THE EQUIVARIANT COHOMOLOGY OF[1] $Fl_n(\mathbb{C})$

Let $Fl_n(\mathbb{C})$ be the complex flag manifold. This can be regarded either as the space of all chains $V_1 \subset \ldots \subset V_n = \mathbb{C}^n$, where $\dim V_k = k$, or the space of all sequences $(L_1, \ldots, L_n)$, where L_j is perpendicular to L_k, for any two j, k, with $j \neq k$. We denote by $T = (S^1)^n$ the maximal torus of $U(n)$, so that $Fl_n(\mathbb{C}) \simeq U(n)/T$. The following theorem which describes the T-equivariant cohomology[2] of $Fl_n(\mathbb{C})$ was proved by Rebecca Goldin in [1, 2].

Theorem. (Goldin) *We have*

$$H^*_T(Fl_n(\mathbb{C})) \simeq \mathbb{Z}[x_1, \ldots, x_n, u_1 \ldots, u_n]/\langle \prod(1 + x_j) = \prod(1 + u_j)\rangle.$$

Here $x_j = c_1^T(\mathcal{L}_j) \in H^2_T(Fl_n(\mathbb{C}))$ is the T-equivariant Chern class of the line bundle $\mathcal{L}_j$; also $u_j \in H^*_T(BT)$ is the T-equivariant Euler class of the line bundle $ET \times_T \mathbb{C}e_j$ over BT, so that we have

$$H^*(BT) = \mathbb{Z}[u_1, \ldots, u_n].$$

In what follows I just wanted to spell out the details of the proof that can be found in the two aforementioned works.

I first recall that, by definition, $H^*_T(Fl_n(\mathbb{C})) = H^*(ET \times_T Fl_n(\mathbb{C}))$. One has the identification

$$ET \times_T Fl_n(\mathbb{C}) \simeq Fl(ET \times_T \mathbb{C}^n), \tag{1}$$

which is explained as follows: $\pi : ET \times_T \mathbb{C}^n \to BT$ is the rank n vector bundle whose fiber over $[e] \in BT = ET/T$ is

$$\pi^{-1}([e]) := \{[e, v] \mid v \in \mathbb{C}^n\} \simeq \mathbb{C}^n,$$

where the last (linear) isomorphism, call it $\imath_{[e]}$, maps v to $[e, v]$, for all $v \in \mathbb{C}^n$ (it is an easy exercise to show that $\imath_{[e]}$ really depends only on the coset of e); then

$$\sigma : Fl(ET \times_T \mathbb{C}^n) \to BT$$

is the associated flag bundle, whose fiber over $[e] \in BT$ is the flag manifold

$$Fl(\pi^{-1}([e])) = \{(L'_1, \ldots, L'_n) \mid L'_j \subset \pi^{-1}([e]), \dim L'_j = 1, L'_j \perp L'_k \text{ if } j \neq k\}.$$

The isomorphism (1) is given by

$$[e, (L_1, \ldots, L_n)] \mapsto ([e], \imath_{[e]}(L_1), \ldots, \imath_{[e]}(L_n)), \tag{2}$$

for all $(L_1, \ldots, L_n) \in Fl_n(\mathbb{C})$.

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[2] All cohomology rings are with coefficients in $\mathbb{Z}$.

The manifold $Fl(ET \times_T \mathbb{C}^n)$ is a split manifold [Bott-Tu, section 21] for the vector bundle $ET \times_T \mathbb{C}^n$, in the sense that one has the following splitting of the pull-back vector bundle

$$\sigma^{-1}(ET \times_T \mathbb{C}^n) = \bigoplus_{j=1}^{n} \mathcal{L}'_j$$

(where the line bundles $\mathcal{L}_j$ over $Fl(ET \times_T \mathbb{C}^n)$ are obvious) and also the map

$$\sigma^* : H^*(BT) \to H^*(Fl(ET \times_T \mathbb{C}^n))$$

is injective (this follows from the fact that the cohomology of both the base space and the fiber of the bundle σ vanish in all odd dimensions). According to [Bott-Tu, p. 284], the ring $H^*(Fl(ET \times_T \mathbb{C}^n))$ is generated by $\sigma^* H^*(BT) \simeq H^*(BT)$ together with the Chern classes $x'_j := c_1(\mathcal{L}'_j)$, $1 \leq j \leq n$, which satisfy the relations

$$\prod_{j=1}^{n}(1 + x'_j) \stackrel{!}{=} c(ET \times_T \mathbb{C}^n) = \prod_{j=1}^{n}(1 + u_j),$$

where the last equality uses that

$$ET \times_T \mathbb{C}^n = \bigoplus_{j=1}^{n} ET \times_T \mathbb{C}e_j.$$

To prove Theorem 1.1, I only need to show that the identification (1) can be completed to a line bundle isomorphism between $\hat{\mathcal{L}}_j$ over $Fl(ET \times_T \mathbb{C}^n)$ and $ET \times_T \mathcal{L}_j$ over $ET \times_T Fl_n(\mathbb{C})$. Indeed, the fiber of the former bundle over the point $([e], L'_1, \ldots, L'_n)$ is L'_j (which is a line in $\pi^{-1}([e])$). This implies that the fiber over the image of $[e, (L_1, \ldots, L_n)]$ (see (2)) is $\imath_{[e]}(L_j)$. So the line bundle isomorphism I was looking for is the natural identification between $\{[e, v] \mid v \in L_j\}$ and $\imath_{[e]}(L_j)$.

References

[1] R. Goldin, *The cohomology ring of weight varieties and polygon spaces*, Adv. Math. **160** (2001), 175-204

[2] R. Goldin, *The cohomology of weight varieties*, Doctoral Dissertation, MIT, 1999