DIFFERENTIAL GEOMETRY OF CURVES AND SURFACES

8. Minimal Surfaces

8.1. Definition, Characterization, Examples.

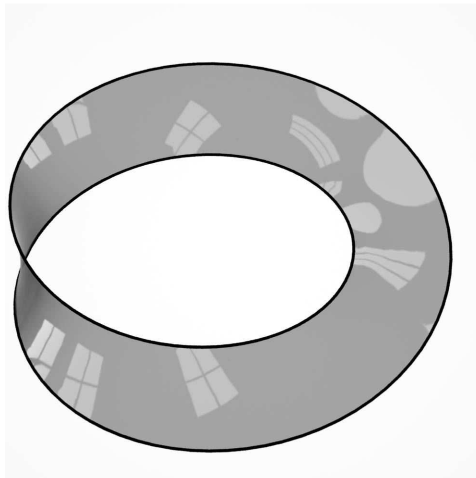

Figure 1. An example of a soap film (it looks very much like a Möbius strip, but it's not).

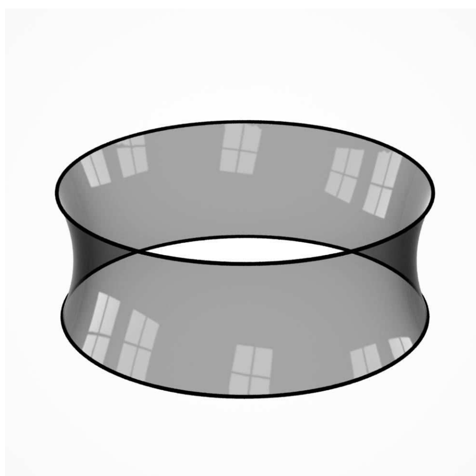

Figure 2. Another soap film, which is a piece of the catenoid (the top and bottom frames are circles).

Motivation. By dipping a wire frame into a soap solution and withdrawing it, we obtain a soap film: see Figures 1 and 2. Physical considerations (or just your intuition) are saying that this surface is "exactly the one which is bounded by the wire and whose *area is minimal*". The quotation signs are due to the fact that the assertion is not completely true. There are two ways of expressing the assertion in a more precise way:

(a) if we consider the function A which assigns to each surface bounded by the wire its area, then the soap film is a *local* minimum of A (like in Calculus I); by this we mean that if we deform the soap film *slightly*, the area will become larger.
(b) if we isolate a "sufficiently" small piece of the surface, then any variation of that small piece results into an increase of the area.

In this chapter we will discuss about surfaces with property (b). Before getting further, it is worth looking at figure 3 and try to understand the idea of the rigorous definition which will come shortly.

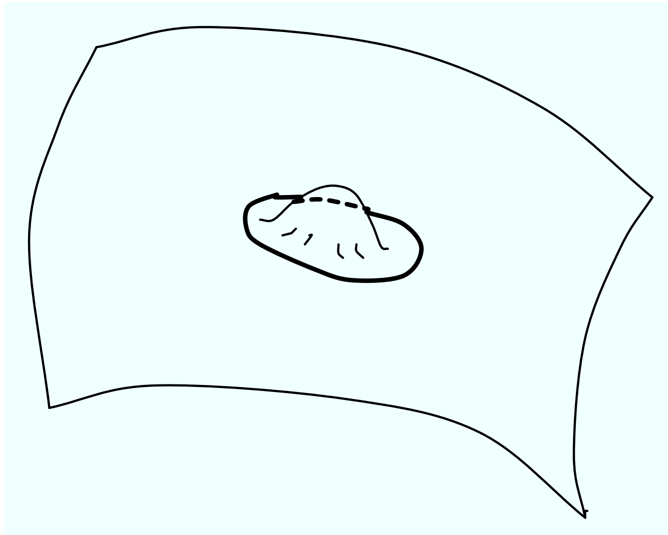

FIGURE 3. Understanding point (b) from above: Fix a "sufficiently" small contour on the surface; you are allowed to deform the surface, but only inside the contour; this should result in an increase of the area of the surface.

The notion of surface we will use in this chapter is slightly different from the one we have used in the previous ones.

Definition 8.1.1. A *regular parametrized surface* is a differentiable map $\varphi : U \to \mathbb{R}^3$, where U is an open subset of $\mathbb{R}^2$ such that for any Q in U the vectors $\varphi'_u(Q)$ and $\varphi'_v(Q)$ are linearly independent.

So we will consider (images of) local parametrizations which are not necessarily injective (you may want to look again at Definition 3.1.1). For example, Figure 4 is the image of a regular parametrized surface.

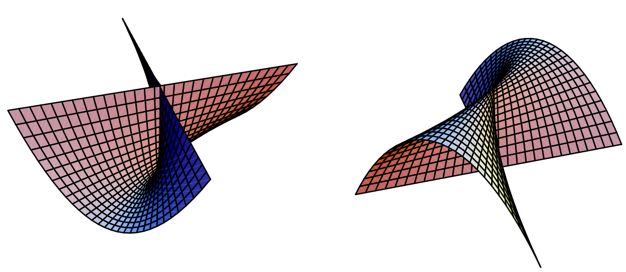

FIGURE 4. Two views of the Whitney umbrella, which is the image of $\varphi(u, v) = (uv, u, v^2)$. This is *not* a surface in the sense of Definition 3.1.1, since φ is not injective — this gives the self-intersections in the figure. Nevertheless, it is a surface in the sense of this chapter (see Definition 8.1.1.)

We now return to the considerations from the beginning of the section and define "soap film" surfaces.

Definition 8.1.2. A regular parametrized surface $\varphi : U \to \mathbb{R}^3$ is called *minimal* if any point in U has a compact neighborhood R in U, with the property that for any variation $\varphi_\lambda : U \to \mathbb{R}^3$, with $-\epsilon < \lambda < \epsilon$ satisfying the conditions

- $\varphi_0 = \varphi$
- $\varphi_\lambda = \varphi$ on $U \setminus R$, for all λ

we have

$$A(\varphi(R)) \le A(\varphi_\lambda(R))$$

for any λ.

The following proposition gives an equivalent definition.

Proposition/Definition 8.1.3. *A surface $\varphi : U \to \mathbb{R}^3$ is minimal if and only if for any Q in U we have*

$$H(Q) = 0,$$

where H denotes the mean curvature.

We give the (idea of the proof of the) " $\Rightarrow$ " implication. Let R be a region in U like in Definition 8.1.2. We consider a special kind of variation φ_λ, namely a *normal variation.* More precisely, this is

$$\varphi_\lambda(Q) = \varphi(Q) + \lambda h(Q)N(Q) \tag{1}$$

for any Q in U. Here $N(Q)$ denotes as usually the unit normal vector

$$N(Q) := \frac{\varphi'_u(Q) \times \varphi'_v(Q)}{\|\varphi'_u(Q) \times \varphi'_v(Q)\|} \tag{2}$$

and $h : U \to \mathbb{R}$ is a function with

$$h(Q) = 0, \text{ for any } Q \text{ not in } R.$$

Lemma 8.1.4. *If we denote*

$$A(\lambda) := A(\varphi_\lambda(R))$$

then we have

$$A'(0) = -\int\int_R 2hH\sqrt{EG - F^2}dA.$$

Proof (sketch). We have

$$(\varphi_\lambda)'_u = \varphi'_u + \lambda h'_u N + \lambda h N'_u$$
$$(\varphi_\lambda)'_v = \varphi'_v + \lambda h'_v N + \lambda h N'_v.$$

We denote by E, F, G the coefficients of the first fundamental form of φ and by $E^\lambda, F^\lambda, G^\lambda$ the coefficients of the first fundamental form of φ_λ. We use the formulas for e, f, g, and H (coefficients of the second fundamental form, respectively mean curvature of φ) given in chapter 5, page 12 (see equations (3) and (4)). We deduce[1] the following formula, which is crucial for the proof:

$$E^\lambda G^\lambda - (F^\lambda)^2 = (EG - F^2)(1 - 4\lambda hH) + O(\lambda)$$

where $O(\lambda)$ is a multiple of λ^2. Consequently, the area of $\varphi_\lambda(R)$ is

$$A(\lambda) = \int\int_R \sqrt{E^\lambda G^\lambda - (F^\lambda)^2}dA = \int\int_R \sqrt{(EG - F^2)(1 - 4\lambda hH) + O(\lambda)}dA.$$

[1]The details can be found in [dC], p. 198.

Because $(d/d\lambda)|_{\lambda=0}$ and $\int\int_R$ commute with each other, and

$$\frac{d}{d\lambda}|_{\lambda=0}\sqrt{(EG-F^2)(1-4\lambda hH)+O(\lambda)} = -2hH\sqrt{EG-F^2}$$

we obtain the desired formula. The lemma is proved. □

Now we are ready to justify " $\Rightarrow$ " in Proposition 8.1.3. Because $\lambda = 0$ is a minimum of $A(\lambda)$, we have $A'(0) = 0$ for any normal variation φ_λ (see (1)). From Lemma 8.1.4 we deduce that $H(Q) = 0$ for all Q in R. Because R was chosen arbitrary, we deduce that $H(Q) = 0$ for all Q in U.

The opposite implication in Proposition 8.1.3 is proved for instance in [Es] section 8.2 (this book is not published yet and is written in German, but I can provide a copy of that section, with translation and explanations, to anyone interested).

Examples of minimal surfaces. The following surfaces are minimal. We will only justify this *later*.

1. The *helicoid* given by

$$\varphi_1(u,v) = (u\cos v, u\sin v, v)$$

where u and v are in $\mathbb{R}$ (see Figure 5 and remember HW no. 4, question 5).

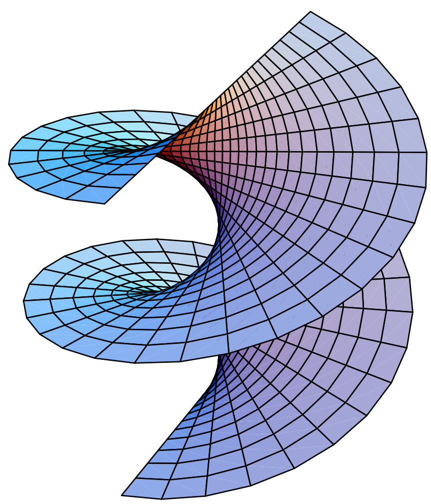

FIGURE 5. The helicoid.

2. The *catenoid* given by

$$\varphi_2(u,v) = (\cosh u\cos v, \cosh u\sin v, u)$$

where u is in $\mathbb{R}$ and $0 < v < 2\pi$ (see Figure 6 and remember again HW no. 4, question 5).

3. *Enneper's minimal surface*, given by

$$\varphi(u,v) = (u - \frac{u^3}{3} + uv^2, -v + \frac{v^3}{3} - vu^2, u^2 - v^2),$$

where u, v are in $\mathbb{R}$ (see Figure 7, but it's a good idea to go to `http://mathworld.wolfram.com/EnnepersMinimalSurface.html` and drag the cursor to rotate the surface and see it better).

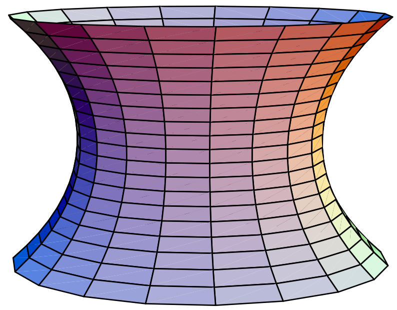

FIGURE 6. The catenoid.

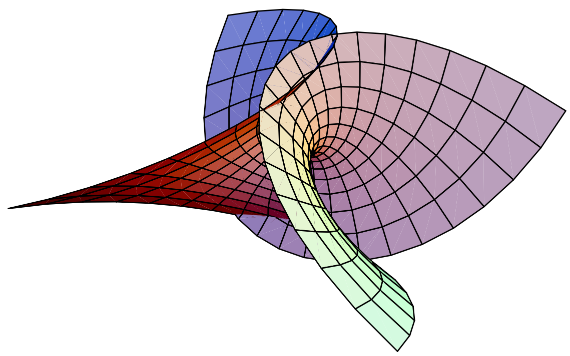

FIGURE 7. The Enneper minimal surface: it has lots of self-intersections, unlike the helicoid and the catenoid.

You can go online to `http://rsp.math.brandeis.edu/3d-xplormath/surface/gallery_m.html` and see many other minimal surfaces. The main goal of the next section is to describe the general framework which leads to those examples — namely the Weierstrass-Enneper representation.

8.2. Constructing Minimal Surfaces: the Weierstrass-Enneper Representation. Our goal here is to describe a method of constructing minimal surfaces. It is convenient to use isothermal surfaces, in the sense of the following definition.

Definition 8.2.1. A regular parametrized surface $\varphi : U \to \mathbb{R}^3$ is said to be <u>isothermal</u> if the coefficients E, F, G of the first fundamental form satisfy

$$E = G \text{ and } F = 0.$$

The reason we prefer this kind of surfaces is that they give a simple formula for the mean curvature H, as follows.

Proposition 8.2.2. *If $\varphi : U \to \mathbb{R}^3$ is an isothermal surface with $E = G =: \lambda^2$, normal vector N (see (2)), and mean curvature H then we have*

$$\varphi''_{uu} + \varphi''_{vv} = 2\lambda^2 H N.$$

Proof. We have
$$\varphi'_u \cdot \varphi'_u = \varphi'_v \cdot \varphi'_v, \ \varphi'_u \cdot \varphi'_v = 0.$$
By differentiation, this implies
$$\varphi''_{uu} \cdot \varphi'_u = \varphi''_{vu} \cdot \varphi'_v = -\varphi''_{vv} \cdot \varphi'_u$$
Consequently $\varphi''_{uu} + \varphi''_{vv}$ is perpendicular to both φ'_u and φ'_v, so it is parallel to N. On the other hand, from chapter 5 we have
$$H = \frac{eG - 2fF + gE}{2(EG - F^2)} = \frac{g + e}{2\lambda^2}$$
which implies
$$2\lambda^2 H = g + e = N \cdot (\varphi''_u + \varphi''_v)$$
and this finishes the proof. □

Although this will play no role here, we mention that on any surface one can find around any point an isothermal parametrization (a proof of this result can be found for instance in [Sp, Ch. 9, Addendum 1]).

The *Laplacian* of a function $f : U \to \mathbb{R}$ is
$$\Delta(f) = f''_{uu} + f''_{vv}.$$
We say that a function $f : U \to \mathbb{R}$ is *harmonic* if
$$\Delta(f) = 0.$$
A straightforward consequence of the previous proposition is as follows.

Corollary 8.2.3. *An isothermal surface $\varphi : U \to \mathbb{R}^3$ of the form*
$$\varphi(u, v) = (\varphi^1(u, v), \varphi^2(u, v), \varphi^3(u, v)),$$
(u, v) in U, is minimal if and only if the components φ^1, φ^2, and φ^3 are harmonic functions.

It's not an easy task to produce three functions $\varphi^1(u, v), \varphi^2(u, v), \varphi^3(u, v)$ which are all harmonic and such that $\varphi = (\varphi^1, \varphi^2, \varphi^3)$ is regular. The task will become more handy if we use tools from complex analysis. First we identify $\mathbb{R}^2$ with the complex plane $\mathbb{C}$, by
$$\mathbb{R}^2 \ni (u, v) = u + iv =: z \in \mathbb{C}.$$
So U is now an open subset of $\mathbb{C}$. We recall that a function $h : U \to \mathbb{C}$, of the form
$$h(z) = a(u, v) + ib(u, v)$$
is holomorphic[2] if and only if the following equations (of Cauchy and Riemann) are satisfied:
$$a'_u = b'_v$$
$$a'_v = -b'_u$$
everywhere on U. Moreover, the derivative of h is given by
$$h'(z) = a'_u(u, v) - ia'_v(u, v).$$

Notations. (a) If $f : U \to \mathbb{R}$ is differentiable, we denote
$$f_z := \frac{1}{2}(f'_u - if'_v),$$
which is a function $U \to \mathbb{C}$.

(b) If $\varphi : U \to \mathbb{R}^3$, of the form $\varphi(u, v) = (\varphi^1(u, v), \varphi^2(u, v), \varphi^3(u, v))$, we define
$$\varphi_z := (\varphi^1_z, \varphi^2_z, \varphi^3_z),$$

[2] By definition, this means that h has derivative at any point z in U.

which is a function $U \to \mathbb{C}^3$.

Proposition 8.2.4. *A. Let $\varphi = (\varphi^1, \varphi^2, \varphi^3) : U \to \mathbb{R}^3$ be an arbitrary map.*

(a) *φ is isothermal if if and only if*

$$(\varphi_z^1)^2 + (\varphi_z^2)^2 + (\varphi_z^3)^2 = 0 \tag{3}$$

everywhere on U.

(b) *if (3) is satisfied, φ is a regular parametrized surface if and only if*

$$|\varphi_z^1|^2 + |\varphi_z^2|^2 + |\varphi_z^3|^2 \neq 0 \tag{4}$$

everywhere on U.

(c) *if (3) and (4) are satisfied, φ is a minimal surface if and only if the functions φ_z^1, φ_z^2, and φ_z^3 are holomorphic.*

B. Conversely, let $\psi_1, \psi_2, \psi_3 : U \to \mathbb{C}$ be holomorphic functions such that

$$(\psi_1)^2 + (\psi_2)^2 + (\psi_3)^2 = 0 \text{ and } |\psi_1|^2 + |\psi_2|^2 + |\psi_3|^2 \neq 0. \tag{5}$$

If U is simply connected[3] *then there exists a regular minimal isothermal surface $\varphi = (\varphi^1, \varphi^2, \varphi^3) : U \to \mathbb{R}^3$ such that*

$$\varphi_z^1 = \psi_1, \ \varphi_z^2 = \psi_2, \ \varphi_z^3 = \psi_3.$$

More precisely,

$$\varphi^1 = \mathrm{Re} \int \psi_1(z)dz, \ \varphi^2 = \mathrm{Re} \int \psi_2(z)dz, \ \varphi^3 = \mathrm{Re} \int \psi_3(z)dz.$$

Proof. A. (a) The assertion follows from the equation

$$\begin{aligned}
&(\varphi_z^1)^2 + (\varphi_z^2)^2 + (\varphi_z^3)^2 \\
&= \frac{1}{4}\left[((\varphi^1)'_u - i(\varphi^1)'_v)^2 + ((\varphi^2)'_u - i(\varphi^2)'_v)^2 + ((\varphi^3)'_u - i(\varphi^3)'_v)^2\right] \\
&= \frac{1}{4}(E - G - 2iF)
\end{aligned}$$

(b) We have

$$\begin{aligned}
&|\varphi_z^1|^2 + |\varphi_z^2|^2 + |\varphi_z^3|^2 \\
&= \frac{1}{4}\left[((\varphi^1)'_u)^2 + (\varphi^1)'_v)^2 + ((\varphi^2)'_u)^2 + ((\varphi^2)'_v)^2 + ((\varphi^3)'_u)^2 + ((\varphi^3)'_v)^2\right] \\
&= \frac{1}{4}(E + G) \\
&= \frac{1}{2}G
\end{aligned}$$

Take Q in U arbitrary. If $\varphi'_u(Q) = r\varphi'_v(Q)$ for some number $r \neq 0$, then we have

$$0 = F(Q) = \varphi'_u(Q) \cdot \varphi'_v(Q) = rG$$

which implies $G = 0$, which is not true.

(c) If (3) and (4) are satisfied, φ is a regular parametrized isothermal surface. By Corollary 8.2.3, it is minimal if and only if

$$(\varphi^k)''_{uu} + (\varphi^k)''_{vv} = 0$$

[3] There is a result in Complex Analysis saying that if U is simply connected, then any holomorphic function $h : U \to \mathbb{C}$ has an antiderivative, denoted $\int h(z)dz$.

everywhere on U, for $k = 1, 2, 3$. This is equivalent to the Cauchy-Riemann equations for φ^k_z.

B. The statement follows from the following general fact: if U is an open simply connected subset of $\mathbb{C}$ and $g : U \to \mathbb{C}$ a holomorphic function, then g has an antiderivative[4] h, denoted

$$h := \int g(z)dz$$

with the property that

$$h'(z) = g(z).$$

Moreover, if $h(z) = a(z) + ib(z)$, where $a(z), b(z)$ are real numbers, then

$$a_z = g.$$

□

We are especially interested in the following aspect, described at point B. of the theorem: if U is simply connected, we can construct minimal surfaces from U to $\mathbb{R}^3$ by picking a triple of holomorphic functions ψ_1, ψ_2, ψ_3 satisfying (5), integrate each of them and take each time the real part: the three resulting functions are components of a minimal surface, call it $\varphi : U \to \mathbb{R}^3$. It is interesting to note that if ψ_1, ψ_2, ψ_3 satisfy (5), then for any θ in $\mathbb{R}$, the triple $e^{i\theta}\psi_1, e^{i\theta}\psi_2, e^{i\theta}\psi_3$ satisfies (5) as well; if we integrate the new triple and take the real parts, we obtain a new minimal surface, call it $\varphi_\theta : U \to \mathbb{R}^3$. We obviously have $\varphi_0 = \varphi$. The family $\{\varphi_\theta\}_\theta$ is called the *associated family* of φ. The surface $\varphi_{\frac{\pi}{2}}$ is called the *conjugate* of φ. Note that in fact the latter is given by

$$\varphi^1_{\frac{\pi}{2}} = -\mathrm{Im} \int \psi_1(z)dz,\ \varphi^2_{\frac{\pi}{2}} = -\mathrm{Im} \int \psi_2(z)dz,\ \varphi^3_{\frac{\pi}{2}} = -\mathrm{Im} \int \psi_3(z)dz$$

We can easily check that the coefficients of the first fundamental form of φ_θ are independent of θ: we say that $\{\varphi_\theta\}_\theta$ is an isometric deformation of φ. Finally, one can show that the Gauss map does not change during the deformation: by this we mean that if we fix $Q = (u, v)$ in U, then

$$N(\varphi(Q)) = N(\varphi_\theta(Q)),$$

for all θ.

The following theorem is the main result of the section.

Theorem 8.2.5. (The Weierstrass Representation Theorem) *Let U be simply connected and $h, g : U \to \mathbb{C}$ two holomorphic functions with $h(z) \neq 0$ everywhere on U. Then $\varphi = (\varphi^1, \varphi^2, \varphi^3) : U \to \mathbb{R}^3$ given by*

$$\varphi^1 := \mathrm{Re} \int \frac{1}{2}h(z)(1 - g(z)^2)dz$$

$$\varphi^2 := \mathrm{Re} \int \frac{i}{2}h(z)(1 + g(z)^2)dz$$

$$\varphi^3 := \mathrm{Re} \int h(z)g(z)dz$$

is a (isothermal) minimal surface.

Proof. The functions

$$\psi_1(z) = \frac{1}{2}h(z)(1 - g(z)^2),\ \psi_2(z) = \frac{i}{2}h(z)(1 + g(z)^2),\ \psi_3(z) = h(z)g(z)$$

[4] This is unique up to adding a constant.

satisfy equations (5): the first equation is obvious; for the second one, we note that

$$\begin{aligned}
&|\psi_1(z)|^2 + |\psi_2(z)|^2 + |\psi_3(z)|^2 \\
&= \frac{1}{4}|h(z)|^2(|1-g(z)^2|^2 + |1+g(z)^2|^2 + 4|g(z)|^2) \\
&= \frac{1}{2}|h(z)|^2(1+|g(z)|^2)^2.
\end{aligned}$$

Here we have used the identity

$$|1-w^2|^2 + |1+w^2|^2 + 4|w|^2 = 2(1+|w|^2)^2,$$

where w is any complex number. We use Proposition 8.2.4, point B. □

The theorem is telling us how to produce minimal surfaces out of two holomorphic functions h and g. We will do a few examples.

Examples. 1. Take

$$h(z) = -e^{-z} \text{ and } g(z) = -e^{z}.$$

The corresponding minimal surface φ has

$$\begin{aligned}
\varphi^1(u,v) &= \frac{1}{2}\mathrm{Re}\int(-e^{-z}+e^{z})dz \\
&= \frac{1}{2}\mathrm{Re}(e^{-z}+e^{z}) \\
&= \frac{1}{2}\mathrm{Re}\left(e^{-u}(\cos v - i\sin v) + e^{u}(\cos v + i\sin v)\right) \\
&= \cosh u \cos v.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\varphi^2(u,v) &= \frac{1}{2}\mathrm{Re}\int i(-e^{-z}-e^{z})dz \\
&= \frac{1}{2}\mathrm{Re}\left(i(e^{-z}-e^{z})\right) \\
&= \frac{1}{2}\mathrm{Re}\left(i\left(e^{-u}(\cos v - i\sin v) - e^{u}(\cos v + i\sin v)\right)\right) \\
&= \cosh u \sin v.
\end{aligned}$$

and

$$\varphi^3(u,v) = \mathrm{Re}\int 1dz = \mathrm{Re}z = u$$

So

$$\varphi(u,v) = (\cosh u\cos v, \cosh u \sin v, u)$$

which describes the catenoid (see HW no. 4, question 5).

2. The surface conjugate to the one from above is obtained by formally taking "$-$Im" instead of "Re" everywhere (this was pointed out above). This gives now

$$\varphi(u,v) = (-\sinh u\sin v, \sinh u\cos v, -v)$$

We change the variables (actually reparametrize the surface) by

$$\tilde{u} = \sinh u,\ \tilde{v} = v + \frac{\pi}{2},$$

and obtain

$$\tilde{\varphi}(u,v) = (\tilde{u}\cos\tilde{v}, \tilde{u}\sin\tilde{v}, -\tilde{v}+\frac{\pi}{2}).$$

This surface (strictly speaking, its trace) is the helicoid we saw in HW no. 4, question 5, up to the transformation of $\mathbb{R}^3$ described by

$$(x, y, z) \mapsto (x, y, -z + \frac{\pi}{2}),$$

which is downward (vertical) translation with $\frac{\pi}{2}$ followed by a reflection about the xy (horizontal) plane. Now we can understand better the result mentioned in HW no. 4, question 5, which says that the catenoid and the helicoid are locally isometric (that is, they have the same first fundamental forms). In fact, there is an isometric deformation (family of minimal surfaces $\{\varphi_\theta\}_\theta$) from the catenoid to the helicoid. This deformation is the one we can actually see in the wonderful animations we can see online and which I have indicated on the course home page.

3. Take

$$h(z) = 1, \text{ and } g(z) = z.$$

The corresponding minimal surface has

$$\begin{aligned}
\varphi^1(u, v) &= \frac{1}{2}\mathrm{Re}\int(1 - z^2)dz\frac{1}{2}\mathrm{Re}(z - \frac{z^3}{3}) = \frac{1}{2}(u - \frac{1}{3}u^3 + uv^2) \\
\varphi^2(u, v) &= \frac{1}{2}\mathrm{Re}\left(i\int(1 + z^2)dz\right) = \frac{1}{2}\mathrm{Re}\left(i(z + \frac{z^3}{3})\right) = \frac{1}{2}(-v + \frac{1}{3}v^3 - u^2v), \\
\varphi^3(u, v) &= \mathrm{Re}\int zdz = \frac{1}{2}\mathrm{Re}(z^2) = \frac{1}{2}(u^2 - v^2).
\end{aligned}$$

This is Enneper's surface (see Example 3 at the end of the previous section), up to the factor 1/2

Remarks. a) It is in general hard to decide if for given h and g the resulting minimal surface is with or without self-intersections (that is, φ is injective or not). We just note that only examples 1 and 2 (not 3) gave no self-intersecting surfaces. Minimal surfaces with no self-intersections are very rare, though. That's why the surface discovered by Costa in 1984 (see below) was a surprise for the specialists.

b) We can enlarge the class of examples described in Theorem 8.2.5 by allowing g to have poles (that is, to be meromorphic); but then we need to assume that if z_0 is a pole[5] of g of order k, then z_0 is a zero of h of order at least $2k$ — so that the functions which are integrated in Theorem 8.2.5 are still holomorphic.

Example 4 (Costa's minimal surface). We choose

$$f(z) = \wp(z), \ g(z) = \frac{2\sqrt{2\pi}e}{\wp'(z)}$$

where $\wp(z)$ is (a certain choice of) the Weierstrass elliptic function. Although it won't bring too much understanding[6], we just mention that, by definition, this is

$$\wp(z) = \frac{1}{z} + \sum_{m,n \text{ in } \mathbb{Z}, m^2+n^2\neq 0}\left(\frac{1}{(z-m-in)^2} - \frac{1}{(m+in)^2}\right)$$

One can see that it is a meromorphic function, with poles of order 2 at any point $m + in$, with m, n integers. We will not display the parametrization of the surface resulting via Theorem 8.2.5: it can be found in [Gr-Abb-Sa], Theorem 22.43. Figure 8 is intended to give you a (vague) idea of what the surface looks like: but if you want to see it better, go to `http://rsp.math.brandeis.edu/3d-xplormath/Surface/costa/costa_lg1.html` and

[5] Recall that z_0 is a pole of order k of the function f if it is a zero of order k of the function $1/f$.

[6] The curious reader is referred to [Gr-Abb-Sa, section 22.7].

drag the cursor to rotate the surface. As we already mentioned, this surface has no self-intersections.

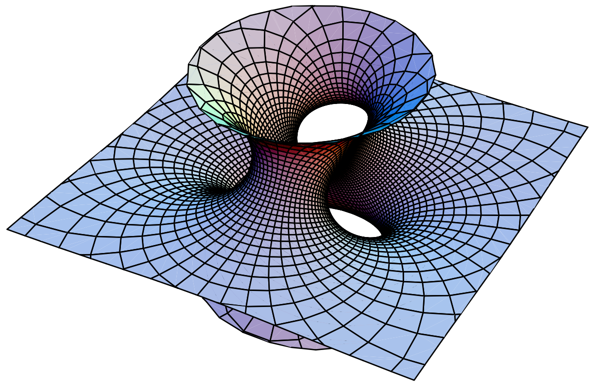

Figure 8. Costa's minimal surface.

Finally, I strongly recommend you to look at the examples of minimal surfaces displayed at

- `http://rsp.math.brandeis.edu/3d-xplormath/Surface/gallery_m.html`
- `http://www.indiana.edu/ minimal/gallery/index/index.html`

Observe that none of them is bounded. The explanation is very simple: they all arise from the process described in this section, so all three components of φ are harmonic functions; it is known that a harmonic function $\mathbb{R}^2 \to \mathbb{R}$ is not bounded.

References

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