EQUIVARIANT COHOMOLOGY OF QUATERNIONIC FLAG MANIFOLDS

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ABSTRACT. The main result of the paper is a Borel type description of the $Sp(1)^n$-equivariant cohomology ring of the manifold $Fl_n(\mathbb{H})$ of all complete flags in $\mathbb{H}^n$. To prove this, we obtain a Goresky-Kottwitz-MacPherson type description of that ring.

1. INTRODUCTION

In this paper we study the quaternionic flag manifold $Fl_n(\mathbb{H})$, which is the space of all sequences $(V_1, \ldots, V_n)$ where V_ν is a ν-dimensional quaternionic vector subspace (that is, left $\mathbb{H}$-submodule) of $\mathbb{H}^n$, for $1 \le \nu \le n$, such that $V_1 \subset V_2 \subset \ldots \subset V_n$. We can see that $Fl_n(\mathbb{H})$ has a transitive action of the symplectic group $Sp(n)$, with stabilizer

$$K := Sp(1)^n.$$

We are interested in the K-equivariant cohomology[1] of $Fl_n(\mathbb{H})$. We describe this ring in terms of the canonical vector bundles $\mathfrak{V}_\nu$ over $Fl_n(\mathbb{H})$, where $0 \le \nu \le n$. More precisely, we prove the following theorem.

Theorem 1.1. *There is a ring isomorphism*

$$H^*_K(Fl_n(\mathbb{H}); \mathbb{Z}) \simeq \frac{\mathbb{Z}[x_1, \ldots, x_n, u_1, \ldots, u_n]}{\langle (1+x_1)\ldots(1+x_n) = (1+u_1)\ldots(1+u_n)\rangle},$$

where $x_\nu = e_K(\mathfrak{V}_\nu/\mathfrak{V}_{\nu-1}) \in H^4_K(Fl_n(\mathbb{H}); \mathbb{Z})$ is the K-equivariant Euler class of $\mathfrak{V}_\nu/\mathfrak{V}_{\nu-1}$ and $u_1, \ldots, u_n$ are copies of the generator of $H^4(BSp(1); \mathbb{Z})$, so that

$$H^*(BK; \mathbb{Z}) = H^*(BSp(1)^n; \mathbb{Z}) = \mathbb{Z}[u_1, \ldots, u_n].$$

We use the following strategy. First we prove a Kirwan injectivity type result for the K action on $Fl_n(\mathbb{H})$; that is, the restriction map $\imath^* : H^*_K(Fl_n(\mathbb{H})) \to H^*_K(Fl_n(\mathbb{H})^K)$ is injective, where $Fl_n(\mathbb{H})^K$ denotes the fixed point set of the K action. Then we prove the Goresky-Kottwitz-MacPherson (henceforth GKM) type characterization of the image of $\imath^*$. Next we compare what we have obtained with the GKM description of the T-equivariant cohomology of $Fl_n(\mathbb{C})$, where $T = (S^1)^n$ is the maximal torus of $U(n)$. We observe that there exists an abstract isomorphism between the two cohomology rings, which doubles degrees. Finally, we use the well-known Borel type description of $H^*_T(Fl_n(\mathbb{C}))$.

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[1] All cohomology rings in this paper will be with coefficients in $\mathbb{Z}$ (unless otherwise specified).

Remarks. 1. The proof of the GKM presentation of $H^*_K(Fl_n(\mathbb{H}))$ we will give in Section 3 uses the methods of Tolman and Weitsman [To-We] (see also Harada and Holm [Ha-Ho, Section 2]).

2. The space $Fl_n(\mathbb{H})$ with the action of $Sp(1)^n$ is an important example in our previous papers [Ma1] (where we have proved that the action is equivariantly formal) and [Ma2] (where we have proved a certain Kirwan surjectivity type result, see Example 5.4). In the former paper we have also obtained a presentation of the usual cohomology ring of $Fl_n(\mathbb{H})$ (the result had been originally proved by Hsiang, Palais, and Terng [Hs-Pa-Te]). Theorem 1.1 above gives the equivariant "deformation" of that presentation.

3. The flag manifold $Fl_n(\mathbb{H})$ is a real locus (that is, fixed point set of an anti-symplectic involution) of a $SU(2n)$-coadjoint orbit (see for instance [Ma1, Remark following Theorem 1.1] and [Ma2, Example 5.4]). Consequently, some of the results of this paper, like the GKM description given in Proposition 3.6, have "mod $2\mathbb{Z}$-versions" (for $(\mathbb{Z}/2\mathbb{Z})^n$-equivariant cohomology with coefficients in $\mathbb{Z}/2\mathbb{Z}$) which can be deduced from theorems of Biss, Guillemin, and Holm [Bi-Gu-Ho].

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2. The quaternionic flag manifold

The goal of this section is to give a few alternative presentations of the manifold $Fl_n(\mathbb{H})$, which will be used later.

Let

$$\mathbb{H} = \{a + bi + cj + dk \mid a, b, c, d \in \mathbb{R}\}$$

be the skew field of quaternions. The space $\mathbb{H}^n$ is a $\mathbb{H}$ module with respect to scalar multiplication from the left. We equip it with the scalar product $(\ , \ )$ given by

$$(h, k) = \sum_{\nu=1}^{n} h_\nu \bar{k}_\nu,$$

for all $h = (h_1, \dots, h_n)$, $k = (k_1, \dots, k_n)$ in $\mathbb{H}^n$. Any linear transformation of $\mathbb{H}^n$ is described by a matrix $A \in \mathrm{Mat}^{n\times n}(\mathbb{H})$ according to the formula

$$Ah := h \cdot A^*,$$

where $h = (h_1, \dots, h_n) \in \mathbb{H}^n$. Here $\cdot$ denotes the matrix multiplication and the superscript $*$ indicates the transposed conjugate of a matrix. We denote by $Sp(n)$ the group of linear transformations A of $\mathbb{H}^n$ with the property that

$$(Ah, Ak) = (h, k),$$

for all $h, k \in \mathbb{H}^n$. Alternatively, $Sp(n)$ consists of all $n \times n$ matrices A with entries in $\mathbb{H}$ with the property that $A \cdot A^* = I_n$.

The flag manifold $Fl_n(\mathbb{H})$ defined in the introduction can also be described as the space of all sequences $(L_1, \dots, L_n)$ of 1-dimensional $\mathbb{H}$-submodules of $\mathbb{H}^n$ such that L_ν is perpendicular to L_μ for all $\mu, \nu \in \{1, 2, \ldots, n\}$, $\mu \neq \nu$. The group $Sp(n)$ acts transitively on the latter space. Indeed, let us consider the canonical basis $e_1 = (1, 0, \ldots, 0), \ldots, e_n = (0, \ldots, 0, 1)$ of $\mathbb{H}^n$; if $h^1, \ldots, h^n$ is an orthonormal system in $\mathbb{H}^n$, then the matrix A whose columns are $(h^1)^*, \ldots, (h^n)^*$ is in $Sp(n)$ and satisfies $Ae_\nu = h^\nu$, $1 \leq \nu \leq n$. The $Sp(n)$ stabilizer of the flag $(\mathbb{H}e_1, \ldots, \mathbb{H}e_n)$ consists of diagonal matrices, that is, it is equal to $Sp(1)^n$. In this way we obtain the identification

$$Fl_n(\mathbb{H}) = Sp(n)/Sp(1)^n. \tag{1}$$

Yet another presentation of the quaternionic flag manifold can be obtained by considering the conjugation action of $Sp(n)$ on the space

$$\mathcal{H}_n := \{X \in \mathrm{Mat}^{n\times n}(\mathbb{H}) \mid X = X^*,\ \mathrm{Trace}(X) = 0\}.$$

We pick n distinct real numbers $r_1, \ldots, r_n$ with $r_1 < r_2 < \ldots < r_n$ and $\sum_{\nu=1}^n r_\nu = 0$, and consider the orbit of the diagonal matrix $\mathrm{Diag}(r_1, \ldots, r_n)$. For any element X of the orbit there exist mutually orthogonal lines $L_1, \ldots, L_n$ such that $X|_{L_\nu}$ is the multiplication by r_ν, for all $1 \leq \nu \leq n$. This gives the identification

$$Fl_n(\mathbb{H}) = Sp(n).\mathrm{Diag}(r_1, \ldots, r_n). \tag{2}$$

Finally, we also mention that $Fl_n(\mathbb{H})$ is an s-orbit or a real flag manifold. More precisely, it is a principal isotropy orbit of the symmetric space $SU(2n)/Sp(n)$. It turns out that the isotropy representation of the latter space is just the conjugation action of $Sp(n)$ on $\mathcal{H}_n$ mentioned above. Via this identification, the metric on $\mathcal{H}_n$ turns out to be the standard one, given by

$$\langle X, Y\rangle := \mathrm{Re}(\mathrm{Trace}(XY)),$$

$X, Y \in \mathcal{H}_n$. Moreover, a maximal abelian subspace of $\mathcal{H}_n$ is the space $\mathfrak{d}$ of all diagonal matrices with real entries and trace 0. For the details, we refer the reader to [Ma2, Example 5.4]. Let us consider an element $A = \mathrm{Diag}(a_1, \ldots, a_n)$ of $\mathfrak{d}$, where a_ν are real numbers such that $a_1 < a_2 < \ldots < a_n$ and $\sum_{\nu=1}^n a_\nu = 0$. The corresponding height function

$$h_A(X) := \langle A, X\rangle,$$

$X \in Fl_n(\mathbb{H})$, will be an important instrument in our paper. We will use the following result, a proof of which can be found in the last section of this paper.

Proposition 2.1. (i) *The critical set* $\mathrm{Crit}(h_A)$ *can be identified with the symmetric group* S_n *via*

$$S_n \ni w = \mathrm{Diag}(r_{w(1)}, \ldots, r_{w(n)}) = (\mathbb{H}e_{w(1)}, \ldots, \mathbb{H}e_{w(n)}), \tag{3}$$

$w \in S_n$ (see also equation (2)). All critical points are non-degenerate.

(ii) *Take $w \in S_n$ and $v := s_{pq}w$, for some $1 \leq p < q \leq n$, where s_{pq} denotes the transposition of p and q in the symmetric group S_n. Assume that $h_A(w) > h_A(s_{pq}w)$. Then the subspace*

$$\mathcal{S}_{w,pq} := K_{pq}.w, \tag{4}$$

of $Fl_n(\mathbb{H})$ is a metric sphere of dimension four in $(\mathcal{H}_n, \langle\ ,\ \rangle)$, for which w and $s_{pq}w$ are the north, respectively south pole (with respect to the height function h_A). Here K_{pq} denotes the subgroup of $Sp(n)$ consisting of matrices with all entries zero, except *for those on the diagonal and on the positions pq and qp. The meridians of $\mathcal{S}_{w,pq}$ are gradient lines between w and $s_{pq}w$ for the function $h_A : Fl_n(\mathbb{H}) \to \mathbb{R}$ with respect to the submanifold metric induced by $\langle\ ,\ \rangle$.*

(iii) *The negative space of the Hessian of h_A at w is $\bigoplus_{(p,q)\in\mathcal{I}} T_w\mathcal{S}_{w,pq}$, where by $\mathcal{I}$ we denote the set of all pairs (p,q) with $1 \le p < q \le n$ such that $h_A(w) > h_A(ws_{pq})$.*

The sphere $\mathcal{S}_{w,pq}$ can also be described as the set of all flags $(L_1,\ldots,L_n)$ with the property that $L_\nu = \mathbb{H}e_\nu$, for all $\nu \notin \{p,q\}$ and L_p and L_q are arbitrary (orthogonal) lines in $\mathbb{H}e_{w(p)} \oplus \mathbb{H}e_{w(q)}$. It is obvious that $\mathcal{S}_{w,pq}$ can be identified with the projective line $\mathbb{H}P^1 = \mathbb{P}(\mathbb{H}e_{w(p)} \oplus \mathbb{H}e_{w(q)})$, which is indeed a four-sphere.

3. The GKM type description of $H^*_K(Fl_n(\mathbb{H}))$

We consider the canonical embedding of $U(1) = \{a+bi \mid a^2+b^2 = 1\}$ into $Sp(1) = \{a+bi+cj+dk \mid a^2+b^2+c^2+d^2 = 1\}$. This induces an embedding of $T := U(1)^n$ into $K = Sp(1)^n$ (as spaces of diagonal matrices). We are interested in the fixed points of the action of the groups T and K on $Fl_n(\mathbb{H})$.

Lemma 3.1. *The fixed point sets $Fl_n(\mathbb{H})^T$ and $Fl_n(\mathbb{H})^K$ are both equal to S_n (under the identification (3)).*

Proof. It is obvious that any flag of the form indicated in the lemma is fixed by K. We prove that a flag fixed by T has the form indicated in the theorem. Equivalently, we show that if the vector $h = (h_1,\ldots,h_n) \in \mathbb{H}^n$ has the property that $L := \mathbb{H}h$ is T-invariant, then L must be $\mathbb{H}e_\nu$, for some $\nu \in \{1,\ldots,n\}$. Indeed, for any $A = \mathrm{Diag}(z_1,\ldots,z_n) \in T$ we have $A.h \in L$, that is, there exists $\lambda \in \mathbb{H}$ such that

$$h_1\bar{z}_1 = \lambda h_1, \ldots, h_n\bar{z}_n = \lambda h_n. \tag{5}$$

If $h_\mu \neq 0$ and $h_\nu \neq 0$ for $\nu \neq \mu$, we pick $z_\mu = 1$ and $z_\nu = -1$ and see that there exists no λ satisfying (5). This finishes the proof. □

The following result is a non-abelian version of the so-called Kirwan injectivity theorem. The latter theorem holds for hamiltonian torus actions on compact symplectic manifolds, and it was proved by Kirwan, without being explicitly stated, in [Ki, Chapter 5] (see the proof of Proposition 5.8). Similar localization results for torus actions on manifolds had been obtained previously by Borel [Bo] (a nice account of his theorem can be found in [Gu-Gi-Ka, Appendix C, Section 3.3]) and Frankel (see [Fr, Section 4]).

Proposition 3.2. *The map $\imath^* : H^*_K(Fl_n(\mathbb{H})) \to H^*_K(S_n)$ induced by the inclusion $\imath : S_n \to Fl_n(\mathbb{H})$ is injective.*

Proof. By Proposition 2.1, h_A is a Morse function, whose critical set is S_n. Let us order the latter set as $w_1,\ldots,w_k$ such that $h_A(w_1) < h_A(w_2) < \ldots < h_A(w_k)$. Take $\epsilon > 0$, smaller than the minimum of $h_A(w_\ell) - h_A(w_{\ell-1})$, where $2 \le \ell \le k$. Denote $M_\ell := h_A^{-1}(-\infty, h_a(w_\ell)+\epsilon]$, and $S_n^\ell = S_n \cap M_\ell$. We prove by induction on $\ell \in \{1,\ldots,k\}$ that the map $\imath_\ell^* : H^*_K(M_\ell) \to H^*_K(S_n^\ell)$ is injective. For $\ell = 1$, the result is obviously true, as M_1 is equivariantly contractible to

$\{w_1\}$. We assume the result is true for $\ell - 1$. As in [To-We, diagram 2.5], we have the following commutative diagram.

$$
\begin{tikzcd}
\cdots \arrow[r] & H_K^*(M_\ell, M_{\ell-1}) \arrow[r, "②"] \arrow[d, "\simeq"] & H_K^*(M_\ell) \arrow[r] \arrow[d, "①"] & H_K^*(M_{\ell-1}) \arrow[r] & \cdots \\
& H_K^{*-\mathrm{index}(w_\ell)}(\{w_\ell\}) \arrow[r, "\cup e_\ell"] & H_K^*(\{w_\ell\}) & &
\end{tikzcd}
\tag{6}
$$

Here $\simeq$ denotes the isomorphism obtained by composing the excision map $H_K^*(M_\ell, M_{\ell-1}) \simeq H_K^*(D, S)$ (where D, S are the unit disk, respectively unit sphere in the negative normal bundle of the Hessian of h_A at the point w_ℓ), with the Thom isomorphism $H_K^*(D,S) \simeq H_K^{*-\mathrm{index}(w_\ell)}(\{w_\ell\})$; the map ① is induced by the inclusion of $\{w_\ell\}$ in M_ℓ; the class $e_\ell \in H_K^{\mathrm{index}(w_\ell)}(\{w_\ell\}) = H^{\mathrm{index}(w_\ell)}(BK)$ is the K-equivariant Euler class of the negative space of the Hessian of h_A at w_ℓ. According to Lemma 3.1, w_ℓ is an isolated fixed point of the T action on $Fl_n(\mathbb{H})$. By the Atiyah-Bott lemma (see appendix A of this paper) we deduce that the multiplication by e_ℓ is an injective map. We deduce that the long exact sequence of the pair $(M_\ell, M_{\ell-1})$ splits into short exact sequences of the form

$$0 \longrightarrow H_K^*(M_\ell, M_{\ell-1}) \longrightarrow H_K^*(M_\ell) \longrightarrow H_K^*(M_{\ell-1}) \longrightarrow 0.$$

Let us consider now the following commutative diagram.

$$
\begin{tikzcd}
0 \arrow[r] & H_K^*(M_\ell, M_{\ell-1}) \arrow[r] \arrow[d, "③"'] & H_K^*(M_\ell) \arrow[r] \arrow[d, "\imath_\ell^*"'] & H_K^*(M_{\ell-1}) \arrow[r] \arrow[d, "\imath_{\ell-1}^*"'] & 0 \\
0 \arrow[r] & H_K^*(\{w_\ell\}) \arrow[r] & H_K^*(S_n^\ell) \arrow[r] & H_K^*(S_n^{\ell-1}) \arrow[r] & 0
\end{tikzcd}
$$

where we have identified $H_K^*(S_n^\ell, S_n^{\ell-1}) = H_K^*(\{w_\ell\})$. The map $\imath_{\ell-1}^*$ is injective, by the induction hypothesis. The map ③ is the same as the composition of ① and ② (see diagram (6)), thus it is injective as well. By a diagram chase we deduce that $\imath_\ell^*$ is injective. $\square$

Our next goal is to describe the image of $\imath^*$. To this end we consider

$$N := \bigcup \mathcal{S}_{w,pq},$$

where the union runs over all w, p, q such that $w \in S_n$, $1 \le p < q \le n$, and $h_A(w) > h_A(ws_{pq})$. This is a K-invariant subspace of $Fl_n(\mathbb{H})$, which contains S_n. The idea is to show first that the image of $\imath^*$ is the same as the image of the natural map $H_K^*(N) \to H_K^*(S_n)$: this task will be accomplished by the following series of three technical lemmas. The image of the latter map is then obtained in Proposition 3.6. Let us recall that since $K = Sp(1)^n$, the K-equivariant cohomology of a point pt. is given by

$$H_K^*(\mathrm{pt.}) = H^*(BK) = H^*(BSp(1)^n) = \mathbb{Z}[u_1, \ldots, u_n].$$

Lemma 3.3. *(i) Fix $p \in \{1, \ldots, n\}$ and consider the action of K on the vector bundle $\mathbb{H}$ over a point* pt. *given by $\gamma.h := \gamma_p h$, for any $\gamma = (\gamma_1, \ldots, \gamma_n) \in K$ and any $h \in \mathbb{H}$. The K-equivariant Euler class of the bundle is u_p.*

(ii) Fix $p, q \in \{1, \ldots, n\}$ and consider the action of K on the vector bundle $\mathbb{H}$ over a point pt. *given by $\gamma.h := \gamma_p h \bar{\gamma}_q$, for any $\gamma = (\gamma_1, \ldots, \gamma_n) \in K$ and any $h \in \mathbb{H}$. The K-equivariant Euler class of the bundle is $u_p - u_q$.*

Proof. (i) Let $E = EK = E(Sp(1))^n$ be the total space of the classifying bundle of K. By definition (see for instance Section 6.2 of [Gu-Gi-Ka]), the equivariant Euler class is

$$e_K(\mathbb{H}) = e(E \times_K \mathbb{H}),$$

where $E \times_K \mathbb{H}$ is a vector bundle with fiber $\mathbb{H}$ over $E/K = BK = B(Sp(1))^n$. Thus $e_K(\mathbb{H})$ is an element of degree 4 in $H^*(BK) = \mathbb{R}[u_1, \ldots, u_n]$, hence a degree one polynomial in $u_1, \ldots, u_n$. We denote $BK = B(Sp(1))^n = \mathbb{H}P_1^\infty \times \ldots \mathbb{H}P_n^\infty$, where $\mathbb{H}P_\nu^\infty$, $1 \le \nu \le n$, are copies of $\mathbb{H}P^\infty$ (see e.g. [Hu, Chapter 8, Theorem 6.1]). The latter is the space of all $\mathbb{H}$ lines (that is, one dimensional $\mathbb{H}$-submodules) in $\mathbb{H}^\infty$, where the scalar multiplication is *from the right*. By this we mean that if $e \in \mathbb{H}^\infty$ and $h \in \mathbb{H}$, then, by definition, $h.e := e\bar{h}$. It is sufficient to show that if $\imath_\nu$ is the inclusion of $\mathbb{H}P_\nu^\infty$ into BK, then we have

$$\imath_\nu^*(e(E \times_K \mathbb{H})) = \begin{cases} 0, & \text{if } \nu \neq p \\ x, & \text{if } \nu = p. \end{cases}$$

Here $x \in H^4(\mathbb{H}P^\infty)$ denotes the Euler class of the tautological vector bundle over $\mathbb{H}P^\infty$. To show this, we note first that if $\nu \neq p$, then the restriction of the bundle $E \times_K \mathbb{H}$ to $\mathbb{H}P_\nu^\infty$ is the trivial bundle. Also, the restriction of $E \times_K \mathbb{H}$ to $\mathbb{H}P_p^\infty$ is the bundle $ESp(1) \times_{Sp(1)} \mathbb{H}$ over $ESp(1)/Sp(1) = \mathbb{H}P^\infty$ (here the action of $Sp(1)$ on $\mathbb{H}$ is by left multiplication). The latter bundle is actually the tautological bundle

$$\tau = \{(v, L) \mid L \text{ is a } \mathbb{H} \text{ line in } \mathbb{H}^\infty, v \in L\}$$

over $\mathbb{H}P^\infty$. Indeed, we take into account that $ESp(1)$ is the unit sphere S^∞ in $\mathbb{H}^\infty$ (again by [Hu, Chapter 8, Theorem 6.1]), thus we can identify $ESp(1) \times_{Sp(1)} \mathbb{H}$ with τ via

$$[e, h] \mapsto (eh, e\mathbb{H}),$$

for all $e \in ESp(1)$ and $h \in \mathbb{H}$. We conclude that $\imath_p^*(e(E \times_K \mathbb{H})) = e(\tau) = x$.

(ii) Now we must check that

$$\imath_\nu^*(e(E \times_K \mathbb{H})) = \begin{cases} 0, & \text{if } \nu \neq p \text{ or } q \\ x, & \text{if } \nu = p \\ -x, & \text{if } \nu = q. \end{cases}$$

The cases $\nu \neq q$ have been discussed before, at point (i). To analyze the case $\nu = q$, we note that the restriction of $E \times_K \mathbb{H}$ to $\mathbb{H}P_q^\infty$ is $ESp(1) \times_{Sp(1)} \mathbb{H}$, where $Sp(1)$ acts on $\mathbb{H}$ from the right. The map

$$ESp(1) \times_{Sp(1)} \mathbb{H} \ni [e, h] \mapsto (e\bar{h}, e\mathbb{H})$$

is an isomorphism between $ESp(1) \times_{Sp(1)} \mathbb{H}$ and the rank four vector bundle over $\mathbb{H}P^\infty$, call it $\bar{\tau}$, whose fiber over L is $\bar{L} := \{\bar{v} \mid v \in L\}$. Because the linear automorphism of $\mathbb{R}^4 = \mathbb{H}$ given by

$$(x_1, x_2, x_3, x_4) = x_1 + x_2 i + x_3 j + x_4 k \mapsto x_1 - x_2 i - x_3 j - x_4 k = (x_1, -x_2, -x_3, -x_4)$$

is orientation reversing (its determinant is equal to -1), we deduce that

$$i_q^*(e(E\times_K \mathbb{H})) = e(\bar{\tau}) = -e(\tau) = -x.$$

The claim, and also the lemma, are completely proved. □

Lemma 3.4. *Fix $w \in S_n$, $\epsilon > 0$ strictly smaller than $|h_A(w) - h_A(v)|$, for any $v \in S_n$, $v \neq w$, and set $N_+ := N \cap h_A^{-1}(-\infty, h_A(w)+\epsilon]$, $S_n^- = S_n \cap h_A^{-1}(-\infty, h_A(w)-\epsilon]$. Let η be an element of $H_K^*(N^+)$ which vanishes when restricted to S_n^-. Then the restriction $\eta|_w$ is a multiple of e_w, where e_w is the K-equivariant Euler class of the negative space of the Hessian of h_A at w.*

Proof. By (iii) of Proposition 2.1, the negative space of the Hessian of h_A at w is $\bigoplus_{(p,q)\in\mathcal{I}} T_w\mathcal{S}_{w,pq}$. For each $(p,q)\in\mathcal{I}$, the class $\eta_1 := \eta|_{\mathcal{S}_{w,pq}} \in H_K^*(\mathcal{S}_{w,pq})$ vanishes when restricted to ws_{pq}, which is the south pole of $\mathcal{S}_{w,pq}$. Consequently, η_1 vanishes when restricted to $h_A^{-1}(-\infty, h_A(w)-\epsilon]\cap\mathcal{S}_{w,pq}$, since the latter can be equivariantly retracted onto ws_{pq}. The long exact sequence of the pair $(\mathcal{S}_{w,pq}, h_A^{-1}(-\infty,h_A(w)-\epsilon]\cap\mathcal{S}_{w,pq})$ is

$$\ldots \to H_K^*(\mathcal{S}_{w,pq}, h_A^{-1}(-\infty,h_A(w)-\epsilon]\cap\mathcal{S}_{w,pq}) \to H_K^*(\mathcal{S}_{w,pq}) \to H_K^*(h_A^{-1}(-\infty,h_A(w)-\epsilon]\cap\mathcal{S}_{w,pq}) \to \ldots.$$

Here we can replace $H_K^*(\mathcal{S}_{w,pq}, h_A^{-1}(-\infty,h_A(w)-\epsilon]\cap\mathcal{S}_{w,pq})$ with $H_K^*(D,S)$ where D and S are a 4-disk and 3-sphere around w (the north pole of $\mathcal{S}_{w,pq}$). In turn, $H_K^*(D,S)\simeq H_K^{*-4}(\{w\})$. As in the proof of Proposition 3.2 (see diagram (6)), we deduce that $\eta_1|_w$, which is the same as $\eta|_w$, is a multiple of $e_K(T_w(\mathcal{S}_{w,pq}))$, the K-equivariant Euler class of the tangent space $T_w(\mathcal{S}_{w,pq})$. Now we recall that w is a fixed point of the K action and $\mathcal{S}_{w,pq} = K_{pq}.w$. Consequently, the tangent space at w to $\mathcal{S}_{w,pq}$ is

$$T_w(\mathcal{S}_{w,pq}) = \mathfrak{k}_{pq}^0.w$$

where the dot indicates the infinitesimal action and

$$\mathfrak{k}_{pq}^0 = \{hE_{pq} + \bar{h}E_{qp} \mid h \in \mathbb{H}\}.$$

Here E_{pq} denotes the $n\times n$ matrix whose entries are all 0, except for the one on position pq, which is equal to 1 (and the same for E_{qp}). This implies that $T_w(\mathcal{S}_{w,pq})$ is K-equivariantly isomorphic to $\mathbb{H}$, with the K action given by

$$(\gamma_1,\ldots,\gamma_n).h := \gamma_p h {\gamma_q}^{-1},$$

for any $(\gamma_1,\ldots,\gamma_n)\in K = (Sp(1))^n$ and any $h\in\mathbb{H}$. By Lemma 3.3, we have

$$e_K(T_w(\mathcal{S}_{w,pq})) = u_p - u_q.$$

Thus the polynomial $\eta|_w$ is divisible by $u_p - u_q$, for all $(p,q)\in\mathcal{I}$. Because the latter polynomials are relatively prime with each other, we deduce that $\eta|_w$ is actually divisible by their product, which is just e_w. □

Lemma 3.5. *If $\imath$, $\jmath$ are the inclusion maps of S_n into $Fl_n(\mathbb{H})$, respectively N, then the images of $\imath^* : H_K^*(Fl_n(\mathbb{H})) \to H_K^*(S_n)$ and $\jmath^* : H_K^*(N) \to H_K^*(S_n)$ are the same.*

Proof. As in the proof of Proposition 3.2, we order $S_n = \{w_1,\ldots,w_k\}$ such that $h_A(w_1) < h_A(w_2) < \ldots < h_A(w_k)$. We choose $\epsilon > 0$ smaller than the minimum of $h_A(w_\ell) - h_A(w_{\ell-1})$, where $\ell = 2,\ldots,k$. We denote $M_\ell = h_A^{-1}(-\infty, h_A(w_\ell)+\epsilon])$, $N_\ell := N\cap M_\ell$, $S_n^\ell = S_n \cap M_\ell$

and will prove by induction on ℓ that the images of the maps $\imath_\ell^*: H_K^*(M_\ell) \to H_K^*(S_n^\ell)$ and $\jmath_\ell^*: H_K^*(N_\ell) \to H_K^*(S_n^\ell)$ are the same. For $\ell = 1$, the assertion is clear, because both M_1 and its subset N_1 can be retracted equivariantly onto w_1. Now we assume that the assertion is true for $\ell - 1$ and we prove that it is true for ℓ. Let us consider the following commutative diagram.

$$\begin{array}{ccc} H_K^*(N_\ell) & \longrightarrow & H_K^*(N_{\ell-1}) \\ {\scriptstyle \jmath_\ell^*}\downarrow & & \downarrow{\scriptstyle \jmath_{\ell-1}^*} \\ H_K^*(S_n^\ell) & \xrightarrow[r]{} & H_K^*(S_n^{\ell-1}) \end{array} \tag{7}$$

where r is the restriction map. We deduce that r maps $\mathrm{im}(\jmath_\ell^*)$ to $\mathrm{im}(\jmath_{\ell-1}^*)$. From now on, by r we will denote the induced map

$$r: \mathrm{im}(\jmath_\ell^*) \to \mathrm{im}(\jmath_{\ell-1}^*).$$

Let us consider another commutative diagram, namely

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H_K^*(M_\ell, M_{\ell-1}) & \longrightarrow & H_K^*(M_\ell) & \longrightarrow & H_K^*(M_{\ell-1}) & \longrightarrow & 0 \\ & & {\scriptstyle h}\downarrow & & {\scriptstyle \imath_\ell^*}\downarrow & & {\scriptstyle \imath_{\ell-1}^*}\downarrow & & \\ 0 & \longrightarrow & \ker r & \xrightarrow[g]{} & \mathrm{im}(\jmath_\ell^*) & \xrightarrow[r]{} & \mathrm{im}(\jmath_{\ell-1}^*) & \longrightarrow & 0 \end{array}$$

In this diagram we have made two identifications: first, $H_K^*(S_n^\ell, S_n^{\ell-1})$ is identified with $H_K^*(\{w_\ell\})$; then $H_K^*(\{w_\ell\})$ is canonically embedded in $H_K^*(S_n^\ell)$. In this way, h goes from $H_K^*(M_\ell, M_{\ell-1})$ to $H_K^*(S_n^\ell, S_n^{\ell-1}) = H_K^*(\{w_\ell\})$; but one can easily see from the diagram that h actually takes values in the subspace $\ker r$ of the latter space. The map g is just the inclusion map. We will prove that the image of h is the whole $\ker r$. To this end, we take $\eta \in H_K^*(N_\ell)$ such that $r(\jmath_\ell^*(\eta)) = 0$ and we prove that $\jmath_\ell^*(\eta)$ is in the image of $g \circ h$, or, equivalently, that $\eta|_{w_\ell}$ is in the image of h. From the commutative diagram (7) we deduce that the restriction of η to $S_n^{\ell-1}$ is equal to 0. From Lemma 3.4 we deduce that $\eta|_{w_\ell}$ is a multiple of the Euler class e_{w_ℓ}. Now let us consider again the diagram (6). We deduce that $\eta|_{w_\ell}$ is in the image of ① ∘ ②, which is the same as h. In conclusion, we can use that h and $\imath_{\ell-1}^*$ are surjective and, by a chase diagram, deduce that $\imath_\ell^*$ is surjective as well. The proposition is proved. □

Now we are ready to characterize the image of $\imath^*$, by using that it is the same as the image of $\jmath^*$.

Proposition 3.6. *The image of* $\imath^*: H_K^*(Fl_n(\mathbb{H})) \to H_K^*(S_n) = \bigoplus_{w \in S_n} \mathbb{Z}[u_1, \ldots, u_n]$ *is*

$$\{(f_w)_{w \in S_n} \mid u_p - u_q \text{ divides } \ f_w - f_{ws_{pq}}, \forall w \in S_n, \forall 1 \le p < q \le n\}. \tag{8}$$

Proof. Denote by $\mathfrak{im}$ the space described by (8). By Lemma 3.5, it is sufficient to show that the image of $\jmath^*: H_K^*(N) \to H_K^*(S_n)$ is equal to $\mathfrak{im}$. First, let $(f_w)_{w \in S_n}$ be in the image of $\jmath^*$. Pick $w \in S_n$ and p, q integers such that $1 \le p < q \le n$ and $h_A(w) > h_A(ws_{pq})$. The pair $(f_w, f_{ws_{pq}})$ is in the image of the restriction map $H_K^*(\mathcal{S}_{w,pq}) \to H_K^*(\{w, ws_{pq}\})$.

Claim. The polynomial $f_w - f_{ws_{pq}}$ is divisible by $u_p - u_q$.

Put $U_1 := \mathcal{S}_{w,pq} \setminus \{w\}$, $U_2 := \mathcal{S}_{w,pq} \setminus \{ws_{pq}\}$. The idea is to use the Mayer-Vietoris sequence in equivariant cohomology for the triple $(\mathcal{S}_{w,pq}, U_1, U_2)$. We note that $U_1 \cap U_2 =$

$\mathcal{S}_{w,pq} \setminus \{w, ws_{pq}\}$. The rings $H^*_K(U_1), H^*_K(U_2), H^*_K(U_1 \cap U_2)$ and the corresponding restriction maps can be determined as follows. We start with the identification

$$\mathcal{S}_{w,pq} = P(\mathbb{H}e_p + \mathbb{H}e_q) = \mathbb{H}P^1 = (\mathbb{H} \oplus \mathbb{H})/\mathbb{H}^*,$$

described after Proposition 2.1. The action of $K = Sp(1)^n$ is given by

$$(\gamma_1, \ldots, \gamma_n).[h_1, h_2] = [h_1\bar{\gamma}_p, h_2\bar{\gamma}_q],$$

for all $(\gamma_1, \ldots, \gamma_n) \in K$ and all $[h_1, h_2] \in \mathcal{S}_{w,pq}$. The fixed points are $w = [0, 1]$ and $ws_{pq} = [1, 0]$. In this way we can further identify $U_1 = \mathbb{H}$ (and then $U_1 \cap U_2 = \mathbb{H}^*$) via

$$[1, h] \mapsto h,$$

for any $[1, h] \in U_1$. We have

$$H^*_K(U_1 \cap U_2) = H^*_K(\mathbb{H}^*),$$

where K acts on $\mathbb{H}^*$ via

$$(\gamma_1, \ldots, \gamma_n).h = \bar{\gamma}_p^{-1} h\bar{\gamma}_q, \tag{9}$$

for all $(\gamma_1, \ldots, \gamma_n) \in K$ and $h \in \mathbb{H}^*$. It will be convenient to consider the group $G := (\mathbb{H}^*)^n$ and use the fact that for any G-space X we have $H^*_G(X) = H^*_K(X)$ (this is because G/K is contractible). In particular, if we let G act on $\mathbb{H}^*$ via

$$(\gamma_1, \ldots, \gamma_n).h = \gamma_p h\gamma_q^{-1},$$

(cf. equation (9)) then

$$H^*_K(\mathbb{H}^*) = H^*_G(\mathbb{H}^*).$$

This means that we have to compute the cohomology of $E^n \times_{(\mathbb{H}^*)^n} \mathbb{H}^*$, where $E := E\mathbb{H}^*$. We note that $E^n \times_{(\mathbb{H}^*)^n} \mathbb{H}^*$ is homeomorphic to $E^n/\Delta((\mathbb{H}^*)^n)$, where

$$\Delta((\mathbb{H}^*)^n) := \{(\gamma_1, \ldots, \gamma_n) \in G \mid \gamma_p = \gamma_q\}.$$

The homeomorphism is given by

$$[(e_1, \ldots, e_p, \ldots, e_q, \ldots, e_n), h] \mapsto [e_1, \ldots, e_p h, \ldots, e_q, \ldots, e_n].$$

We also observe that $E^n = E((\mathbb{H}^*)^n) = E(\Delta((\mathbb{H}^*)^n))$, so that we have

$$H^*_G(U_1 \cap U_2) = H^*(E(\Delta((\mathbb{H}^*)^n)/\Delta((\mathbb{H}^*)^n))) = H^*(\Delta((E/\mathbb{H}^*)^n)),$$

where we have denoted

$$\Delta((E/\mathbb{H}^*)^n) = \{(a_1, \ldots, a_n) \in (E/\mathbb{H}^*)^n \mid a_p = a_q\}.$$

We now turn to the equivariant cohomology of U_1. This is

$$H^*_K(U_1) = H^*_K(\mathbb{H}) = H^*_G(\mathbb{H}) = H^*((E/\mathbb{H}^*)^n) = \mathbb{Z}[u_1, \ldots, u_n],$$

where we have used that the space $\mathbb{H}$ is equivariantly contractible. Moreover, the restriction map $H^*_K(U_1) \to H^*_K(U_1 \cap U_2)$ is the same as the map

$$H^*((E/\mathbb{H}^*)^n) \to H^*(\Delta((E/\mathbb{H}^*)^n)), \tag{10}$$

induced by the inclusion

$$\Delta((E/\mathbb{H}^*)^n) \hookrightarrow (E/\mathbb{H}^*)^n.$$

It is a simple exercise that the kernel of the map described by (10) is the ideal of $H^*((E/\mathbb{H}^*)^n) = \mathbb{Z}[u_1, \ldots, u_n]$ generated by $u_p - u_q$, hence

$$H^*(\Delta((E/\mathbb{H}^*)^n)) = \mathbb{Z}[u_1, \ldots, u_n]/\langle u_p - u_q \rangle.$$

In the same way, also the restriction map $H^*_K(U_2) \to H^*_K(U_1 \cap U_2)$ coincides with the canonical projection

$$\mathbb{Z}[u_1, \ldots, u_n] \to \mathbb{Z}[u_1, \ldots, u_n]/\langle u_p - u_q \rangle.$$

The Mayer-Vietoris sequence of the triple $(\mathcal{S}_{w,pq}, U_1, U_2)$ is as follows.

$$\ldots \to H^*_K(\mathcal{S}_{w,pq}) \to H^*_K(U_1) \oplus H^*_K(U_2) \xrightarrow{g} H^*_K(U_1 \cap U_2) \to \ldots. \tag{11}$$

The claim follows from the fact that $g(f_w, f_{ws_{pq}}) = f_w - f_{ws_{pq}} \mod \langle u_p - u_q \rangle$ is equal to 0.

Now we prove that $\mathfrak{im}$ is contained in the image of $\jmath^*$. Let $(f_w)_{w \in S_n}$ be an element of $\mathfrak{im}$. From the exact sequence (11), we deduce that for each $w \in S_n$ and each pair p, q with $1 \le p < q \le n$, $h_A(w) > h_A(ws_{pq})$, there exists $\alpha_{w,pq} \in H^*_K(\mathcal{S}_{w,pq})$ with $\alpha_{w,pq}|_w = f_w$ and $\alpha_{w,pq}|_{ws_{pq}} = f_{ws_{pq}}$. A simple argument (using again a Mayer-Vietoris sequence) shows that there exists $\alpha \in H^*_K(N)$ such that $\alpha|_{\mathcal{S}_{w,pq}} = \alpha_{w,pq}$. This implies $(f_w)_{w \in S_n} = \jmath^*(\alpha)$. □

Remark. It is likely that the main results of this section, namely Proposition 3.2 and Proposition 3.6, can be proved with the methods of [Ha-He-Ho].

The following technical result is needed to complete the proof of Theorem 1.1. We consider the tautological vector bundles $\mathcal{L}_\nu$ over $Fl_n(\mathbb{H})$, $1 \le \nu \le n$. That is, the fibre of $\mathcal{L}_\nu$ over $(L_1, \ldots, L_n) \in Fl_n(\mathbb{H})$ is L_ν.

Lemma 3.7. *Take $w \in S_n = (Fl_n(\mathbb{H})^K)$, and identify $H^*_K(\{w\}) = H^*(BK) = \mathbb{Z}[u_1, \ldots, u_n]$. Then the equivariant Euler class $e_K(\mathcal{L}_\nu)$ restricted to w is equal to $u_{w(\nu)}$, for all $\nu \in \{1, \ldots, n\}$.*

Proof. The cohomology class $e_K(\mathcal{L}_\nu)|_w$ is the equivariant Euler class of the space $\mathcal{L}_\nu|_w = \mathbb{H}$ with the K action

$$(\gamma_1, \ldots, \gamma_n).h = \gamma_{w(\nu)} h,$$

for all $(\gamma_1, \ldots, \gamma_n) \in K = Sp(1)^n$ and $h \in \mathbb{H}$. The result follows from Lemma 3.3. □

We are now ready to prove our main result.

Proof of Theorem 1.1. Let $Fl_n(\mathbb{C})$ be the space of flags in $\mathbb{C}^n$, which can be equipped in a natural way with the action of the torus $T := (S^1)^n$. The idea of the proof is to compare the equivariant cohomology rings $H^*_T(Fl_n(\mathbb{C}))$ and $H^*_K(Fl_n(\mathbb{H}))$. The first one can be computed using the GKM theory (cf. e.g. [Ho-Gu-Za]), as follows. Like in the quaternionic case explained here, the fixed point set $Fl_n(\mathbb{C})^T$ can be identified with the symmetric group S_n, and the restriction map $H^*_T(Fl_n(\mathbb{C})) \to H^*_T(S_n)$ is injective. Moreover, the image of the latter map consists of all sequences of polynomials $(f_w)_{w \in S_n}$ such that $f_w - f_{s_{pq}w}$ is divisible by $\tilde{u}_p - \tilde{u}_q$, for all $w \in S_n$ and $1 \le p < q \le n$, where we have identified

$$H^*(BT) = \mathbb{Z}[\tilde{u}_1, \ldots, \tilde{u}_n]$$

(compare with Proposition 3.6). The equivariant Euler classes $e_T(\tilde{\mathcal{L}}_\nu)$ of the tautological complex line bundles over $Fl_n(\mathbb{C})$ have the property that $e_T(\tilde{\mathcal{L}}_\nu)|_w = \tilde{u}_{w(\nu)}$, for all $1 \le \nu \le n$

(compare with Lemma 3.7). On the other hand, we have the Borel type description of $H^*_T(Fl_n(\mathbb{C}))$, namely

$$H^*_T(Fl_n(\mathbb{C})) \simeq \frac{\mathbb{Z}[x_1,\ldots,x_n,\tilde{u}_1,\ldots,\tilde{u}_n]}{\langle (1+x_1)\ldots(1+x_n)=(1+\tilde{u}_1)\ldots(1+\tilde{u}_n)\rangle},$$

via $e_T(\tilde{\mathcal{L}}_\nu)\mapsto x_\nu$, $1\le\nu\le n$. This is a standard result, which can be proved for instance by identifying $ET\times_T Fl_n(\mathbb{C})$ with the total space of the flag bundle associated to the vector bundle $ET\times_T\mathbb{C}^n$ over BT, and then using a theorem which describes the integer cohomology of such manifolds, cf. e.g. [Bo-Tu, Section 21], to deduce

$$H^*_T(Fl_n(\mathbb{C})) = H^*(ET\times_T Fl_n(\mathbb{C})) \simeq \frac{H^*(BT)[x_1,\ldots,x_n]}{\langle (1+x_1)\ldots(1+x_n) = c(ET\times_T\mathbb{C}^n)\rangle}.$$

The Chern class $c(ET\times_T\mathbb{C}^n)$ of the vector bundle $ET\times_T\mathbb{C}^n\to BT$ can be calculated by splitting the latter as $\bigoplus_{\nu=1}^n ET\times_T\mathbb{C}e_\nu$, the first Chern class of the summand $ET\times_T\mathbb{C}e_\nu$ being $\tilde{u}_\nu$. Theorem 1.1 follows. □

4. Appendix A. The Atiyah-Bott lemma

We will prove the following version of [At-Bo, Proposition 13.4]. It is worth noting that in general, the latter result holds only for cohomology with rational coefficients. Nevertheless, in the following lemma all cohomology rings have integer coefficients: the reason why this works is that $H^*(B(Sp(1)^n);\mathbb{Z})$ has no torsion, thus any non-zero element is not a zero divisor.

Lemma 4.1. *Let V be an even-dimensional real vector space, with the linear action of the group $K:=Sp(1)^n$. Assume that the only fixed point of the action of $T:=(S^1)^n\subset(Sp(1))^n$ on V is 0. If one regards V as a vector bundle over the point 0, then the equivariant Euler class $e_K(V)\in H^*_K(\{0\})=H^*(BK)$ is different from zero (hence it is not a zero divisor).*

Proof. If we denote $E:=EK=ET$, then the natural map $BT=E/T\to E/K=BK$ induces the ring homomorphism $H^*_K(\{0\})\to H^*_T(\{0\})$. The latter one maps $e_K(V)$ to $e_T(V)$ (because the line bundle $E\times_T V$ over E/T is the pullback of $E\times_K V$ over E/K via the natural map $E/T\to E/K$). So it is sufficient to show that $e_T(V)\in H^*_T(\{0\})=\mathbb{Z}[u_1,\ldots,u_n]$ is different from zero. Since the representation of T on V has no nonzero fixed points, we have $V=\oplus_{i=1}^m L_i$, where L_i are 1-dimensional complex representations of T. Thus we have

$$e_T(V)=c_m^T(V)=c_m^T(\oplus_{i=1}^m L_i)=c_1^T(L_1)\ldots c_1^T(L_m),$$

where c_m^T and c_1^T denote the T-equivariant Chern classes. Each Chern class $c_1^T(L_i)$ is different from zero, since the 1-dimensional complex representations of T are labeled by the character group $\mathrm{Hom}(T,S^1)$, and the map $\mathrm{Hom}(T,S^1)\to H^2(BT)$ given by $L\mapsto c_1^T(L)$ is a linear isomorphism (see for instance [Hu, Chapter 20, Section 11]). This finishes the proof. □

5. Appendix B. Height functions on isoparametric submanifolds

The goal of this section is to provide a proof of Proposition 2.1, and also to achieve a better understanding of the spheres $\mathcal{S}_{w,pq}$, which are important objects of this paper. We will place ourselves in the more general context of isoparametric submanifolds. We recall (see for instance [Pa-Te, Chapter 6]) that an n-dimensional submanifold $M \subset \mathbb{R}^{n+k}$ which is closed, complete with respect to the induced metric, and full (i.e. not contained in any affine subspace) is called *isoparametric* if any normal vector at a point of M can be extended to a parallel normal vector field ξ on M with the property that the eigenvalues of the shape operators $A_{\xi(x)}$ (i.e. the principal curvatures) are independent of $x \in M$, as values and multiplicities. It follows that for $x \in M$, the set $\{A_{\xi(x)} : \xi(x) \in \nu(M)_x\}$ is a commutative family of self-adjoint endomorphisms of $T_x(M)$, and so it determines a decomposition of $T_x(M)$ as a direct sum of common eigenspaces $E_1(x), E_2(x), ..., E_r(x)$. There exist normal vectors $\eta_1(x), \eta_2(x), ..., \eta_r(x)$ such that

$$A_{\xi(x)}|_{E_i(x)} = \langle \xi(x), \eta_i(x) \rangle \mathrm{id}_{E_i(x)},$$

for all $\xi(x) \in \nu_x(M)$, $1 \le i \le r$. By parallel extension in the normal bundle we obtain the vector fields $\eta_1, \dots, \eta_r$. The eigenspaces from above give rise to the distributions $E_1, \dots, E_r$ on M, which are called the *curvature distributions*. The numbers

$$m_i = \mathrm{rank} E_i,$$

$1 \le i \le r$, are the *multiplicities* of M. We fix a point $x_0 \in M$ and we consider the normal space

$$\nu_0 := \nu_{x_0}(M).$$

In the affine space $x_0 + \nu_0$ we consider the hyperplanes

$$\ell_i(x_0) := \{x_0 + \xi(x_0) : \xi(x_0) \in \nu_0, \langle \eta_i(x_0), \xi(x_0) \rangle = 1\},$$

$1 \le i \le r$; one can show that they have a unique intersection point, call it c_0, which is independent of the choice of x_0. Moreover, M is contained in a sphere with center at c_0. We do not lose any generality if we assume that M is contained in the unit sphere S^{n+k-1}, hence c_0 is just the origin 0 and $x_0 + \nu_0 = \nu_0$ (because $x_0 \in \nu_0$). One shows that the group of linear transformations of ν_0 generated by the reflections about $\ell_1(x_0), \dots, \ell_r(x_0)$ is a Coxeter group. We denote it by W and call it the *Weyl group* of M. We have

$$\nu_0 \cap M = W.x_0.$$

For each $i \in \{1, \dots, r\}$, the distribution E_i is integrable and the leaf through $x \in M$ of E_i is a round distance sphere $S_i(x)$, of dimension m_i, whose center is the orthogonal projection of x on $\ell_i(x)$. These are called the *curvature spheres*. We note that $S_i(x)$ contains x and $s_i x$ as antipodal points. We have the following result.

Proposition 5.1. *Let $a \in \nu_0$ contained in (the interior of) the same Weyl chamber as x_0 and let $h_a : M \to \mathbb{R}$, $h_a(x) = \langle a, x \rangle$ be the corresponding height function. The following is true.*

(a) *The gradient of h_a at any $x \in M$ is $a^{T_x(M)}$, that is, the orthogonal projection of a on the tangent space $T_x M$.*
(b) $\mathrm{Crit}(h_a) = W.x_0$ *and the negative space of the hessian of h_a at $x \in \mathrm{Crit}(h_a)$ is $\bigoplus_i E_i(x)$, where i runs over those indices in $\{1, \dots, r\}$ with the property $h_a(x) > h_a(s_i x)$.*
(c) *For any $x \in \mathrm{Crit}(h_a)$, and $i \in \{1, \dots, n\}$ such that $h_a(x) > h_a(s_i x)$, the meridians of $S_i(x)$ going from x to $s_i x$ are gradient lines of the function $h_a : M \to \mathbb{R}$ with respect to the submanifold metric on M.*

Proof. The points (a) and (b) are proved in [Pa-Te, Chapter 6]. We only need to prove (c). This follows immediately from the fact that for any $z \in S_i(x)$, the vector $\nabla(h_a)(z) = a^{T_zM}$ is tangent to $S_i(z)$ (because $a \in \nu_{x_0}(M) = \nu_x(M)$ is perpendicular to $\bigoplus_{j\neq i} E_j(x) = \bigoplus_{j\neq i} E_j(z)$, cf. [Pa-Te, Theorem 6.2.9 (iv) and Proposition 6.2.6]). □

Finally, Proposition 2.1 can be deduced from Proposition 5.1 by noting that $Fl_n(\mathbb{H})$ is an isoparametric submanifold of $\mathcal{H}_n$ with the following properties (cf. [Pa-Te, Example 6.5.6] for the symmetric space $SU(2n)/Sp(n)$):

- the normal space to $x_0 := \mathrm{Diag}(r_1, \ldots, r_n)$ is $\nu_0 = \mathfrak{d}$
- the Weyl group W is S_n acting in the obvious way on $\mathfrak{d}$
- the curvature spheres through wx_0 are the orbits $K_{pq}.w$, where $1 \leq p < q \leq n$; thus all multiplicities are equal to 4.

References

[At-Bo] M. Atiyah and R. Bott, *The Yang-Mills equations over Riemann surfaces*, Phil. Trans. Royal Soc. London **A 308** (1982), 523–615

[Bi-Gu-Ho] D. Biss, V. Guillemin, and T. Holm, *The mod 2 equivariant cohomology of real loci*, Adv. Math. **185** (2004), 370–399

[Bo] A. Borel, *Seminar on Transformation Groups*, Annals of Math. Studies, Princeton University Press, 1960

[Bo-Sa] R. Bott and H. Samelson, *Applications of the theory of Morse to symmetric spaces*, Amer. J. Math. **80** (1958), 964–1029

[Bo-Tu] R. Bott and L. Tu, *Differential Forms in Algebraic Topology*, Springer-Verlag 1982

[Fr] T. Frankel, *Fixed points and torsion on Kähler manifolds*, Ann. of Math. **70** (1959), 1–8

[Gu-Gi-Ka] V. Guillemin, V. Ginzburg, and Y. Karshon, *Moment Maps, Cobordism, and Hamiltonian Group Actions*, Amer. Math. Soc., Mathematical Surveys and Monographs **98**, 2002

[Ha-Ho] M. Harada and T. Holm, *The equivariant cohomology of hypertoric varieties and their real loci*, Comm. Anal. Geom **13** (2005), 645–677

[Ha-Ho-He] M. Harada, A. Henriques, and T. Holm, *Computation of generalized equivariant cohomologies of Kac-Moody flag varieties*, Adv. Math. **197** (2005), 198–221

[Ho-Gu-Za] T. Holm, V. Guillemin, and C. Zara, *A GKM description of the equivariant cohomology ring of a homogeneous space*, Jour. Alg. Comb. **23** (2006), 21–41

[Hs-Pa-Te] W.Y. Hsiang, R. Palais, and C.-L. Terng, *Topology of isoparametric submanifolds*, J. Diff. Geom. **27** (1988), 423–460

[Hu] D. Husemoller, *Fibre Bundles (Third Edition)*, Springer-Verlag, 1994

[Ki] F. C. Kirwan, *Cohomology of Quotients in Symplectic and Algebraic Geometry*, Princeton University Press, 1984

[Ma1] A.-L. Mare, *Equivariant cohomology of real flag manifolds*, Diff. Geom. Appl. **23** (2006), 223–229

[Ma2] A.-L. Mare, *Connectivity and Kirwan surjectivity for isoparametric submanifolds*, Inter. Math. Res. Not., 2005, Issue 55, 3427-3443

[Pa-Te] R. Palais and C.-L. Terng, *Critical Point Theory and Submanifold Geometry*, Lecture Notes in Mathematics **1353**, Springer Verlag 1988

[To-We] S. Tolman and J. Weitsman, *On the cohomology rings of Hamiltonian T-spaces*, Proc. of the Northern California Symplectic Geometry Seminar, A.M.S. Translation Series 2, 1999, 251-258

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