basic facts of the theory of isoparametric submanifolds, adapted to our particular case; for more details, the reader can consult [11]. Let us denote by ξ the unit normal vector field on M. The eigenspace decomposition of TM_p relative to the compact, selfadjoint operator $A_{\xi(p)}, p \in M$ arbitrary, furnishes a countable family of distributions $\{E_i\}_{i \in \mathbb{Z}}$, where $E_0(p)$ denotes the kernel of $A_{\xi(p)}$ and $E_i(p) = \mathrm{Ker}(A_{\xi(p)} - t_i \mathrm{id})$ is finite dimensional, for $i \neq 0$. The punctual spectrum

$\{t_i\}_{i\in\mathbb{Z}}$ is obviously independent of p. We arrange the subscripts in such a way that

$$\cdots\frac{1}{t_{-2}}<\frac{1}{t_{-1}}<0<\frac{1}{t_1}<\frac{1}{t_2}<\cdots.$$

It turns out that every distribution E_i, $i\neq 0$, is integrable and its leaf through p is the full sphere $S_i(p)$ centered at $p+(1/t_i)\xi(p)$, of radius $1/t_i$, in the affine space $p+E_i(p)\oplus\nu(M)_p$. For a fix $j\neq 0$, the reflection φ_j of the affine line $p+\nu(M)_p$ at $p+(1/t_j)\xi(p)$ leaves invariant the set $\{p+(1/t_i)\xi(p): i\neq 0\}$ of the centers of all curvature spheres through p. Consequently, the group generated by $\{\varphi_j\}_{j\geqslant 1}$ is an affine Weyl group of rank 1 and the curvature radii can be expressed as

$$\frac{1}{t_i}=\begin{cases}\alpha+(i-1)l, & \text{if } i>0,\\ -\beta-(i+1)l, & \text{if } i<0,\end{cases}$$

where α, β, l are positive numbers, $\alpha+\beta=l$, l being the distance between two focal points on the line $p+\nu(M)_p$. Notice that the Weyl group W defined above acts in a obvious way on M. The intersection of M with $p+\nu(M)_p$ is just the W-orbit of p. Also notice that W acts on the set of all curvature spheres in such a way that the latter ones can be partitioned as the disjoint union $W.S_{-1}(p)\sqcup W.S_{+1}(p)$. Hence, if m_i denotes $\dim(S_i)=\mathrm{rank}(E_i)$, then

$$m_{-1}=m_2=m_{-3}=m_6=m_{-5}=\cdots:=m_-$$

and

$$m_1=m_{-2}=m_3=m_{-6}=m_5=\ldots:=m_+.$$

Fix p a point of M and choose a arbitrarily in the interior of the Weyl chamber

$$\left[p+\frac{1}{t_{-1}}\xi(p), p+\frac{1}{t_1}\xi(p)\right].$$

The distance function

$$f_a\colon M\to\mathbb{R},\ f_a(x)=\|x-a\|^2,$$

is then nondegenerate and its critical set is the intersection of $p+\nu(M)_p$ with M, i.e. the set $W.p$. Any critical point x of f_a is of linking type: from the point of view of the Morse theory, this means that the cell attached to the sublevel set after passing the critical level $f_a^{-1}(x)$ can be extended to a full cycle in the upper sublevel set; consequently, the only change which occurs in the homology module is the addition of a new one-dimensional direct summand, namely those generated by that cycle. A beautiful geometric construction of Hsiang, Palais and Terng furnishes the linking cycle we are referring to as a so-called cycle of Bott–Samelson type. More precisely, this is of the form (N,φ), where N is a closed manifold of a dimension equal to the index of x and $\varphi\colon N\to M$ is a smooth map such that $f_a\circ\varphi$ has a unique maximum, which is nondegenerate, at the point $\varphi^{-1}(x)$. The linking cycle will be then $\varphi_*([N])$, where $[N]$ is the fundamental homology cycle of N. Of course, homology can be considered with coefficients in the ring $\mathcal{R} = \mathbb{Z}$ only if N is orientable; otherwise, we shall always take $\mathcal{R} = \mathbb{Z}_2$.

For more details concerning the topic from above, one can consult the original paper of Hsiang, Palais and Terng [5]. We would like to sketch, in a few words their construction, adapted to our particular situation. Counting from a to x, we consider the focal points of M contained in the line segment $[ax]$: suppose they are of the form $x + (1/t_i)\xi(x)$, $i = k, k-1, \dots, 1$. Then define

$$N = \{(y_1, \dots, y_k): y_1 \in S_k(x), y_2 \in S_{k-1}(y_1), \dots, y_k \in S_1(y_{k-1})\}$$

and

$$u: N \to R, \;\; u(y_1, \dots, y_k) = y_k.$$

It is not difficult to see that $d(u)_{(x,\dots x)}$ maps $(TN)_{(x,\dots,x)}$ isomorphically onto the negative space of f_a at x. Now, we can show that if $(y_1, \dots, y_k) \in N$, then $f_a(y_k) \leqslant f_a(x)$, with equality only when $(y_1, \dots, y_k) = (x, \dots x)$. Namely, we notice that the broken polygonal line

$$\left[a, x + \frac{1}{t_k}\xi(x)\right] \cup \left[y_1 + \frac{1}{t_k}\xi(y_1), y_1 + \frac{1}{t_{k-1}}\xi(y_1)\right] \cup \dots \cup$$
$$\left[y_{k-1} + \frac{1}{t_2}\xi(y_{k-1}), y_{k-1} + \frac{1}{t_1}\xi(y_{k-1})\right] \cup \left[y_k + \frac{1}{t_1}\xi(y_k), y_k\right]$$

has the same length as $[ax]$ and joins a to y_k, hence the inequality from above is clear. At the end of the construction we notice that N is an iterated sphere bundle whose initial base space and fibers are spheres of dimensions m_- and m_+, so N is orientable if m_- and m_+ are both greater than 1. This is the reason why the ring $\mathcal{R}$ of coefficients will be $\mathbb{Z}$ in the latter situation and $\mathbb{Z}_2$ otherwise. The module $H_*(M, \mathcal{R})$ is then free, with a basis consisting of the cycles associated to all critical points of f_a. We shall usually identify $H_*(M, \mathcal{R})$ with the module freely generated by the elements of W.

The main goal of the section is to show how we can replace the homology cycles considered by Hsiang, Palais and Terng in their paper by some others, which will make our future computations possible. Fix once again $x \in M$ a critical point of f_a and take $w \in W$ determined by $x = w.p$. The reduced expression of w is in our case, unique, say $w = \varphi_{i_k}\varphi_{i_{k-1}} \dots \varphi_{i_1}$, with $i_j \in \{-1, 1\}$. We define the *F-cycle* associated to x as the pair (F, u') with

$$F = \{(z_1, \dots, z_k): z_1 \in S_{i_1}(p), z_2 \in S_{i_2}(z_1), \dots, z_k \in S_{i_k}(z_{k-1})\}$$

and $u': F \to M, (z_1, \dots, z_k) \mapsto z_k$. For sake of simplicity, we shall usually denote F by $S_{i_1}(p)S_{i_2} \dots S_{i_p}$.

PROPOSITION 2.1. *The map* $\Phi: F \to N$,

$$(z_1, \dots, z_k) \mapsto (\varphi_{i_k}\varphi_{i_{k-1}} \dots \varphi_{i_2} z_1, \dots, \varphi_{i_k}(z_{k-1}), z_k)$$

is a diffeomorphism which satisfies $u \circ \Phi = u'$. *Consequently,* (F, u') *is another cycle of the Bott–Samelson type of* f_a *at* x, *which induces in homology the cycle* $u'_*[F] = u_*[N]$.

Proof. Since Φ is an automorphism of $M^{\times r}$, it is enough to show that Φ maps F into N and its inverse maps N into F. But these facts are obvious. □

3. The Action of W on $H_{lower}(M)$

In the previous section we describe the action of the Weyl group $W = \langle \varphi_-, \varphi_+ | \varphi_-^2 = \varphi_+^2 = 1 \rangle$ on M (from now on, + means +1 and − means −1). For reasons that will become clear later, we need a complete description of the representation induced by W on the $\mathcal{R}$-module

$$H_{lower}(M, \mathcal{R}) := \begin{cases} H_{m_-}(M, \mathcal{R}) \oplus H_{m_+}(M, \mathcal{R}), & \text{if } m_- \neq m_+, \\ H_m(M, \mathcal{R}), & \text{if } m_- = m_+. \end{cases}$$

A basis of this module consists in the cycles $b_+ = [S_+(p)]$, $b_- = [S_-(p)]$ carried by the two fundamental curvature spheres. What we need is $\varphi_{i*}(b_j)$ as a linear combination of b_- and b_+, for $i \neq j$, having in mind that $\varphi_{i*}(b_i) = (-1)^{m_i+1} b_i$, $i = +, -$. We shall get that in terms of the Euler, respectively Stiefel–Whitney classes of the distributions E_- and E_+. As already specified in the introduction, in the case $\mathcal{R} = \mathbb{Z}$ we fix arbitrary orientations on E_- and E_+. The spheres S_- and S_+ are automatically oriented.

PROPOSITION 3.1. *It holds* $\varphi_{+*}(b_-) = b_- - d_1 b_+$, $\varphi_{-*}(b_+) = b_+ - d_2 b_-$, *where*

$$(d_1, d_2) = \begin{cases} (0, 0), & \text{if } m_- \neq m_+, \\ (e(E_+|_{S_-}), e(E_-|_{S_+})), & \text{if } m_- = m_+ > 1, \\ (w_1(E_+|_{S_-}), w_1(E_-|_{S_+})), & \text{if } m_- = m_+ = 1. \end{cases}$$

Proof. Consider $F_2 = S_-(p)S_+$, which is the total space of the S_+-bundle $\pi_2 : F_2 \to S_-(p)$, $(z_1, z_2) \mapsto z_1$. The bundle admits a section, namely $s_2 : S_-(p) \to F_2$, $z \mapsto (z, z)$. It is not difficult to see that

$$H_*(F_2, \mathcal{R}) = \mathrm{Ker}\pi_{2*} \oplus \mathrm{Im}s_{2*}, \mathrm{Ker}\pi_{2*} = \mathcal{R}[a_2] \quad \text{and} \quad \mathrm{Im}s_{2*} = \mathcal{R}[a_1],$$

where we have denoted

$$a_1 = \{(z, z) : z \in S_-(p)\} \qquad a_2 = \{(p, z_2) : z_2 \in S_+(p)\}.$$

The map

$$\overline{\varphi}_+ : F_2 \to F_2, \overline{\varphi}_+(z_1, z_2) = (z_1, \varphi_+(z_2))$$

is an involutive automorphism of degree $(-1)^{m_++1}$ of F_2 with the property that $\pi_2 \circ \overline{\varphi}_+ = \pi_2$. It follows that $\pi_{2*}\overline{\varphi}_{+*}(a_1) = \pi_{2*}(a_1)$, hence $\overline{\varphi}_{+*}(a_1) - a_1 \in \mathrm{Ker}\pi_{2*}$.

Let $d_1 \in \mathcal{R}$ be given by

$$\overline{\varphi}_{+*}(a_1) = a_1 - d_1 a_2. \tag{1}$$

Notice now that $\varphi_+ \circ u_2 = u_2 \circ \overline{\varphi_+}$. Since $u_{2*}(a_1) = b_-$ and $u_{2*}(a_2) = b_+$, it follows from (1) that

$$\varphi_{+*}(b_-) = b_- - d_1 b_+. \tag{2}$$

It now remains to calculate d_1. It it quite clear that $d_1 = 0$ when $m_- \neq m_+$. Suppose from now on that $m_- = m_+$. Consider the basis (ξ_1, ξ_2) of $H^{lower}(F_2, \mathcal{R})$ dual to (a_1, a_2). By the theorem of Leray–Hirsch, the set $\{1, \xi_1, \xi_2, \xi_1\xi_2\}$ is a basis of $H^*(F_2, \mathcal{R})$. There exists a natural orientation of F_2 which leads to $\xi_2\xi_1[F_2] = 1$ (of course, we are referring only to the case $\mathcal{R} = \mathbb{Z}$). In order to describe this orientation, we notice first that the submanifolds a_1 and a_2 of F_2 have a unique intersection at (p, p) and the tangent space at this point is

$$(TF_2)_{(p,p)} = \{(X, X + Y): X \in E_-(p),\ Y \in E_+(p)\},$$

the direct sum of

$$(Ta_1)_{(p,p)} = \{(X, X): X \in E_-(p)\} \quad \text{and} \quad (Ta_2)_{(p,p)} = \{(0, Y): Y \in E_+(p)\}.$$

The orientation we give to F_2 associates to the point (p, p) the direct sum orientation $(Ta_1)_{(p,p)} \oplus (Ta_2)_{(p,p)}$ (a_1 and a_2 are already oriented). Take $l \in \mathbb{Z}$ given by $\xi_1\xi_2[F_2] = (-1)^l$. Since $\xi_1^2 = 0$, it follows that $[F_2] \cap \xi_1 = (-1)^l a_2$, hence $(-1)^l a_2$ is the Poincaré dual of ξ_1 in F_2. The submanifolds a_1 and a_2 have the intersection number +1. By using, for instance, [3], Prop. 31.7, we have $(-1)^l \xi_1(a_1) = (-1)^{m^2} = (-1)^m$, hence $l \equiv m (\text{mod } 2)$. Finally, we obtain $\xi_2\xi_1[F_2] = 1$.

If α denotes the Poincaré dual of ξ_2 in F_2, then from $[F_2] \cap \xi_2 = \alpha$ it follows $\alpha \cap \xi_1 = 1$. But $\xi_1 = \pi_2^*(\xi_-)$, where ξ_- is the generator of $H^{m_-}(S_-(p), \mathcal{R})$, hence $\alpha \cap \pi_2^*(\xi_-) = 1$ and further $\pi_{2*}(\alpha) \cap \xi_- = 1$, i.e. $\pi_{2*}(\alpha) = [S_-(p)]$. In this way we get $\alpha = a_1 + q a_2$, for a certain $q \in \mathcal{R}$.

We have now to anticipate Lemma 4.5 and deduce $\xi_2 \overline{\varphi}_+^*(\xi_2) = 0$, which is by (1) equivalent to $(-1)^m \xi_2^2 = -d_1 \xi_2 \xi_1$. If $m > 1$ is odd, then clearly $d_1 = 0$. Suppose from now on that $m > 1$ is even or $m = 1$. Return to the relation $[F_2] \cap \xi_2 = a_1 + q a_2$ and deduce that

$$q = \xi_2([F_2] \cap \xi_2) = [F_2] \cap \xi_2^2 = -d_1[F_2] \cap \xi_2\xi_1 = -d_1,$$

hence $\alpha = \overline{\varphi}_{+*}(a_1)$. Consequently, $-d_1 = \overline{\varphi}_{+*}(a_1) \cap \xi_2$, that is the intersection number of $\overline{\varphi}_{+*}(a_1)$ with itself. Denote by $PD: H_*(F_2, \mathcal{R}) \to H^{2m-*}(F_2, \mathcal{R})$ the inverse of the Poincaré duality operator. Since $\overline{\varphi}_+$ is a diffeomorphism of degree -1 over $\mathcal{R}$, we have $\overline{\varphi}_+^* \circ PD = -PD \circ \overline{\varphi}_{+*}$. By also using the fact that $\overline{\varphi}_+$ is involutive, it follows that $\xi_2 = PD(\overline{\varphi}_{+*}(a_1)) = -\overline{\varphi}_+^*(PD(a_1))$, consequently

$$\begin{aligned} d_1 &= \overline{\varphi}_{+*}(a_1) \cap \overline{\varphi}_+^*(PD(a_1)) = \overline{\varphi}_{+*}(\overline{\varphi}_{+*}^2(a_1) \cap PD(a_1)) \\ &= \overline{\varphi}_{+*}(a_1 \cap PD(a_1)) = a_1 \cap PD(a_1) = PD(a_1)(a_1), \end{aligned}$$

hence d_1 is the intersection number of a_1 with itself in F_2.

On the other hand, the $\mathcal{R}$-Poincaré dual of the submanifold a_1 is the Euler class (respectively, the first Stiefel–Whitney class) of the normal bundle of a_1 in F_2 (we have in mind that both the dimension and codimension in F_2 of a_1 are even numbers). For any $(z, z) \in a_1$, we have $(\nu a_1)_{(z,z)} = \{(0, Y): Y \in E_+(z)\}$. The mapping $(a_1, \nu(a_1)) \to (S_-(p), E_+|_{S_-(p)})$ given by $((z, z), (0, Y)) \mapsto (z, Y)$ is an orientation preserving vector bundle isomorphism, hence the Euler (respectively, Stiefel–Whitney) numbers of $\nu(a_1)$ and $E_+|_{S_-(p)}$ coincide. We can conclude now that

$$\begin{aligned} &d_1 = PD(a_1)(a_1) = e(\nu a_1)(a_1) = e(E_+|_{S_-(p)})([S_-(p)]) = e(E_+|_{S_-(p)}), \\ &\text{if} \quad m_- = m_+ > 1 \end{aligned}$$

and similarly, $d_1 = w_1(E_+|_{S_-(p)})$ if $m_- = m_+ = 1$. □

Remark. If $m_- = m_+$ is odd and greater than 1, then we have again $d_1 = d_2 = 0$.

4. General Estimations

A complete description of the cohomology ring of an isoparametric hypersurface M in the Hilbert space V in terms of multiplicities and the numbers d_1 and d_2 described above will be obtained in this section.

Fix p a point in M. For any positive integer k, take

$$F_k = \underbrace{S_+(p) S_- S_+ \ldots}_{k \text{ elements}},$$

together with $u_k: F_k \to M$, $u_k(z_1, \ldots, z_k) = z_k$. By convention, we take $F_0 = \{p\}$ and $u_0(p) = p$. Denote by $s_k: F_{k-1} \hookrightarrow F_k$ the map given by $(z_1, \ldots, z_{k-1}) \to (z_1, \ldots, z_{k-1}, z_{k-1})$, which is obviously an imbedding and will play for us the role of the inclusion map. It is obvious that$u_k \circ s_k = u_{k-1}$. Consider now $F = \bigcup_{k \leqslant 0} F_k$, with the direct limit topology and also $u: F \to M$,

$$u = \lim_{\overrightarrow{k}} u_k.$$

Then

$$H_*(F, \mathcal{R}) = \lim_{\overrightarrow{k}} H_*(F_k, \mathcal{R})$$

and

$$u_* = \lim_{\overrightarrow{k}} u_{k*}: H_*(F) \to H_*(M).$$

As an (infinite) iterated sphere bundle, F is very well manageable from the point of view of the homology module and even cohomology ring. More precisely, the map $\pi_k: F_k \to F_{k-1}$, $(z_1, \ldots, z_k) \mapsto (z_1, \ldots, z_{k-1})$ is an $S^{m_{i_k}}$- bundle (as usual $i_k = (-1)^{k+1}$) and $s_k: F_{k-1} \to F_k$ is a section of it. By a simple argument involving the Gysin sequence, we can see that the module $H_*(F, \mathcal{R})$ is free. More precisely, associate to any $J = \{j_1 < \ldots < j_r\} \subset \mathbb{N} \setminus \{0\}$ the space

$$a_J = \{(z_1, z_2, \ldots) \in F: z_1 = \cdots z_{j_1-1} = p, z_{j_1} = \cdots z_{j_2-1} \in S_{i_{j_1}}(p),$$
$$z_{j_2} = \cdots z_{j_3-1} \in S_{i_{j_2}}(z_{j_1}), \ldots, z_{j_{r-1}} = \cdots z_{j_r-1} \in S_{i_{j_{r-1}}}(z_{j_{r-2}}), z_{j_r} = \cdots \in S_{i_{j_r}}(z_{j_{r-1}})\}.$$

It is itself an iterated sphere bundle, orientable if both multiplicities are greater then 1. The orientation of any a_J (inclusively any F_r) is obviously completely determined once we have oriented S_- and S_+. Namely, we consider recursively the bundle $\pi_r: F_r \to F_{r-1}$, $r \geqslant 1$, and we orient F_r by demanding that $s_r(F_{r-1})$ and the fiber have intersection number +1 inside F_r. The set $\{[a_J]: J \subset \mathbb{N} \setminus \{0\}$ finite subset$\}$ is a basis of $H_*(F, \mathcal{R})$.

To any finite subset $J \subset \mathbb{N} \setminus \{0\}$ we associate the diffemorphism $g(J) = \varphi_{i_{j_r}} \varphi_{i_{j_{r-1}}} \ldots \varphi_{i_{j_1}}$ which is in the same time an element of W. With respect to the generators φ_+, φ_- of W, the length $l(g(J))$ of $g(J) \in W$ is at most equal to the cardinality of J.

LEMMA 4.1. *By identifying W with a basis of $H_*(M, \mathcal{R})$, the linear mapping $u_*: H_*(F, \mathcal{R}) \to H_*(M, \mathcal{R})$ is completely determined by the relationship:*

$$u_*(a_J) = \begin{cases} g(J), & \textit{if } l(g(J)) = Card(J), \\ 0, & \textit{if contrary}. \end{cases}$$

In particular, u_* is surjective.

Proof. Suppose that $g(J)$ is not reduced. This means that there exists $t \in \{1, \ldots, r-1\}$ with $i_{j_t} = i_{j_{t+1}}$. If F' denotes the space

$$\{(y_1, \ldots, y_r): y_1 \in S_{i_{j_1}}(p), y_2 \in S_{i_{j_2}}(y_1), \ldots, y_r \in S_{i_{j_r}}(y_{r-1})\}$$

and $u': F' \to M$, $(y_1, \ldots, y_r) \mapsto y_r$, then $u_*[a_J] = u'_*[F']$. We also consider

$$F'' = \{(y_1, \ldots, y_{t-1}, y_{t+1}, \ldots, y_r): y_1 \in S_{i_{j_1}}(p), \ldots, y_{t+1} \in S_{i_{j_t}}(y_{t-1}), \ldots,$$
$$y_r \in S_{i_{j_r}}(y_{r-1})\},$$

together with $u'': F'' \to M$, given by $(y_1, \ldots, y_{t-1}, y_{t+1}, \ldots y_r) \mapsto y_r$. The map $\Psi: F' \to F''$, $(y_1, \ldots, y_r) \mapsto (y_1, \ldots, y_{t-1}, y_{t+1}, \ldots, y_r)$ is well-defined and satisfies $u'' \circ \Psi = u'$. Hence, $u'_*[F'] = u''_* \Psi_*[F'] = 0$, since $\dim F' > \dim F''$. □

We are now interested in the cohomology of F and M and the relationship that u induces between them. We start with a general viewpoint: suppose that the homology module $H_*(X, \mathcal{R})$ of the space X is free and $\{\alpha_k\}_{k \geqslant 1}$ is a basis of it ($\mathcal{R}$ here can be $\mathbb{Z}$ or $\mathbb{Z}_2$). If α_k^* denotes the homomorphism of $H_*(X, \mathcal{R})$ dual to α_k, then the module $H^*(X, \mathcal{R})$ can be identified with the set of formal series of the form $\varphi = \sum_{k \geqslant 1} n_k \alpha_k^*$, $n_k \in \mathcal{R}$.

Regarding the ring structure of $H^*(X, \mathcal{R})$, it is completely determined by the coefficients of the decompositions

$$\alpha_i^* \cup \alpha_j^* = \sum_{k \geqslant 1} c_{ij}^k \alpha_k^*, \tag{3}$$

$i, j \geqslant 1$, if the folowing assumptions are fullfilled:

ASSUMPTION 1. For all subscripts i, j, the sum given by (3) is finite.
ASSUMPTION 2. For any $k \geqslant 1$, there exist only finitely many pairs (i, j) with $(\alpha_i^* \cup \alpha_j^*)(\alpha_k) \neq 0$.

Both conditions are satisfied by F, as we shall prove in the sequel. First of all, we construct in a convenient manner the element of $H^*(F, \mathcal{R})$ dual to a_J, $J \subset \mathbb{N} \setminus \{0\}$ arbitrary finite subset. Namely, for any $r \geqslant 1$, we take the bundle $\pi_r : F_r \to F_{r-1}$ and $\xi_r \in H^{m_{i_r}}(F_r)$ uniquely determined by $\xi_r(a_{\{r\}}) = 1$ and $\xi_r(a_{\{j\}}) = 0$ for any $1 \leqslant j \leqslant r-1$ with $m_{i_j} = m_{i_r}$. The section $s_r : F_{r-1} \to F_r$ satisfies $s_r^*(\xi_r) = 0$. It follows from the theorem of Leray–Hirsch that $\pi_r^* : H^*(F_{r-1}) \to H^*(F_r)$ is injective and we have the decomposition:

$$H^*(F_r) = \pi_r^* H^*(F_{r-1}) \oplus \xi_r \pi_r^* H^*(F_{r-1}).$$

Via the monomorphisms of the form π_{k+1}^*, we can regard the rings $H^*(F_k)$, $1 \leqslant k < r$, as subrings of $H^*(F_r)$. In particular, any ξ_j with $1 \leqslant j \leqslant r$ is regarded as an element of $H^*(F_r)$.

We shall need in the sequel the following algebraic notion ($\mathcal{R}$ will be always $\mathbb{Z}$ or $\mathbb{Z}_2$).

DEFINITION. Let A^* be a skew-commutative infinite-dimensional graduated $\mathcal{R}$-algebra (i.e. $xy = (-1)^{deg(x)deg(y)} yx$ for any two x, $y \in A^*$). For any $J = \{j_1 < j_2 < \ldots < j_t\} \subset \mathbb{N} \setminus \{0\}$, denote $x_J := x_{j_1} x_{j_2} \ldots x_{j_t}$. We say that A^* is *simply generated* by its elements x_1, x_2, ... and denote that by $A^* = \Delta(x_1, x_2, \ldots)$ if, regarded as an $\mathcal{R}$-module, A^* is the set of all formal homogeneous series of the form $\sum_{J \subset \mathbb{N}} n_J x_J$, with n_J integer numbers.

Now take an algebra A^* which is simply generated by its elements x_1, x_2, ... Suppose that we know how to express all squares of the type x_j^2, $j \geqslant 1$ as series of the type $\sum_{j_1 < j_2} c_{j_1 j_2} x_{j_1} x_{j_2}$, with $c_{j_1 j_2} \in \mathcal{R}$. The structure of algebra of A^* is then completely determined if the following assumption is satisfied:

ASSUMPTION 3. For any $K \subset \mathbb{N} \setminus \{0\}$, there exist only finitely many pairs (I, J), $I, J \subset \mathbb{N} \setminus \{0\}$ with the property that x_K occurs as summand with nonzero coefficient in the homogeneous series corresponding to $x_I x_J$.

Let us return now to our previous considerations. Our next goal is to show that $H^*(F, \mathcal{R}) = \Delta(\xi_1, \xi_2, \ldots)$. The following two propositions are intended to present the dual of a_J in $H^*(F, \mathcal{R})$. Denote by $\mathcal{I}_r$ the set $\{1, 2, \ldots, r\}$, for r positive integer.

PROPOSITION 4.2. *For any $J \subset \mathcal{I}_r$, we have $\xi_J(a_J) = (-1)^{l(J)}$, where:*

$$l(J) = \begin{cases} 0, & \text{if } J \text{ consists of only one element,} \\ \sum_{j_1, j_2 \in J, j_1 < j_2} m_{j_1} m_{j_2}, & \text{otherwise.} \end{cases}$$

Proof. It will be sufficient to prove that $\xi_{\mathcal{I}_r}([F_r]) = (-1)^{l_r}$, where l_r denotes 0 if $r = 1$, respectively $\sum_{1 \leqslant j_1 < j_2 \leqslant r} m_{j_1} m_{j_2}$, otherwise. We shall do that by an inductive argument. For $r = 1$, the relation is obvious.

We now show that if ξ simply denote $\xi_{\mathcal{I}_{r-1}}$, then from $\xi([F_{r-1}]) = (-1)^{l_{r-1}}$ it follows $\xi_{\mathcal{I}_r}([F_r]) = \pi_r^*(\xi)\xi_r([F_r]) = (-1)^{l_r}$. The homological cycle carried by

$$a_r := \{(z_1, z_2, \ldots, z_r) \in F_r : z_1 = \cdots = z_{r-1} = p, z_r \in S_{i_r}\}$$

generates the homology of the fiber for the bundle $\pi_r : F_r \to F_{r-1}$. Let ξ' denote the Poincaré dual of a_r in $H^*(F_r)$. As already noticed at the beginning of the section, the submanifold $s_r(F_{r-1})$ and a fiber can intersect at only one point, the intersection number being +1. By using, for instance, [3], Prop. 31.7, we have

$$\xi'(s_{r*}[F_{r-1}]) = (-1)^{q_r}, \tag{4}$$

where $q_r = \dim a_r \cdot \dim F_{r-1} = m_{i_r}(\sum_{j=1}^{r-1} m_{i_j})$.

On the other hand, $\pi_{r*}([F_r] \cap \pi_r^*(\xi)) = 0$, hence $[F_r] \cap \pi_r^*(\xi) = k a_r$, with $k \in \mathcal{R}$. It follows that

$$k = \xi_r([F_r] \cap \pi_r^*(\xi)) = (\pi_r^*(\xi) \cup \xi_r)([F_r]) = (-1)^l,$$

for a certain integer l (we used in the last step the theorem of Leray–Hirsch). Consequently, $\pi_r^*(\xi) = (-1)^l \xi'$ and by (4), $l = q_r + l_{r-1}$ i.e. $l = l_r$. □

PROPOSITION 4.3. *For any I, J different subsets of $\mathcal{I}_r$ with* $\dim a_I = \dim a_J$*, it holds* $\xi_J(a_I) = 0$.

Proof. Once again we shall use an inductive argument by $r \geqslant 1$. For $r = 1$, the assertion is trivial. Let us suppose it true for all $1 \leqslant k < r$ and prove it for r.

Case 1: $r \notin J$. If $r \notin I$, then we can use the induction hypothesis. If $r \in I$, then $\pi_{r*}(a_I) = 0$, hence $\xi_J(a_I) = \pi_{r*}(\xi_J)(a_I) = \xi_J(\pi_{r*} a_I) = 0$.

Case 2: $r \in J$. Take J of the form $J = J' \bigcup \{r\}$, with $J' \subseteq \mathcal{I}_{r-1}$. We have to prove that $\xi_J(a_I) = \pm \xi_r \cup \xi_{J'}(a_I) = 0$.

If $J' \not\subseteq I$, then notice that

$$\xi_J(a_I) = \pm \xi_r \cup \xi_{J'}(i_{I*} a_I) = \pm (i_I)^*(\xi_r) \cup (i_I)^*(\xi_{J'})(a_I),$$

where $i_I: a_I \hookrightarrow F_r$ is the inclusion map. The idea is to show that the element $(i_I)^*(\xi_{J'})$ of $H^*(a_I)$ is zero. As long as the module $H_*(a_I)$ is freely generated by the cycles a_K, with $K \subseteq I$, we only need to take a such K with $\dim a_K = \dim a_{J'}$ and to observe that $(i_I)^*(\xi_{J'})(a_K) = \xi_{J'}(a_K)$ is zero, by using Case 1 (notice that it is impossible to have $K = J'$).

If $J' \subseteq I$ and $r \notin I$, then

$$(\xi_r \cup \xi_{J'})(a_I) = (\xi_r \cup \xi_{J'})(s_r^* a_I) = s_r^*(\xi_r) \cup s_r^*(\xi_{J'})(a_I) = 0,$$

since $s_r^*(\xi_r) = 0$ (remember that ξ_r has been chosen in such a way that it vanishes every m_{i_r}-dimensional element of $\mathrm{Im} s_{r*}$).

From $J' \subseteq I$ and $r \in I$ would follow $J \subset I$ and $J \neq I$, hence $\dim a_J < \dim a_I$, impossible. □

We can resume the last two propositions by saying that, for $J \subset \mathbb{N}$, the monomial ξ_J is, up to a sign, the dual of $[a_J] \in H_*(F, \mathcal{R})$. More precisely,

$$\xi_I(a_J) = \begin{cases} 0, & \textit{if } I \neq J, \\ (-1)^{l(J)}, & \textit{if } I = J. \end{cases}$$

Consequently, $H^*(F, \mathcal{R}) = \Delta(\xi_1, \xi_2, \ldots)$ in the sense of the definition above, with $\deg \xi_j = m_{i_j}$, $j \geqslant 1$.

As already mentioned, the ring structure of $H^*(F, \mathcal{R})$ will be completely determined after getting every ξ_r^2, $r \geqslant 1$, as a (actually finite) series of type $\sum_{j_1 < j_2} c_{j_1 j_2} \xi_{j_1} \xi_{j_2}$, with $c_{j_1 j_2} \in \mathcal{R}$ (we shall also check later Assumption 3).

Consider to this end the involutive automorphism $\overline{\varphi}_r$ of F_r, which associates to an arbitrary $(z_1, \ldots, z_r) \in F_r$ the element $(z_1, \ldots, z_{r-1}, \varphi_{i_r}(z_r))$ of F_r. Notice that $\pi_r \circ \overline{\varphi_r} = \pi_r$. It follows that for any generator a_j of the module $H_{lower}(F_r)$, $1 \leqslant j \leqslant r$, it holds that $\pi_{r*}\overline{\varphi}_{r*}(a_j) = \pi_{r*}(a_j)$, i.e. $\overline{\varphi}_{r*}(a_j) - a_j$ belongs to $\mathrm{Ker}\pi_{r*}|_{H_{lower}(F_r)}$. But the latter one is $\mathcal{R}a_r$, and in this way we obtain:

$$\overline{\varphi}_{r*}(a_j) = a_j + \beta_{jr} a_r,$$

for any $1 \leqslant j \leqslant r$, where β_{jr} are certain coefficients in $\mathcal{R}$. It can be immediately seen that $\varphi_{i_r} \circ u_r = u_r \circ \overline{\varphi}_r$. Hence from $u_{r*}(a_j) = b_{i_j}$ it follows $\varphi_{i_r *}(b_{i_j}) = b_{i_j} + \beta_{jr} b_{i_r}$. By comparing this relation to Proposition 3.1, we immediately identify the coefficients β_{jr}. More precisely, if we denote as follows:

$$d_{kl} = \begin{cases} d_1, & \text{if } k = - \text{ and } l = +, \\ d_2, & \text{if } k = + \text{ and } l = - \\ 1 + (-1)^{m_l}, & \text{if } k = l \end{cases} = \begin{cases} e(E_l|_{S_k}), & \text{if } \mathcal{R} = \mathbb{Z}, \\ w_1(E_l|_{S_k}), & \text{if } \mathcal{R} = \mathbb{Z}_2, \end{cases} \tag{5}$$

then we have

$$\overline{\varphi}_{r*}(a_j) = a_j - d_{i_j i_r} a_r, \tag{6}$$

for any $1 \leqslant j \leqslant r$. Since the basis $\{\xi_j : 1 \leqslant j \leqslant r\}$ of $H_{lower}(F_r)$ is dual to the basis

$\{a_j : 1 \leqslant j \leqslant r\}$ of $H_{lower}(F_r)$, a simple calculation leads to

$$\overline{\varphi}_r^*(\xi_r) = (1 - d_{i_r i_r})\xi_r - \sum_{j=1}^{r-1} d_{i_j i_r}\xi_j. \tag{7}$$

The following technical result will be needed later:

LEMMA 4.4. *Let $\pi: E \to S^1$ be a circle bundle over the circle S^1 and $s: S^1 \to E$ a section of it. Let $\overline{\varphi}$ denote the involutive automorphism of the total space E induced by the antipodal maps of the fibers. Let ξ_2 be the element of $H^1(E, \mathbb{Z}_2)$ uniquely determined by the fact that it generates, together with the unit, the cohomology of the fiber and it vanishes $s_* H_*(S^1, \mathbb{Z}_2)$. Then it holds that $\xi_2 \overline{\varphi}^*(\xi_2) = 0$.*

Proof. One knows that there are only two bundles of the form described in the enouncement, namely the trivial one and the bundle induced by the Klein bottle (see for instance [12], §26).

In the first case, $E = S^1 \times S^1$ and $\overline{\varphi}$ associates to any pair (z_1, z_2) of $S^1 \times S^1$ the pair $(z_1, -z_2)$, hence $\overline{\varphi}$ is obviously homotopic to the identity map. It follows that $\xi_2 \overline{\varphi}^*(\xi_2) = \xi_2^2 = 0$, since this time ξ_2 is simply the generator of $H^1(S^1)$.

Now let E be the Klein bottle regarded as a circle bundle over S^1. Take $s: S^1 \to E$ as a section of this bundle, $a_1 = s_*([S^1])$ and a_2 the homology cycle in $H_*(E)$ carried by the fiber. The same argument as in the proof of Proposition 3.1 shows that $\overline{\varphi}_*(a_1) = a_1 + da_2$, where d is the Stiefel–Whitney number of the line bundle obtained by taking the tangent spaces to the fibers along $s(S^1)$. But this line bundle is obviously nonorientable, hence $d = 1$, i.e.:

$$\overline{\varphi}_*(a_1) = a_1 + a_2. \tag{8}$$

Let $\{\xi_1, \xi_2\}$ be the basis of $H^1(E, \mathbb{Z}_2)$ dual to $\{a_1, a_2\}$. A simple algebraic reasoning shows that (8) implies

$$\overline{\varphi}^*(\xi_2) = \xi_1 + \xi_2. \tag{9}$$

By using the description of the ring $H^*(E, \mathbb{Z}_2)$ (see for instance [8], Example 2, p. 295), we easily see that $\xi_2^2 = \xi_2\xi_1$ and the lemma is completely proved.

We come back now to the general context and prove the following lemma:

LEMMA 4.5. *It holds that $\xi_r \overline{\varphi}_r^*(\xi_r) = 0$, for any $r \geqslant 1$.*

Proof. For any subset I of $\{1, \ldots, r\}$ which do not contain r, it holds that $\xi_r \overline{\varphi}_r^*(\xi_r)(a_I) = \xi_r \overline{\varphi}_r^*(\xi_r)(s_{r*} a_I) = 0$, since $s_{r*}(\xi_r) = 0$. It now follows that we have a description of the form:

$$\xi_r \overline{\varphi}_r^*(\xi_r) = \sum_{j=1}^{r-1} c_j \xi_j \xi_r, \tag{10}$$

with $c_j \in \mathcal{R}$. We shall show that all coefficients c_j involved in (10) vanish, by separately taking the following situations:

Case 1. $m_{i_r} \geqslant 2$. Take $J = \{1 \leqslant j \leqslant r : m_{i_j} = m_{i_r}\}$ and $J' = J \setminus \{r\}$. Then the restriction map

$$\pi_r : a_J \to a_{J'} \tag{11}$$

is also a $S^{m_{i_r}}$-bundle, which is clearly orientable (since $a_{J'}$ is simply connected). The restriction map

$$s_r : a_{J'} \to a_J \tag{12}$$

is a section of this bundle. If $\imath : a_J \to F_p$ denotes the inclusion map, having in mind (10), it is sufficient to prove that $\imath^*(\xi_r \overline{\varphi}_r^*(\xi_r)) = (\imath^*(\xi_r))\overline{\varphi}_r^*(\imath^*(\xi r)) = 0$. To this end, we restrict ourselves to the oriented $S^{m_{i_r}}$-bundle given by (11) with the section described by (12) and the involutive automorphism $\overline{\varphi}_r$ restricted to a_J. Associate to this bundle the Gysin sequence, which has the form:

$$\cdots \to H^q(a_{J'}) \xrightarrow{\pi_r^*} H^q(a_J) \xrightarrow{\tau} H^{q-m_{i_r}}(a_{J'}) \to H^{q+1}(a_{J'}) \to \cdots.$$

By using thm. 21.9 of [6], $\pi_r^* \tau((\imath^*\xi_r)\overline{\varphi}_r^*(\imath^*\xi_r)) = 0$. But in our case π_r^* is injective, hence there exists $u \in H^*(a_{J'})$ with $(\imath^*\xi_r)\overline{\varphi}_r^*(\imath^*\xi_r) = \pi_r^*(u)$. By applying s_r^* to both terms of the last identity, we get $u = 0$, since $s_r^*(\imath^*\xi_r) = 0$ and $\pi_r \circ s_r = \mathrm{id}$.

Case 2. $m_{i_r} = 1$. We have only to take an arbitrary $1 \leqslant j < r$ with $m_{i_j} = 1$ and to notice that if $\jmath : a_{\{j,r\}} \to F_r$ is the inclusion map, then

$$\xi_r \overline{\varphi}_r^*(\xi_r)(a_{\{j,r\}}) = \xi_r \overline{\varphi}_r^*(\xi_r)(\jmath_* a_{\{j,r\}}) = (\jmath^*\xi_r)\overline{\varphi}_r^*(\jmath^*\xi_r)(a_{\{j,r\}}) = 0,$$

because of Lemma 4.4. A simple algebraic calcutation now proves the formula for ξ_k^2 given by Theorem 1.1.

Assumption 1 from above is now obviously fullfilled for F. As about Asumption 2, we can prove even something more:

LEMMA 4.6. *If* $(\xi_I \cup \xi_J)(a_K)$ *makes sense and is nonzero, then* $I \bigcup J \subset K$.

Proof. Let L denote $I \bigcup J \bigcup K$. Suppose we could find $i \in I$ with $i \notin K$. Take $s : a_{L\setminus\{i\}} \to a_L$ the natural inclusion, i.e. the map given by $(z_1, \ldots, z_m) \mapsto (z_1, \ldots, z_{i-1}, z_{i-1}, z_i, \ldots, z_m)$. If $\mu := \xi_{I\cup J\setminus\{i\}}$, then

$$(\xi_I \cup \xi_J)(a_K) = \pm(\xi_i \cup \mu)(s_* a_K) = \pm(s^*\xi_i \cup s^*\mu)(a_K) = 0,$$

since $s^*(\xi_i) = 0$. □

The cohomology of F is now completely determined:

COROLLARY 4.7. *We have* $H^*(F, \mathcal{R}) = \Delta(\xi_1, \xi_2, \ldots)$ *under Assumption 3, with* ξ_r^2 *given by Theorem 1.1 (b)).*

We are now almost finished: Since $u_* : H_*(F, \mathcal{R}) \to H_*(M, \mathcal{R})$ is surjective, its dual $u^* : H^*(M, \mathcal{R}) \to H^*(F, \mathcal{R})$ is injective. We have already seen in which way the Weyl group W can be identified with a basis of $H_*(M, \mathcal{R})$. To any $w \in W$ we associate its dual element $w^* \in H^*(M, \mathcal{R})$. By Lemma 4.1, Proposition 4.2 and 4.3, it follows now Theorem 1.2.

This is what we like to call the quantitative side of our estimations. Unfortunately, it seems impossible to manage with these formulae in order to get $H^*(M, \mathcal{R})$ in terms of generators and relations. The last assertion of Theorem 1.2 will be anyway very useful: it is, of course, sufficient to take, for a certain choice of m_-, m_+, d_1 and d_2, a prototype which corresponds to this data and is easily manageable. As we shall see in the next section, such prototypes can be obtained by lifting equifocal hypersurfaces in symmetric spaces.

5. Lifts of Equifocal Hypersurfaces

A very important class of examples of isoparametric hypersurfaces in Hilbert space (actually, the only ones which are known so far) is given by the lifts of equifocal hypersurfaces in symmetric spaces. We would like to start the section by reviewing the main notions and results concerning equifocal hypersurfaces in symmetric spaces and their lifts. For more details, the reader can consult the basic paper of Terng and Thorbergsson [14].

Let us fix N a simply connected, compact symmetric space, $M \subset N$ a connected compact hypersurface and ξ the unitary normal field on M. We say that $t \in \mathbb{R}$ is a *focal radius* of multiplicity m at the point x_0 if $\exp_{x_0}(t\xi(x_0))$ is a focal point of multiplicity m, i.e. the end-point map $\eta\colon \nu(M) \to N$, $\eta(x, v) = \exp_x(v)$ is singular at the point $(x_0, t\xi(x_0))$. We say that M is *equifocal* if the focal radii are constant along M, as values and multiplicities. For instance, it is not difficult to see that for hypersurfaces in the sphere, constancy of focal radii is the same as constancy of principal curvatures, i.e. equifocal is the same as isoparametric. The theory of isoparametric hypersurfaces in the sphere is a very rich one (see the foundational work of E. Cartan [1] and also the papers of Münzner, [9, 10], of Ferus, Karcher and Münzner, [2], etc.).

The notion of equifocal submanifold was defined by Terng and Torbergsson in [14]. In order to study the geometry of this spaces, they used as the basic instrument, the lifts to isoparametric submanifolds in Hilbert space. In the following we shall sketch the construction of these lifts.

First take N of the form G/K, with G a compact, connected, semisimple and simply connected Lie group and K the fix-point set of an involutive automorphism of G.

THEOREM 5.1 (cf. [14]). *Let $N = G/K$ be a symmetric space of compact type and $\pi\colon G \to G/K$ the corresponding Riemannian submersion. If $M \subset N$ is an equifocal hypersurface, then $M^* := \pi^{-1}(M)$ is an equifocal hypersurface of G. Let ξ^* be the horizontal lift of ξ. Then t is the focal radius of multiplicity m in direction ξ if and only if t is the focal radius of multiplicity m in direction ξ^*.*

Now take $\mathcal{G}$ the Lie algebra of G and consider $V = H^0([0,1], \mathcal{G})$ the Hilbert space of L^2-integrable paths in $\mathcal{G}$. Also take $H^1([0,1], G) = \{g: [0,1] \to G: g' \text{ is } L^2\text{-integrable}\}$ and its subspace $P(G, e \times G) := \{g \in H^1([0,1], G): g(0) = e\}$. Define $E: H^0([0,1], \mathcal{G}) \to P(G, e \times G)$ which associates to the path u the unique solution $E(u)$ of the differential equation:

$$E(u)^{-1}E(u)' = u, \qquad E(u)(0) = e..$$

The map E is an isometry. The action of $P(G, e \times G)$ on V by gauge transformations is described by

$$g \star u := gug^{-1} - g'g^{-1},$$

$g \in P(G, e \times G)$, $u \in V$. It is transitive and free. If $\phi: V \to G$ is defined by $\phi(u) = E(u)(1)$, then ϕ is a $\Omega(G)$-principal bundle, hence a Riemannian submersion.

THEOREM 5.2 (cf. [14]). *If $M^* \subset G$ is an equifocal hypersurface, then $\tilde{M} := \phi^{-1}(M^*)$ is an isoparametric hypersurface in V. If $\tilde{\xi}$ is the horizontal lift of ξ^*, then the following three conditions are equivalent:*

(a) *t is a focal radius of multiplicity m in direction ξ^*;*
(b) *t is a focal radius of multiplicity m in direction $\tilde{\xi}$;*
(c) *$1/t$ is a principal curvature in direction $\tilde{\xi}$.*

We notice now a quite simple fact which will later play an important role: if $p: P(G, e \times G) \to G$ is the end-point map, then obviously $p \circ E = \phi$, hence the lift $\tilde{M}$ of M^* is homeomorphic, via E, to $p^{-1}(M^*) = P(G, e \times M^*)$.

If $M \subset G/K$ is an equifocal hypersurface, we shall call $\tilde{M} := \phi^{-1}(\pi^{-1}(M))$ the *isoparametric lift* of M. The goal of this section is the study of isoparametric lifts from the point of view of their cohomology rings. Some more information about equifocal hypersurfaces is still needed to this end:

THEOREM 5.3 (cf. [14]). *Let M be an equifocal hypersurface in the simply connected symmetric space of compact type N. Then:*

(a) *the normal geodesics to M are all closed and of the same length l;*
(b) *there exist integers m_1, m_2, an even number $2g$ and $\theta \in (0, l/2g)$ such that*
    (1) *the focal points on the normal circle $T_x = \exp(\nu(M)_x)$ are of the form $x_j = \exp(t_j\xi(x))$, $1 \leqslant j \leqslant 2g$, with $t_j = \theta + ((j-1)l/2g)$, and their multiplicities are m_+, if j is odd, respectively m_-, if j is even,*
    (2) *the group generated by the end-point maps $\varphi_j = \eta_{t_j\xi}$, $1 \leqslant j \leqslant 2g$, is isomorphic to the dihedral group W with $2g$ elements which operates freely on M;*

(c) *$M \bigcap T_x = W.x$;*

(d) *the parallel set* $M_t = \eta_{t\xi}(M)$ *is an equifocal hypersurface and* $\eta_{t\xi}$ *maps* M *diffeomorphically onto* M_t *if* $t \in (-(l/2g) + \theta, \theta)$*;*

(e) $M_+ := M_\theta$ *and* $M_- := M_{\theta-(l/2g)}$ *are embedded submanifolds of codimension* $m_+ + 1$, $m_- + 1$ *in* N *and the maps* $\pi_+ := \eta_{\theta\xi}: M \to M_+$ *and* $\pi_- := \eta_{(\theta-(l/2g))\xi}$: $M \to M_-$ *are* S^{m_+}*-, respectively,* S^{m_-}*-bundles;*

(f) $N = D_1 \bigcup D_2$ *and* $D_1 \bigcap D_2 = M$*, where* D_1 *and* D_2 *are diffeomorphic to the normal disk bundles of* M_+ *and* M_- *respectively.*

Before starting the direct approach to our problem, we prove

PROPOSITION 5.4. *Let* $M^n \subset S^{n+1}$ *an isoparametric submanifold,* $\pi: Spin(n+2) \to S^{n+1}$ *the natural Riemannian submersion and* $\phi: V = H^0([0,1], so(n+2)) \to Spin(n+2)$ *the Riemannian submersion considered above. Let* $p \in M$ *and* $p^* \in \phi^{-1}(\pi^{-1}(p))$ *arbitrary.*

(a) *If* t_- *and* t_+ *have the meaning given in Section 2, then* $t_+ = 1/\theta$, $t_- = 1/\theta - (l/2g)$.

(b) *The following diagrams are commutative:*

$$\begin{array}{ccc} \tilde{M} & \xrightarrow{\varphi_+} & \tilde{M} \\ \pi\circ\phi \downarrow & & \downarrow \pi\circ\phi \\ M & \xrightarrow{\varphi_1} & M \end{array} \qquad \begin{array}{ccc} \tilde{M} & \xrightarrow{\varphi_-} & \tilde{M} \\ \pi\circ\phi \downarrow & & \downarrow \pi\circ\phi \\ M & \xrightarrow{\varphi_{2g-1}} & M \end{array}$$

(c) *The sets*

$$S_1(p) := \eta^{-1}_{t_1\xi}(\eta_{t_1\xi}(p)) \quad \text{and} \quad S_2(p) := \eta^{-1}_{t_{2g-1}\xi}(\eta_{t_{2g-1}\xi}(p))$$

are the fundamental curvature spheres of M *through* p*; the map* $\pi \circ \phi$ *establishes a diffeomorphism between* $S_+(p^*)$ *and* $S_1(p)$*, respectively* $S_-(p^*)$ *and* $S_2(p)$.

(d) *The lift* $\tilde{M}$ *is homotopy equivalent to* $M \times \Omega S^{n+1}$.

(e) *If* d_1 *and* d_2 *are the coefficients associated in Proposition 3.1 to* $\tilde{M}$*, then, up to a permutation or a simultaneous change of sign, we have*

$$(d_1, d_2) \in \{(0,0), (-1,-2), (-1,-3), (-2,-2)\}.$$

Proof. Point (a) follows directly from Theorems 5.1, 5.2 and 5.3 above. Regarding (b) and (c), they are direct consequences of Corollary 5.11 and Lemma 6.2 from [14]. (d) Take $p_0 = \pi(e) \in S^{n+1}$ and also

$$\overline{\pi}: P(Spin(n+2), e \times Spin(n+2)) \to P(S^{n+1}, p_0 \times S^{n+1})$$

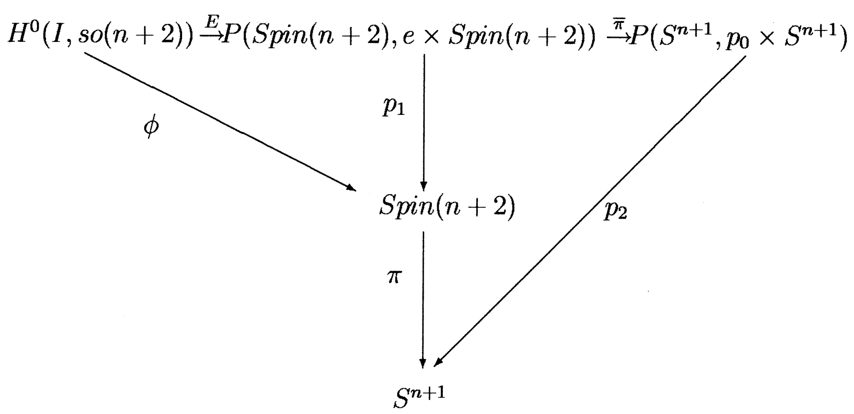
the map naturally induced by π. We can easily see that $\overline{\pi}$ is a $P(Spin(n+1), e \times Spin(n+1))$-fibering. Notice now that if

$$p_1: P(Spin(n+2), e \times Spin(n+2)) \to Spin(n+2)$$

and

$$p_2 \colon P(S^{n+1}, p_0 \times S^{n+1}) \to S^{n+1}$$

are the end-point maps, then the following diagram is commutative:



Since E is an isometry and $\overline{\pi}$ a fibering whose fiber is contractible, $\tilde{M}$ is homotopy equivalent to $p_2^{-1}(M)$. But if p is arbitrary in $S^{n+1} \setminus M$, then $S^{n+1} \setminus \{p\}$ is contractible and because p_2 is a ΩS^{n+1}-fibering, it is quite clear that $p_2^{-1}(M)$ is homotopically equivalent to $M \times \Omega S^{n+1}$.

(e) By Proposition 3.1 applied to $\tilde{M}$, we have

$$\varphi_{+*}[S_-] = [S_-] - d_1[S_+], \;\; \varphi_{-*}[S_+] = [S_+] - d_2[S_-].$$

From (b) and (c) above, we immediately get

$$\varphi_{1*}[S_2] = [S_2] - d_1[S_1], \;\; \varphi_{2*}[S_1] = [S_1] - d_2[S_2]. \tag{13}$$

But the coefficients d_1, d_2 that (13) associates to $M^n \subset S^{n+1}$ isoparametric with at least two distinct principal curvatures have already been determined in [7]. Namely, they must be the nondiagonal entries of one of the Cartan matrices associated with rank 2 root systems (eventually modulo 2, or with a simultaneous change of sign).

It remains the case when we have only one principal curvature: then $M = S^n$ is imbedded in S^{n+1} in the natural way, the spheres $S_1(p)$ and $S_2(p)$ coincide, consequently $(d_1, d_2) = (2, 2)$. The conclusion is now clear. □

Theorem 1.3 is completely proved.

Consider now G an arbitrary semisimple, compact, simply connected Lie group and $M \subset G$ a compact equifocal hypersurface. By Theorem 5.3(f), G is homeomorphic to the double mapping cylinder

$$D = M_- \bigcup_{\pi_-} (M \times I) \bigcup_{\pi_+} M_+ := M_- \sqcup M \times I \sqcup M_+ / \sim,$$

where $\sim$ is defined by

$$\pi_-(x) \sim (x, 1) \quad \text{and} \quad \pi_+(x) \sim (x, 0), \quad x \in M.$$

The inclusion $M \hookrightarrow G$ is then given by $x \mapsto (x, 1/2)$, $x \in M$. The homotopy fiber associated to this inclusion is just $P(G, e \times M)$, the latter one being, as we already remarked, homeomorphic to $\tilde{M}$ (see the remark following Theorem 5.2). The basic tool of Grove and Halperin's paper [4] is the classification, up to the rational homotopy type, of the homotopy fiber associated to the double mapping cylinder of two sphere bundles over the same space. We need, in fact, only a partial result of their paper, namely:

PROPOSITION 5.5 (cf. [4]). *Let* $\pi_1: X \to X_1$, $\pi_2: X \to X_2$ *arbitrary* S^m*-bundles, where* X, X_i *are connected spaces. Let* $D = X_1 \bigcup_{\pi_1} X \bigcup_{\pi_2} X_2$ *be the associated double mapping cylinder and* F *the homotopy fiber of the inclusion* $X \hookrightarrow D$. *If* D *is simply connected and* $m \geqslant 2$, *then the rational cohomology ring of* F *is isomorphic to one of the following rings:*

(a) $H^*(S^m \times S^m \times \Omega S^{2m+1}, \mathbf{Q})$,
(b) $H^*(S^m \times \Omega S^{m+1}, \mathbf{Q})$,
(c) $\mathcal{H}(k, m) \otimes H^*(\Omega S^{mk+1}, \mathbf{Q})$, $k \in \{3, 4, 6\}$, *where* $\mathcal{H}(k, m)$ *is the rational algebra generated by* x, y, *with* $\deg x = \deg y = m$, *subjects of the relations:*

$$\begin{cases} x^k = x^2 + y^2 = 0, & \text{if } k = 4, \\ x^k = x^2 + 3y^2 = 0, & \text{if } k = 3 \text{ or } 6. \end{cases}$$

We are now in a position to prove Theorem 1.4:

Proof of Theorem 1.4. The main idea is to compare the algebraic structures of $H^*(\tilde{M}, \mathbf{Q})$ given in Section 4 and Proposition 5.5 respectively. As already mentioned, the formulae obtained in Section 4 are difficult to manage; but we still can obtain some relations that the generators of lower levels satisfy, as we shall see in the sequel.

Recall that, in our case, $H^*(\tilde{M}, \mathbb{Z})$ has no torsion and

$$\dim H^p(\tilde{M}, \mathbb{Z}) = \begin{cases} 2, & \text{if } p \equiv 0(m), \\ 0, & \text{if contrary.} \end{cases}$$

If $f = i_1 i_2 \ldots i_q$ is a word consisting of alternative signs + and −, we denote by ω_f the element $(\varphi_{i_1} \varphi_{i_2} \ldots \varphi_{i_q})^*$ of $H^*(\tilde{M}, \mathbf{Q})$. Two such possible words of length q furnishes a basis of $H^{qm}(\tilde{M}, \mathbb{Z})$, $q \geqslant 1$ fixed.

A few simple calculations lead to

$$\begin{aligned}
\omega_+^2 &= d_1\omega_{+-},\\
\omega_-^2 &= d_2\omega_{-+},\\
\omega_-\omega_+ &= \omega_{-+} + \omega_{+-},\\
\omega_+^3 &= d_1(d_1d_2 - 1)\omega_{+-+},\\
\omega_-^3 &= d_2(d_1d_2 - 1)\omega_{-+-},\\
\omega_+^2\omega_- &= d_1(d_2\omega_{+-+} + \omega_{-+-}),\\
\omega_+\omega_-^2 &= d_2(d_1\omega_{-+-} + \omega_{+-+}),\\
\omega_+^4 &= d_1^2(d_1d_2 - 1)(d_1d_2 - 2)\omega_{+-+-},\\
\omega_-^4 &= d_2^2(d_1d_2 - 1)(d_1d_2 - 2)\omega_{-+-+}.
\end{aligned}$$

We distinguish now the following cases, dictated by Proposition 5.5:

Case 1. $H^*(\tilde{M}, \mathbf{Q}) \simeq H^*(S^m \times S^m \times \Omega S^{2m+1}, \mathbf{Q})$. Then clearly ω_+^3 and ω_-^3 must both be zero, which implies $d_1d_2 = 1$, or else $d_1 = d_2 = 0$. But the situation $d_1 = d_2 = \pm 1$ cannot occur: it is impossible to find $\omega_+, \omega_- \in H^m(S^m \times S^m, \mathbf{Q})$ with ω_-^2 and ω_+^2 linearly independent.

Case 2.

$$H^*(\tilde{M}, \mathbf{Q}) \simeq H^*(S^m \times \Omega S^{m+1}, \mathbf{Q})$$

or

$$H^*(\tilde{M}, \mathbf{Q}) \simeq \mathcal{H}(k, m) \otimes H^*(\Omega S^{mk+1}, \mathbf{Q}), \quad k \in \{3, 4, 6\}.$$

All these cases have in common that $H^*(\tilde{M}, \mathbf{Q})$ is generated as algebra by its elements of degree m, in our case, by ω_- and ω_+. Moreover, if we compare the relations of degree $2m$ dictated by the two appearances of $H^*(\tilde{M}, \mathbf{Q})$, we shall notice that $d_2\omega_+^2 - d_1d_2\omega_+\omega_- + d_1\omega_-^2$ must be a nonzero multiple of x^2, or $x^2 + y^2$, or $x^2 + 3y^2$. It follows that the discriminant of the quadratic form $d_2\omega_+^2 - d_1d_2\omega_+\omega_- + d_1\omega_-^2$ is less or equal to zero, that is $d_1d_2(d_1d_2 - 4) \leqslant 0$, and the conclusion follows. □

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References

1. Cartan, E.: Sur des familles remarquables d'hypersurfaces isoparamétriques dans les espaces spheriques, *Math. Z.* **45** (1939), 335–367.
2. Ferus, D., Karcher, H. and Münzner, H.-F.: Cliffordalgebren und neue isoparametrische Hyperflächen, *Math. Z.* **177** (1981), 479–502.
3. Greenberg, M. and Harper, J. R.: *Algebraic Topology: A First Course*, Benjamin, New York, 1981.
4. Grove, K. and Halperin, S.: Dupin hypersurfaces, group actions and the double mapping cylinder, *J. Diff. Geom.* **26** (1987), 429–459.
5. Hsiang, W.-Y., Palais, R. S. and Terng, C.-L.: Topology of isoparametric submanifolds, *J. Diff. Geom.* **27** (1988), 423–460.
6. Liao, S. D.: On the theory of obstructions of fiber bundles, *Ann. of Math.* **60**(1) (1954), 146–191.
7. Mare, A.-L.: Topology of isotropy orbits, Doctoral dissertation, Universität Augsburg, 1998.
8. Munkres, J. R.: *Elements of Algebraic Topology*, Benjamin, New York, 1984.
9. Münzner, H. F.: Isoparametrische Hyperflächen in Sphären I, *Math. Ann.* **251** (1980), 57–71.
10. Münzner, H. F.: Isoparametrische Hyperflächen in Sphären II, *Math. Ann.* **256** (1981), 215–232.
11. Palais, R. S. and Terng, C.-L.: *Critical Point Theory and Submanifold Geometry*, Springer-Verlag, 1989.
12. Steenrod, M.: *Topology of Fiber Bundles*, Princeton Univ. Press, 1951.
13. Terng, C.-L.: Proper Fredholm submanifolds of Hilbert space, *J. Diff. Geom.* **29** (1989), 9–47.
14. Terng, C.-L. and Thorbergsson, G.: Submanifold geometry in symmetric spaces, *J. Diff. Geom.* **42** (1995), 665–718.