

DIFFERENTIAL GEOMETRY OF CURVES AND SURFACES

1. CURVES IN THE PLANE

1.1. Points, Vectors, and Their Coordinates. Points and vectors are fundamental objects in Geometry. The notion of point is intuitive and clear to everyone. The notion of vector is a bit more delicate. In fact, rather than saying what a vector *is*, we prefer to say what a vector *has*, namely: direction, sense, and length (or magnitude). It can be represented by an arrow, and the main idea is that *two arrows represent the same vector if they have the same direction, sense, and length*. An arrow representing a vector has a tail and a tip. From the (rough) definition above, we deduce that in order to represent (if you want, to draw) a given vector as an arrow, it is necessary and sufficient to prescribe its tail.

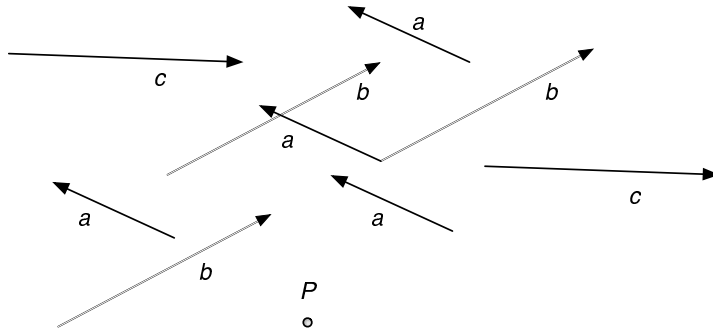


FIGURE 1. We see four copies of the vector a , three of the vector b , and two of the vector c . We also see a point P .

An important instrument in handling points, vectors, and (consequently) many other geometric objects is the Cartesian coordinate system in the plane. This consists of a point O , called the origin, and two perpendicular lines going through O , called coordinate axes. Each line has a positive direction, indicated by an arrow (see Figure 2). We denote by \mathbb{R} the

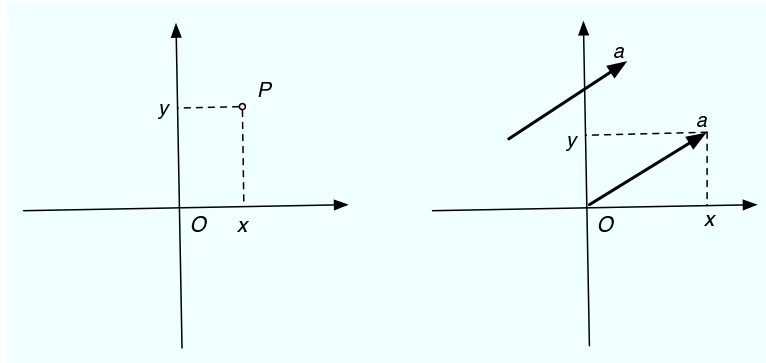


FIGURE 2. The point P has coordinates x, y . The vector a has also coordinates x, y .

set of all real numbers and by \mathbb{R}^2 the set of all *pairs* of numbers, of the form (x, y) , where x, y are in \mathbb{R} . Points are identified with elements \mathbb{R}^2 , as follows: to each point P corresponds the pair (x, y) consisting of the coordinates of the projections of P on the two axes. We say that P has coordinates (x, y) . Also vectors are identified with elements of \mathbb{R}^2 , as follows: if

a is a vector, we move its tail to the origin O , and we take the coordinates of its tip. We say that a has coordinates (x, y) . One of the first advantages of the coordinate system is that we can use it to compute lengths, as follows. The distance between the points $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ is

$$\|P_1P_2\| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

The length of the vector $a = (x, y)$ is

$$\|a\| = \sqrt{x^2 + y^2}.$$

In our course, \mathbb{R}^2 will denote both the set of all points and the set of all vectors (in the plane). It will always be possible to understand from the context if a certain object in \mathbb{R}^2 is a point or a vector.

1.2. Parametrized Curves. A good way of thinking of a curve is as the object which describes the motion of a particle in the plane: at the time t , the particle is at the point in the plane whose coordinates are $(x(t), y(t))$. We stress from the very beginning that what we are interested in is not simply *what* the trajectory of the particle is, but rather *how* the trajectory is run. Now comes the exact definition.

Definition 1.2.1. A parametrized curve in the plane is a differentiable function¹

$$\alpha(t) = (x(t), y(t)),$$

where t satisfies $a < t < b$ (possibly a and/or b can be ∞).

When we say that $\alpha(t) = (x(t), y(t))$ is “differentiable” we mean that both $x(t)$ and $y(t)$ have derivative of any order (we also say that they are C^∞ differentiable).

The standard notation for such an object is

$$\alpha : (a, b) \rightarrow \mathbb{R}^2,$$

where (a, b) is the open interval between a and b . As we already mentioned, it is important to distinguish between the curve α (the assignment which associates to any “time” t the point $(x(t), y(t))$ on the “trajectory”) and the image of the function α (the “trajectory”). The latter is called the trace of the curve α .

Remark. Very often it is possible to describe the trace of a curve $\alpha : (a, b) \rightarrow \mathbb{R}^2$ with $\alpha(t) = (x(t), y(t))$, by an equation of the form $f(x, y) = 0$, where f is a function of variables x, y . If so, we say that

$$x = x(t), y = y(t)$$

are the parametric equations (or explicit equations) of the curve and

$$f(x, y) = 0$$

is the implicit equation of the curve. For example, the trace of the curve

$$x = 1 - 2t, y = 5t$$

is a straight line (see below). The same line can be described by the implicit equation

$$5x + 2y = 5.$$

Also, the circle of radius 1 and centre 0 can be described as the trace of

$$x = \cos t, y = \sin t$$

¹It is worth mentioning that the domain of the function α is the interval $a < t < b$ and the range is \mathbb{R}^2 .

but also by the equation

$$x^2 + y^2 = 1.$$

Examples. 1. The straight line determined by two points P and Q is the trace of the curve

$$\alpha(t) = tP + (1 - t)Q,$$

where $t \in \mathbb{R}$ (see Figure 3). Note that there are several other curves whose traces are the same straight line, like for instance

$$\beta(t) = tQ + (1 - t)P$$

or

$$\gamma(t) = 2tP + (1 - 2t)Q.$$

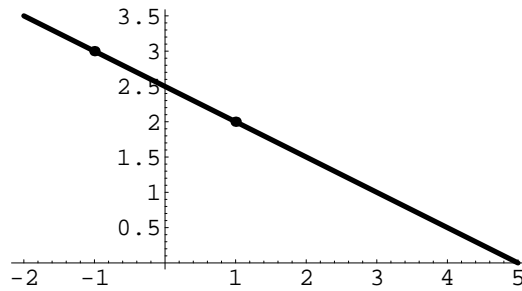


FIGURE 3. The straight line determined by the two indicated points.

2. The circle of center $C = (x_0, y_0)$ and radius r is the trace of the curve

$$\alpha(t) = (x_0 + r \cos(t), y_0 + r \sin(t)),$$

for all t in \mathbb{R} (see Figure 4). It can be described implicitly as

$$(x - x_0)^2 + (y - y_0)^2 = r^2$$

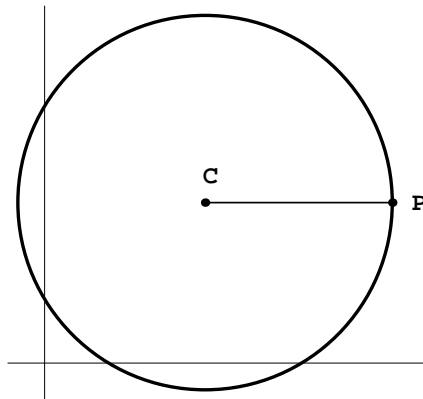


FIGURE 4. A circle of center C .

3. The ellipse is the trace of the curve

$$\alpha(t) = (a \cos(t), b \sin(t)),$$

for all t in \mathbb{R} (see Figure 5). Here a, b are positive numbers. It can be described implicitly as

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

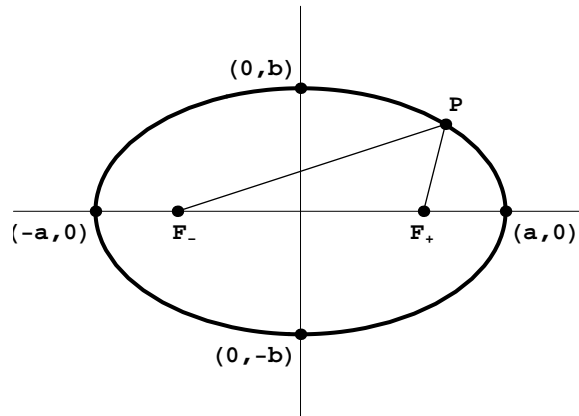


FIGURE 5. The ellipse with its two foci F_- and F_+ .

We have the following geometric (coordinate free) characterization of the ellipse. There exists two points, let's denote them F_- and F_+ , which are called the foci of the ellipse, with the property that for any point P on the ellipse we have

$$\|PF_-\| + \|PF_+\| = \text{constant}.$$

More specifically, F_- has coordinates $(-\sqrt{a^2 - b^2}, 0)$, and F_+ has coordinates $(\sqrt{a^2 - b^2}, 0)$ (you are encouraged to calculate the distances $\|PF_-\|$, $\|PF_+\|$, add them up, and check that the result is constant, that is, independent of t).

4. The parabola is the trace of the curve

$$\alpha(t) = (t, at^2),$$

where a is a number. It can be described implicitly as

$$y = ax^2.$$

Again we have a geometric (coordinate free) characterization. Namely, there exists a point

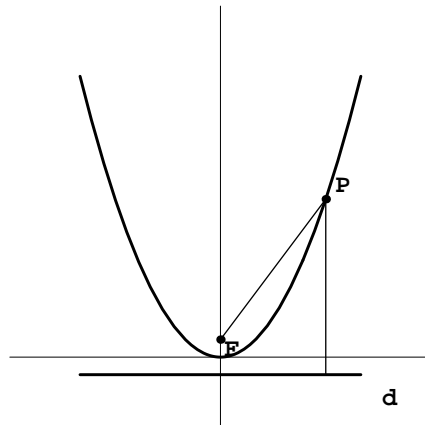


FIGURE 6. The parabola with its focus F and directrix line d .

F , called the *focus*, and a straight line d , called the *directrix*, such that for any point P on the parabola, the distance between P and F is equal to the distance between P and d . For instance, for $\alpha(t) = (t, t^2)$, the focus is $(0, 1/4)$ and the directrix is the horizontal line of equation $y = -1/4$ (check this!).

1.3. Velocity, reparametrizations, and the length of a curve. We start with the following definition.

Definition 1.3.1. If $\alpha(t) = (x(t), y(t))$, with $a < t < b$ is a parametric curve, then the velocity of α at a given t is the vector

$$\alpha'(t) = (x'(t), y'(t)).$$

The speed of α at t is the number $\|\alpha'(t)\|$.

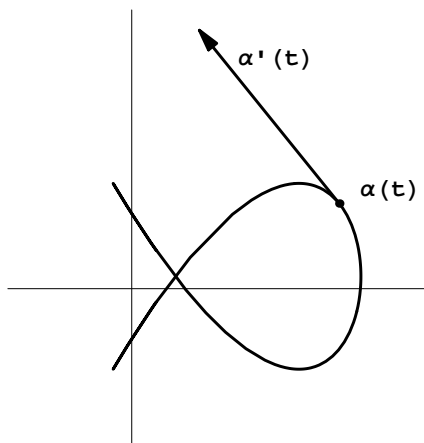


FIGURE 7. The vector $\alpha'(t)$ is tangent to the curve at the point $\alpha(t)$.

The velocity vector is useful because it is tangent to the curve, so it can be used to describe the *tangent line* to the curve (we will get back to this later on). Another reason why the tangent vector is important is that it can be used to compute the length of a piece of a curve. Namely, we have the following definition.

Definition 1.3.2. Let $\alpha(t)$, $a < t < b$ be a parametrized curve and let $[c, d]$ be a closed interval with $a < c < d < b$. The length of α over the interval $[c, d]$ (or from $\alpha(c)$ to $\alpha(d)$) is

$$\ell(\alpha|_{[c,d]}) = \int_c^d \|\alpha'(t)\| dt.$$

To get the idea of this formula, we just say that if we divide the interval $[c, d]$ into n equal pieces given by $c = t_0 < t_1 < \dots < t_{n-1} < t_n = d$, then

$$\ell(\alpha|_{[c,d]}) \approx \sum_{i=1}^n \|\alpha(t_i) - \alpha(t_{i-1})\|,$$

where the right hand side represents the length of the polygonal line in Figure 10. For each i we can approximate

$$\frac{\alpha(t_i) - \alpha(t_{i-1})}{t_i - t_{i-1}} \approx \alpha'(t_{i-1}).$$

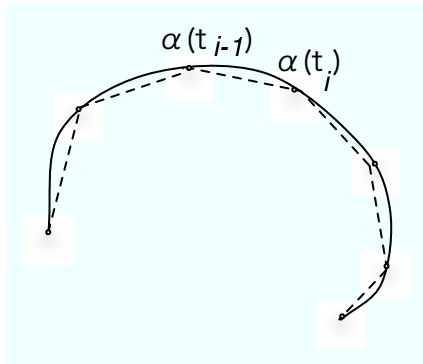


FIGURE 8. The length of the line segment joining $\alpha(t_{i-1})$ and $\alpha(t_i)$ is $\|\alpha(t_i) - \alpha(t_{i-1})\|$.

This is because, by definition, we have

$$\alpha'(t_{i-1}) = \lim_{t \rightarrow t_{i-1}} \frac{\alpha(t) - \alpha(t_{i-1})}{t - t_{i-1}}$$

and t_i is “close” to t_{i-1} . We obtain

$$\ell(\alpha|_{[c,d]}) \approx \sum_{i=1}^n \|\alpha'(t_{i-1})\| \cdot |t_i - t_{i-1}|.$$

In fact the actual length of α over $[c, d]$ is the limit as $n \rightarrow \infty$ of the right hand side. But the latter expression is a Riemann sum of the function $\|\alpha'(t)\|$. The formula given by Definition 1.3.2 follows.

Now we get back to the idea mentioned in section 1.2, example 1. Namely, we said that there are several curves whose trace is a given straight line. In other words, it is possible to parametrize a straight line in several ways. This is a general phenomenon: *any* curve can be reparametrized, in a sense made precise by the following definition.

Definition 1.3.3. Let $\alpha_1 : (a_1, b_1) \rightarrow \mathbb{R}^2$ be a parametrized curve. We say that the curve $\alpha_2 : (a_2, b_2) \rightarrow \mathbb{R}^2$ is a reparametrization of α_1 if there exists $h : (a_1, b_1) \rightarrow (a_2, b_2)$ a smooth function which is differentiable and strictly monotone² such that $\alpha_1(t) = \alpha_2(h(t))$, for all t in (a_1, b_1) .

Note that in the situation described by the definition, the curves α_1 and α_2 have the same trace.

Now the problem is if the length of the curve as described by Definition 1.3.1 depends on the parametrization (of course, it shouldn't, because the length should depend only on the trace of the curve). For example, we can parametrize the half circle in the figure as

$$\alpha_1(t) = (\cos(t), \sin(t)), \quad 0 \leq t \leq \pi$$

but also as³

$$\alpha_2(u) = (\cos(2u), \sin(2u)), \quad 0 \leq u \leq \frac{\pi}{2}.$$

Is $\ell(\alpha_1|_{[0,\pi]})$ the same as $\ell(\alpha_2|_{[0,\frac{\pi}{2}]})$ (and actually equal to π)?

The following theorem shows that the length does not depend on the parametrization.

²That is, h is strictly increasing or strictly decreasing.

³What is h ?

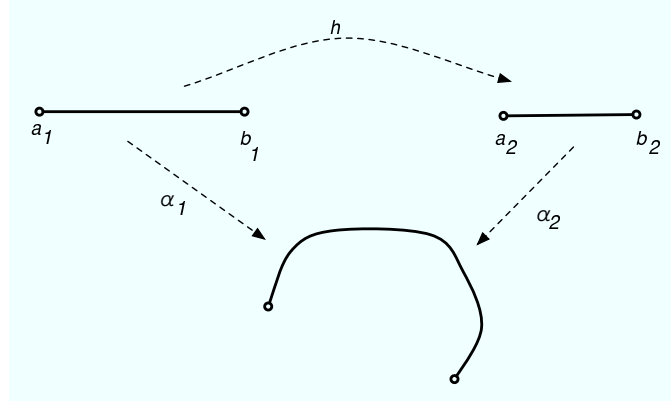


FIGURE 9. The diagram indicates the reparametrization given by $h : (a_1, b_1) \rightarrow (a_2, b_2)$. The little circles indicate that the points indicated do not belong to the curve (line segment).

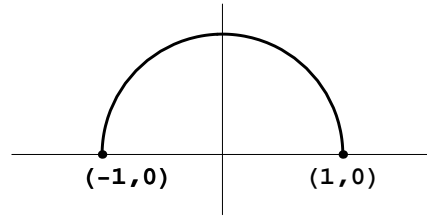


FIGURE 10. The half circle which is at the same time the trace of α_1 and α_2 .

Theorem 1.3.4. *Assume that $\alpha_2 : (a_2, b_2) \rightarrow \mathbb{R}^2$ is a reparametrization of $\alpha_1 : (a_1, b_1) \rightarrow \mathbb{R}^2$ given by $h : (a_1, b_1) \rightarrow (a_2, b_2)$. Take c_1, d_1, c_2, d_2 such that $a_1 < c_1 < d_1 < b_1$ and $a_2 < c_2 < d_2 < b_2$ such that h maps $[c_1, d_1]$ to $[c_2, d_2]$. Then we have*

$$\ell(\alpha_1|_{[c_1, d_1]}) = \ell(\alpha_2|_{[c_2, d_2]}).$$

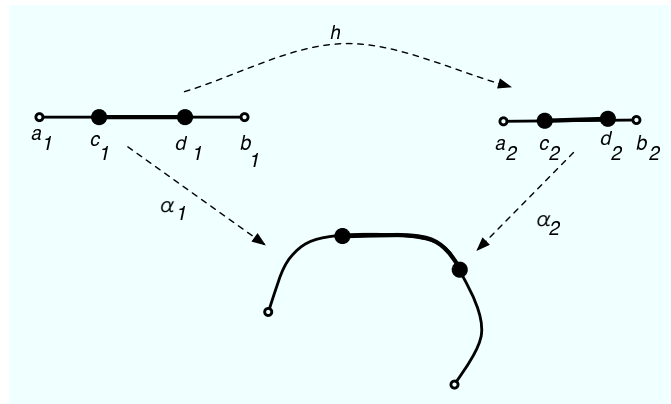


FIGURE 11. The thickened piece of the curve is parametrized in two ways. Is Definition 1.3.1 applied to α_1 and α_2 giving the same result for the length?

Proof. The situation described in the theorem is represented in Figure 11. We have to check that

$$(1) \quad \int_{c_1}^{d_1} \|\alpha'_1(t)\| dt = \int_{c_2}^{d_2} \|\alpha'_2(u)\| du.$$

Let us assume that $h(t)$ is increasing (do the case $h(t)$ strictly decreasing!). We start with the right hand side of (1), where we make the change of variable $u = h(t) \Rightarrow du = h'(t)dt$. Because $h(c_1) = c_2$ and $h(d_1) = d_2$, we have

$$\int_{c_2}^{d_2} \|\alpha'_2(u)\| du = \int_{c_1}^{d_1} \|\alpha'_2(h(t))\| h'(t) dt = \int_{c_1}^{d_1} \|\alpha'_2(h(t))h'(t)\| dt = \int_{c_1}^{d_1} \|(\alpha_2(h(t)))'\| dt,$$

which is just the left hand side of the desired equation (1) (note that on the last step we have used the chain rule for derivatives). \square

From now on we will restrict ourselves to parametric curves with nonzero velocity vectors, which are called regular.

Definition 1.3.5. A curve $\alpha : (a, b) \rightarrow \mathbb{R}^2$ is called regular if $\alpha'(t) \neq 0$, for all t in the interval (a, b) .

Such curves have a convenient parametrization, where the parameter is just the length of the curve. Let us be more precise. We fix a number c with $a < c < b$. We define the arc length function $s(t)$ given by

$$s(t) = \int_c^t \|\alpha'(u)\| du = \begin{cases} -\ell(\alpha|_{[t,c]}), & \text{if } a < t \leq c \\ \ell(\alpha|_{[c,t]}), & \text{if } c \leq t < b \end{cases}$$

From the fundamental theorem of calculus, we have

$$(2) \quad \frac{d}{dt}s(t) = \|\alpha'(t)\|,$$

for all t in (a, b) (as $\alpha'(t) \neq 0$), so the function $s(t)$ is strictly increasing. Denote by $t(s)$ the inverse of this function. Consider the curve

$$\beta(s) = \alpha(t(s)),$$

which is a reparametrization of α . We call it a parametrization by arc length. After a curve has been reparametrized in the way described above, we say that it is parametrized by arc length. We have just shown that

Theorem 1.3.6. *Any regular curve can be parametrized by arc length.*

The advantages of the parametrization by arc length are as follows.

Proposition 1.3.7. *Assume that the curve $\beta = \beta(s)$ is parametrized by arc length. Then we have the following two properties.*

(a) *the length of the velocity vector⁴ $\beta'(s)$ is*

$$\|\beta'(s)\| = 1$$

(b) *$\ell(\beta|_{[s_1, s_2]}) = s_2 - s_1$, for all $s_1 < s_2$.*

Proof. (a) We use the fact that $\beta(s(t)) = \alpha(t)$. Using the chain rule we obtain

$$\alpha'(t) = \frac{d}{dt}\alpha(t) = \frac{d}{ds}\beta(s) \cdot \frac{d}{dt}s(t) = \frac{d}{ds}\beta(s) \cdot \|\alpha'(t)\|,$$

⁴By β' we mean $\frac{d}{ds}\beta$.

where we have used equation (2). We deduce

$$\|\alpha'(t)\| = \left\| \frac{d}{dt} \alpha(t) \right\| = \left\| \frac{d}{ds} \beta(s) \cdot \|\alpha'(t)\| \right\| = \left\| \frac{d}{ds} \beta(s) \right\| \cdot \|\alpha'(t)\|,$$

which implies the desired relation.

(b) We have

$$\ell(\beta|_{s_1, s_2}) = \int_{s_1}^{s_2} \|\beta'(s)\| ds = \int_{s_1}^{s_2} 1 ds = s_2 - s_1.$$

□

Note. The following converse of point (a) above is true. Assume that $\alpha = \alpha(t)$ is a curve, $a < t < b$, such that $\|\alpha'(t)\| = 1$, for all t (such curves are called *unit-speed* curves). To parametrize it by arc length, pick c with $a < c < b$ and calculate

$$s = \int_c^t \|\alpha'(u)\| du = \int_c^t 1 du = t - c.$$

This gives $t = s + c$. So the curve α is already parametrized by arc length, up to the additive constant c . Modulo this ambiguity, we say that a curve is unit-speed if and only if it is parametrized by arc length.

Example. Let's parametrize by arc length the logarithmic spiral

$$\alpha(t) = (ae^{bt} \cos t, ae^{bt} \sin t),$$

where a, b are positive numbers (see Figure 12). To simplify things we make $a = 1$. We have

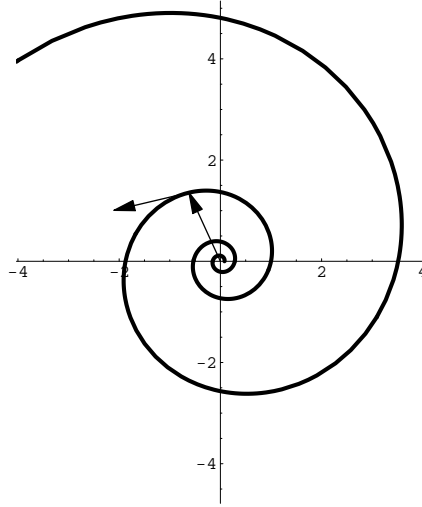


FIGURE 12. The logarithmic spiral is characterized by the fact that at any point on it the angle between the position vector $\alpha(t)$ and the tangent vector $\alpha'(t)$ is the same.

$$\alpha'(t) = (be^{bt} \cos(t) - e^{bt} \sin(t), be^{bt} \sin(t) + e^{bt} \cos(t))$$

which implies

$$\|\alpha'(t)\| = \sqrt{e^{2bt}[(b \cos t - \sin t)^2 + (b \sin t + \cos t)^2]} = e^{bt} \sqrt{1 + b^2}.$$

Then

$$s(t) = \int_0^t e^{bu} \sqrt{1+b^2} du = \frac{\sqrt{1+b^2}}{b} (e^{bt} - 1).$$

To determine $t = t(s)$ we solve

$$s = \frac{\sqrt{1+b^2}}{b} (e^{bt} - 1)$$

with respect to t . We obtain

$$t = \frac{1}{b} \ln \left[\frac{bs}{\sqrt{1+b^2}} + 1 \right].$$

The parametrization by arc length is $\beta(s) = \alpha(t(s))$. To simplify things even more, we make $b = 1$. We obtain

$$\beta(s) = \left(\left(\frac{s}{\sqrt{2}} + 1 \right) \cos \ln \left(\frac{s}{\sqrt{2}} + 1 \right), \left(\frac{s}{\sqrt{2}} + 1 \right) \sin \ln \left(\frac{s}{\sqrt{2}} + 1 \right) \right).$$

1.4. Curvature; the fundamental theorem of plane curves. We would like to measure how “curved” is a curve at a certain point (see Figure 13). In other words, we need an object, called “curvature” of a curve, which measures this. For instance, one would expect the “curvature” of a straight line to be zero and also that the curvature of a circle at any of its points to be the same (the circle is equally “curved” at any of its points).

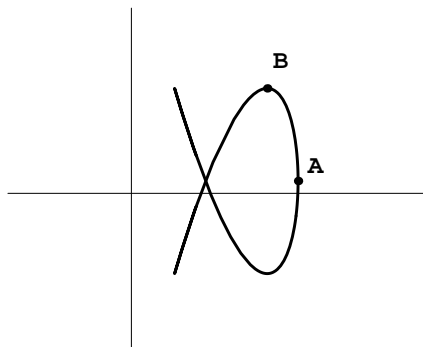


FIGURE 13. The curve in the figure is more “curved” at the point B than at A .

Definition 1.4.1. Let $\alpha(t) = (x(t), y(t))$, $a < t < b$ be a regular curve. The curvature of α at t_0 is⁵

$$(3) \quad \kappa_p(t_0) = \frac{x'(t_0)y''(t_0) - x''(t_0)y'(t_0)}{(x'(t_0)^2 + y'(t_0)^2)^{\frac{3}{2}}}.$$

It is at this moment not clear at all why κ_p given by the strange formula above measures how curved is the curve α . We will address this point later (at the end of this section). Another important question is whether the curvature at a point depends on the parametrization. For instance, if we consider the parametrizations of the unit circle given by

$$\alpha_1(t) = (\cos(t), \sin(t)), \quad \alpha_2(u) = (\cos(2u), \sin(2u))$$

is the curvature of α_1 at, say, $t = \pi/4$ the same as the curvature of α_2 at $t = \pi/8$? The answer is given by the following proposition.

⁵The subscript p stands for “plane”. We will define the curvature κ for curves in the three dimensional space, and that will be slightly different from κ_p . The latter is also called the *signed* curvature, because it can be positive or negative.

Theorem 1.4.2. Assume that $\alpha_2 : (a_2, b_2) \rightarrow \mathbb{R}^2$ is a reparametrization of $\alpha_1 : (a_1, b_1) \rightarrow \mathbb{R}^2$ given by $h : (a_1, b_1) \rightarrow (a_2, b_2)$. Then the curvature of α_1 at t equals the curvature of α_2 at $h(t)$ up to a sign.

Proof. If we denote $\alpha_2(u) = (x(u), y(u))$, then

$$\alpha_1(t) = \alpha_2(h(t)) = (x(h(t)), y(h(t))).$$

We have

$$\frac{d}{dt}x(h(t)) = x'(h(t))h'(t), \quad \frac{d^2}{dt^2}x(h(t)) = x''(h(t))(h'(t))^2 + x'(h(t))h''(t),$$

and similar formulas for y . The curvature of α_1 at t is

$$\frac{x'(h(t))h'(t)(y''(h(t))(h'(t))^2 + y'(h(t))h''(t)) - y'(h(t))h'(t)(x''(h(t))(h'(t))^2 + x'(h(t))h''(t))}{(x'(h(t))^2h'(t)^2 + y'(h(t))^2h'(t)^2)^{\frac{3}{2}}}.$$

This gives

$$\pm \frac{x'(h(t))y''(h(t)) - x''(h(t))y'(h(t))}{(x'(h(t))^2 + y'(h(t))^2)^{\frac{3}{2}}}$$

with a plus sign if $h'(t) > 0$ and minus sign if $h'(t) < 0$.

Now we do a few simple examples and see if κ_p given by (3) responds to our expectations.

Examples. 1. We consider the straight line given by $\alpha(t) = tP + (1-t)Q$, where P, Q are points (see section 1.2). Take $P = (x_1, y_1)$ and $Q = (x_2, y_2)$, then $\alpha(t) = (x(t), y(t))$, where

$$x(t) = tx_1 + (1-t)x_2, \quad y(t) = ty_1 + (1-t)y_2.$$

We can easily see that the formula (3) gives $\kappa_p(t) = 0$ for any t (because $x''(t) = y''(t) = 0$).

2. Let us consider now the circle described by (see again section 1.2)

$$x(t) = x_0 + r \cos(t), \quad y(t) = y_0 + r \sin(t),$$

where t is in \mathbb{R} . We have

$$x'(t) = -r \sin(t), \quad x''(t) = -r \cos(t), \quad y'(t) = r \cos(t), \quad y''(t) = -r \sin(t)$$

which gives

$$\kappa_p(t) = \frac{1}{r}.$$

This is independent of t , that is, the circle is indeed equally curved at any of its points. Also note that this also corresponds to the following intuitive fact: circles with larger radius are less curved than circles with smaller one (for instance, if you go around the Earth along the equator, you don't notice that you travel along a circle, but rather along a straight line, simply because the radius of the Earth at the equator is huge).

3. *Exercise.* Find the curvature of the ellipse at an arbitrary point (see section 1.2, Example 3) and check that if $a > b$ then the ellipse is more curved at $(a, 0)$ than at $(0, b)$.

We get a simpler and more natural description of κ_p if the curve is parametrized by arc length. So let us assume that α is a curve which is (if you want, has been) parametrized by arc length (see Theorem 1.3.6). We denote

$$T(s) = \alpha'(s),$$

which, by Proposition 1.3.7, satisfies

$$\|T(s)\| = 1.$$

For each s we choose the vector $N(s)$ which is uniquely determined by the following three properties:

- $N(s)$ is perpendicular to $T(s)$
- $\|N(s)\| = 1$
- we can bring the pair $(T(s), N(s))$ to (e_1, e_2) by a rotation, where e_1 and e_2 are the vectors of length one pointing in the positive direction on the coordinate axes (see Figure 14).

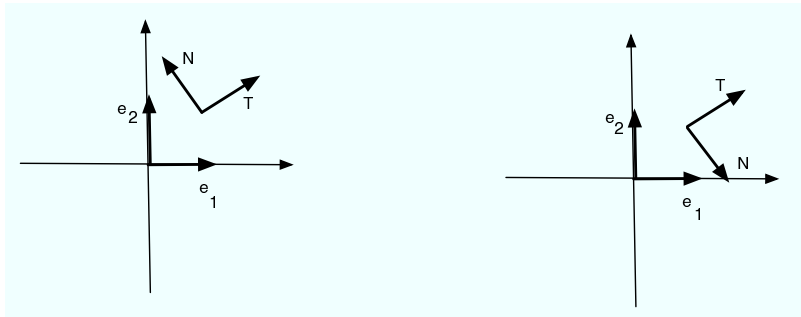


FIGURE 14. In the left-hand side figure, the choice of N is correct, whereas in the right-hand side it's wrong.

In fact we can define $N(s)$ simpler, by saying that it is obtained from $T(s)$ by a 90° rotation in the counterclockwise direction. That is,

$$N(s) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x'(s) \\ y'(s) \end{pmatrix} = \begin{pmatrix} -y'(s) \\ x'(s) \end{pmatrix}.$$

Theorem 1.4.3. (The Frenet Formulas for Plane Curves) *If the curve α is parametrized by arc length and T, N are as above, then we have⁶*

$$\begin{aligned} T'(s) &= \kappa_p(s)N(s), \\ N'(s) &= -\kappa_p(s)T(s) \end{aligned}$$

for all s .

Proof. We can see that

$$T(s) = (x'(s), y'(s)) \Rightarrow T'(s) = (x''(s), y''(s)).$$

Also, from the definition of $N(s)$, we see that this vector is actually obtained from $T(s)$ by a 90° rotation in the counterclockwise direction. This gives

$$N(s) = (-y'(s), x'(s)).$$

From $\|T(s)\| = 1$ we deduce

$$\|T(s)\|^2 = T(s) \cdot T(s) = 1,$$

where \cdot indicates the scalar product of two vectors. We differentiate and obtain

$$T'(s) \cdot T(s) = 0,$$

which says that $T'(s)$ is perpendicular to $T(s)$. Consequently, $T'(s)$ is collinear with $N(s)$, so there exists a number λ with

$$T'(s) = \lambda N(s).$$

This equation gives

$$x''(s) = -\lambda y'(s), \quad y''(s) = \lambda x'(s).$$

⁶The derivative is with respect to s .

We multiply the first equation by $-y'(s)$ and the second one by $x'(s)$ and obtain

$$-x''(s)y'(s) + x'(s)y''(s) = \lambda(x'(s)^2 + y'(s)^2).$$

Because $x'(s)^2 + y'(s)^2 = 1$, we deduce that $\lambda = \kappa_p(s)$, QED. \square

A straightforward consequence of this theorem is as follows (note that it's possible for κ_p to be negative).

Corollary 1.4.4. *If the curve α is parametrized by arc length, then the absolute value of the curvature is given by*

$$|\kappa_p(s)| = \|\alpha''(s)\|.$$

The following proposition gives a geometric characterization of the curvature.

Proposition 1.4.5. *If the curve α is parametrized by arc length, then the curvature is given by*

$$\kappa_p(s) = \frac{d}{ds}\theta(s),$$

where $\theta(s)$ is the angle between Ox and $\alpha'(s)$.

Proof. Since $\|\alpha'(s)\| = 1$, we deduce (see also Figure 15) that the coordinates of $\alpha'(s)$ are

$$\cos \theta(s) = x'(s), \quad \sin \theta(s) = y'(s).$$

This implies that

$$\kappa_p(s) = x'(s)y''(s) - x''(s)y'(s) = (\cos \theta(s))^2\theta'(s) + (\sin \theta(s))^2\theta'(s) = \theta'(s),$$

which is exactly the desired equation. \square

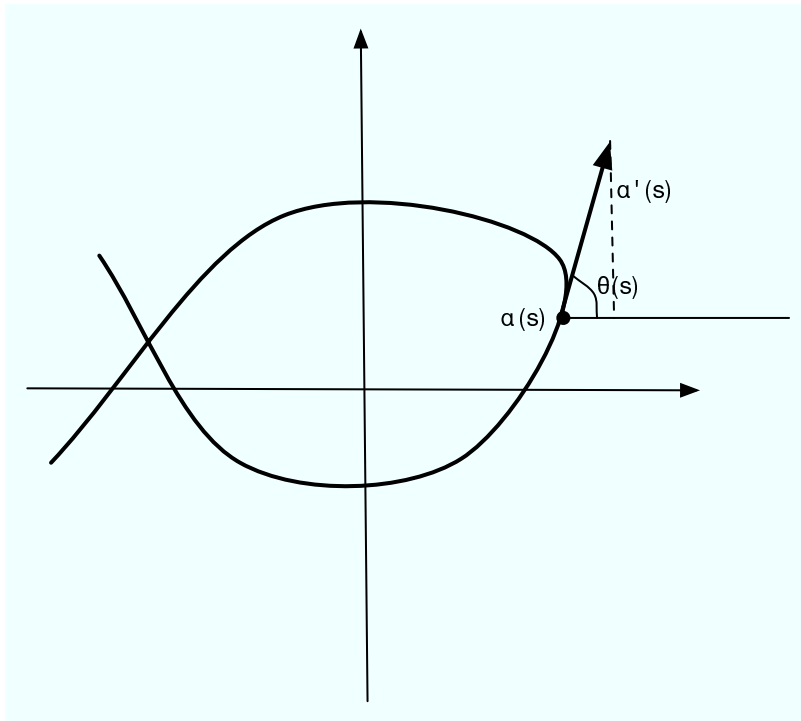


FIGURE 15. The angle between $\alpha'(s)$ and the Ox axis is $\theta(s)$.

Note. The angle θ is called the *turning angle* of the curve $\alpha : (a, b) \rightarrow \mathbb{R}^2$. It is actually a differentiable function $\theta : (a, b) \rightarrow \mathbb{R}$. Its exact definition is given in Section 1.5 of [Gr-Ab-Sa].

The following theorem shows that a curve is uniquely determined (up to rigid motions) by its curvature.

Theorem 1.4.6. (Fundamental Theorem of Plane Curves) *If $\kappa : (a, b) \rightarrow \mathbb{R}$ is an arbitrary differentiable function, then there exists a curve α parametrized by arc length such that the curvature of α is*

$$\kappa_p(s) = \kappa(s).$$

If $\beta : (a, b) \rightarrow \mathbb{R}^2$ is another curve parametrized by arc length with curvature equal to κ , then β differs from α by a translation followed by a rotation. By this we mean that we have

$$\beta(s) = A(\alpha(s)) + X,$$

where A is a matrix of the form

$$A = \begin{pmatrix} \cos \theta_0 & -\sin \theta_0 \\ \sin \theta_0 & \cos \theta_0 \end{pmatrix}$$

which gives the rotation with angle θ_0 and X is a vector of the form

$$X = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

which gives the translation.

Proof. For simplicity, assume that 0 is in the interval (a, b) . From Proposition 1.4.5. we deduce that if a curve has curvature κ then the angle $\theta(s)$ with the Ox axis is given by

$$\theta(s) = \theta_0 + \bar{\theta}(s),$$

where we have denoted

$$\bar{\theta}(s) := \int_0^s \kappa(u) du.$$

Because $x'(s) = \cos \theta(s)$, we deduce that

$$x(s) = x_0 + \int_0^s \cos(\theta_0 + \bar{\theta}(u)) du = x_0 + \cos(\theta_0) \int_0^s \cos \bar{\theta}(u) du - \sin(\theta_0) \int_0^s \sin \bar{\theta}(u) du,$$

and similarly,

$$y(s) = y_0 + \int_0^s \sin(\theta_0 + \bar{\theta}(u)) du = y_0 + \sin(\theta_0) \int_0^s \cos \bar{\theta}(u) du + \cos(\theta_0) \int_0^s \sin \bar{\theta}(u) du.$$

We deduce immediately that

$$\alpha(s) = \left(\int_0^s \cos \bar{\theta}(u) du, \int_0^s \sin \bar{\theta}(u) du \right)$$

is a curve of curvature κ and any curve β with this property differs from α by the rotation of angle θ_0 and the translation given by (x_0, y_0) . \square

1.5. Tangent line, normal line, osculating circle, and evolutes.

We begin with the following definition (see also Figure 16).

Definition 1.5.1. The tangent line to the regular curve $\alpha : (a, b) \rightarrow \mathbb{R}^2$ at $\alpha(t)$ is the straight line which goes through the point $\alpha(t)$ and is parallel to the vector $\alpha'(t)$. The normal line to α at $\alpha(t)$ is the straight line which goes through the point $\alpha(t)$ and is perpendicular to the tangent line.

The tangent line is the one that “approximates the best” a curve at a given point. For instance, if we take again the example of someone who travels around the Earth along the equator, we said already that she has at any moment the impression that she goes along a

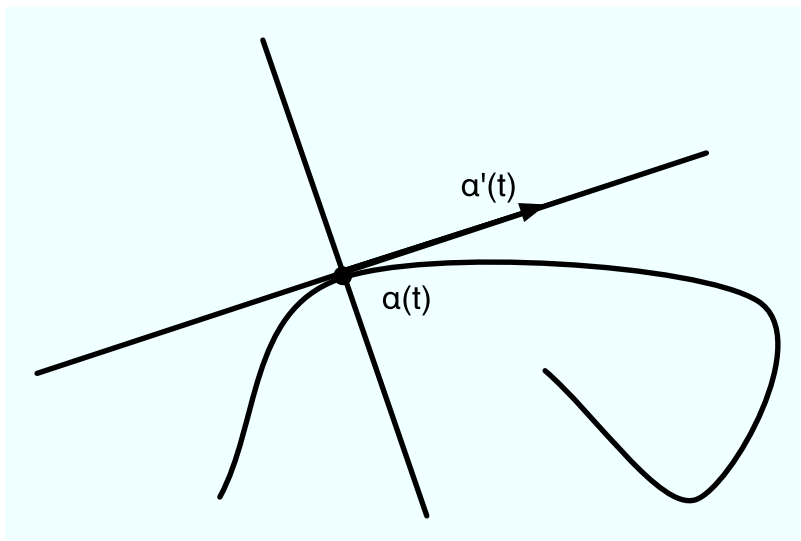


FIGURE 16. The two straight lines are the tangent, respectively normal lines at $\alpha(t)$ to the curve.

straight line, *namely the tangent line*. Now instead of straight lines, we will be trying to approximate a curve by a circle.

Definition 1.5.2. The *osculating circle* to the regular curve $\alpha : (a, b) \rightarrow \mathbb{R}^2$ with $\alpha(t) = (x(t), y(t))$ at the point $\alpha(t)$ is the circle of radius⁷ $\frac{1}{|\kappa_p(t)|}$ and center of coordinates

$$(4) \quad (\bar{x}(t), \bar{y}(t)) := (x(t), y(t)) + \frac{1}{\kappa_p(t)\|\alpha'(t)\|}(-y'(t), x'(t)).$$

The latter point is called the *curvature centre* of α at $\alpha(t)$.

The formulas look messy, but they give the answer to the following approximation question.

Question. Given a point $\alpha(t)$ on the curve, which is the circle going through it and giving “the best” approximation of the curve at $\alpha(t)$?

By “best approximation” of α by a circle we mean the following three things:

- (i) the circle goes through $\alpha(t)$
- (ii) the tangent lines to the circle and the curve α at $\alpha(t)$ are the same
- (iii) the curvature of α at t equals the curvature of the circle, possibly up to a negative sign.

Exercise. Check that the osculating circle given by Definition 1.5.2 satisfies the conditions (i),(ii), and (iii).

It is worth noticing that the conditions (i),(ii), and (iii) do not determine the osculating circle uniquely: there are two possibilities, as one can see in Figure 17. The osculating circle is the one on the *concave* side of the curve, namely the side where the *acceleration* vector $\alpha''(t)$ points. To prove this fact, we only need to notice that the vector $\frac{1}{\kappa_p(t)\|\alpha'(t)\|}(-y'(t), x'(t))$ is perpendicular to the tangent line at $\alpha(t)$ and its angle with $\alpha''(t) = (x''(t), y''(t))$ is acute (as the dot product with the latter vector is positive, in view of Definition 1.4.1. of $\kappa_p(t)$).

⁷If $\kappa_p(t) = 0$, the radius becomes ∞ and the circle becomes a straight line.

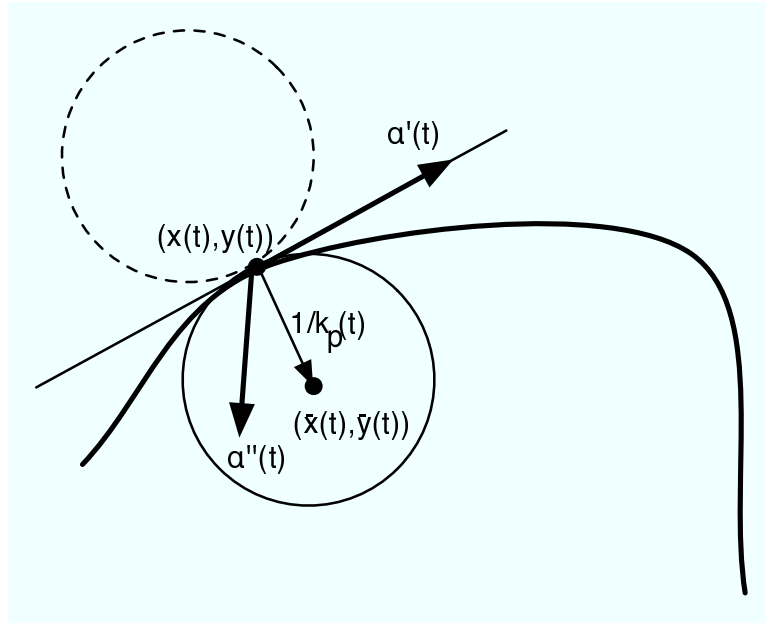


FIGURE 17. Both circles in the figure approximate the curve at $\alpha(t) = (x(t), y(t))$ (in the sense of (i), (ii), and (iii)), but only one of them is the osculating circle. The vector with tail at $\alpha(t)$ and tip at the center of the osculating circle is $\frac{1}{\kappa_p(t)\|\alpha'(t)\|}(-y'(t), x'(t))$.

Let us consider the locus in the plane described by all curvature centres $(\bar{x}(t), \bar{y}(t))$ (see equation (4)). We obtain a curve $\bar{\alpha} : (a, b) \rightarrow \mathbb{R}^2$ given by

$$(5) \quad \bar{\alpha}(t) = (\bar{x}(t), \bar{y}(t)).$$

The curve $\bar{\alpha}$ is called the *evolute* of α .

Exercise. Show that the evolute of the ellipse

$$\alpha(t) = (a \cos t, b \sin(t))$$

is the curve of equation

$$\bar{\alpha}(t) = \left(\frac{(a^2 - b^2) \cos^3 t}{a}, \frac{(b^2 - a^2) \sin^3 t}{b} \right).$$

The latter curve is an *astroid*, see Figure 19.

The evolute has the following interesting property.

Proposition 1.5.3. *Assume that α is a curve parametrized by arc length and let $\bar{\alpha}$ be its evolute. Then the tangent line to $\bar{\alpha}$ at $\bar{\alpha}(s)$ coincides with the normal line to α at $\alpha(s)$, for any s .*

Proof. We use the notation $T(s) = \alpha'(s) = (x'(s), y'(s))$, $N(s) = (-y'(s), x'(s))$, like in section 1.4. Then by (5) and (4) we can write

$$\bar{\alpha}(s) = \alpha(s) + \frac{1}{\kappa_p(s)} N(s).$$

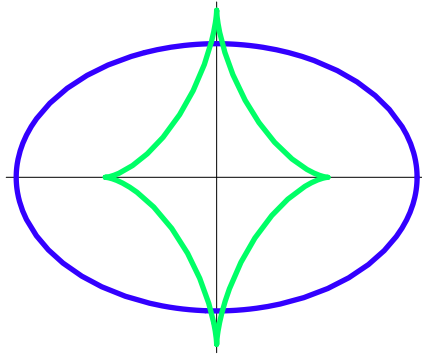


FIGURE 18. The ellipse and its evolute.

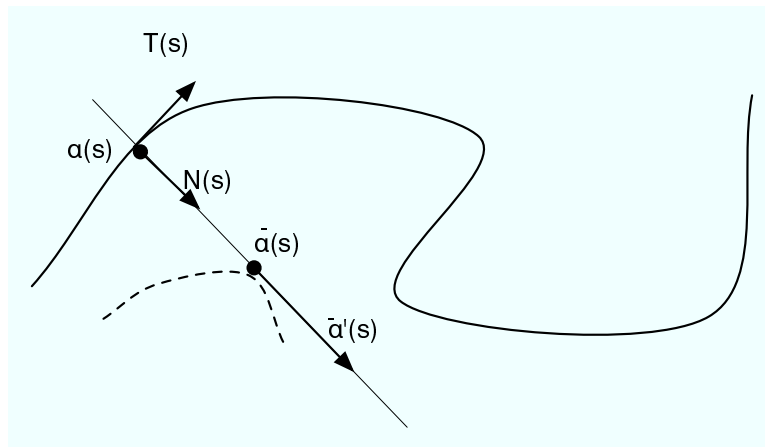
This implies

$$(6) \quad \bar{\alpha}'(s) = \alpha'(s) - \frac{d}{ds} \left(\frac{1}{\kappa_p(s)} \right) N(s) + \frac{1}{\kappa_p(s)} N'(s).$$

Because $N'(s) = -\kappa_p(s)T(s)$ (see Theorem 1.4.3.), we deduce that

$$\bar{\alpha}'(s) = -\frac{d}{ds} \left(\frac{1}{\kappa_p(s)} \right) N(s),$$

which implies that the vectors $\bar{\alpha}'(s)$ and $N(s)$ are collinear. The only thing we still need to check is that the straight line through $\bar{\alpha}(s)$ (see (6)) of direction vector $N(s)$ goes through $\alpha(s)$. But this is obvious. See also Figure 19. \square

FIGURE 19. The dotted curve is the trace of the evolute of α .

We wanted now to raise the following question: given a curve $\bar{\alpha}$, is there any simple way to find a curve α whose evolute is $\bar{\alpha}$? The answer is yes, namely α is the involute of $\bar{\alpha}$ (see the definition below).

Definition 1.5.4. Let $\beta : (a, b) \rightarrow \mathbb{R}^2$ be a curve parametrized by arc length. Pick a number c with $a < c < b$. The *involute* of β starting at $\beta(c)$ is the curve $\alpha : (a, b) \rightarrow \mathbb{R}^2$ given by

$$\alpha(s) = \beta(s) + (c - s)\beta'(s).$$

One can show that if α is the involute of β , then the evolute of α is β . For the details of the proof, one can see [Gr-Abb-Sa], section 4.3.

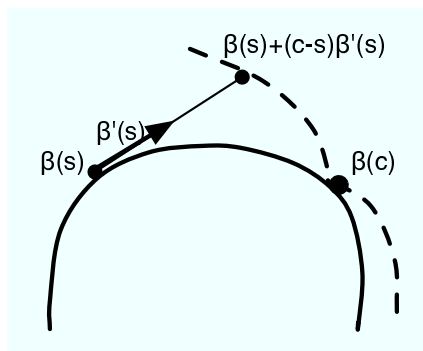


FIGURE 20. The dotted curve is the trace of the involute of the continuous curve.

We also mention the notion of pedal curve.

Definition 1.5.5. Let $\alpha : (a, b) \rightarrow \mathbb{R}^2$ be an arbitrary curve (not necessarily parametrized by arc length). Pick a point P in the plane. The pedal curve of α with respect to P is the curve β , where $\beta(t)$ is the orthogonal projection of P on the tangent line to α at $\alpha(t)$, for all values of t .

Some more details about this can be found for instance in [Gr-Abb-Sa], section 4.6.

1.6. Examples of Curves. Here are a few examples of important plane curves (see also sections 1.2 and 1.3).

1. **The cycloid** is the trajectory described by the motion of a point on a circle which rolls along a straight line.

Exercise. Assuming that the radius of the circle is 1, find a parametrization of the cycloid (as $\alpha(t) = (x(t), y(t))$). Find all points t with $\alpha'(t) = 0$ (these are called *singular* points). You may want to assume that the circle rolls along the x axis and the initial position of the point on the circle is the origin O . Determine the length of the piece of the cycloid which corresponds to a complete (that is, of 360°) rotation of the circle.

2. **The cardioid** is the trajectory described by the motion of a point on a circle which rolls along a (fixed) circle of the same radius.

Exercise. Assuming that the radius of the circles is 1, find a parametrization of the cardioid (as $\alpha(t) = (x(t), y(t))$). Then find the curvature of the cardioid.

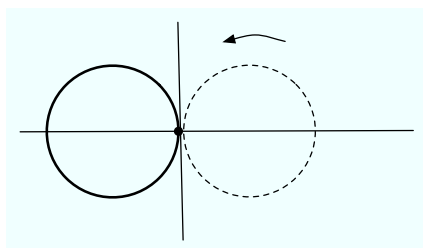


FIGURE 21. The initial position: after that, the dotted circle is rotated around the other one, and the point runs along the cardioid.

3. **The lemniscate of Bernoulli.** By definition, this is the locus of all points in the plane with the property that the product of distances to two given points (called foci) is

constant, equal to the squared half-distance between the foci⁸. In other words, if F_1 and F_2 are the foci, then the lemniscate consists of all points P with

$$\|PF_1\| \cdot \|PF_2\| = \left(\frac{\|F_1F_2\|}{2} \right)^2.$$

There are two ways of describing this curve: implicitly and explicitly (see the remark following Definition 1.2.1). Let's start with the implicit equation. Choose the two foci of coordinates $(-a, 0)$ and $(a, 0)$. Then a point $P(x, y)$ is on the lemniscate if and only if

$$(7) \quad ((x - a)^2 + y^2) ((x + a)^2 + y^2) = a^4.$$

One can see⁹ that this is equivalent to

$$(8) \quad x = \frac{a\sqrt{2} \cos t}{1 + \sin^2 t}, y = \frac{a\sqrt{2} \sin t \cos t}{1 + \sin^2 t},$$

where t is in \mathbb{R} (these give a parametrization of the lemniscate).

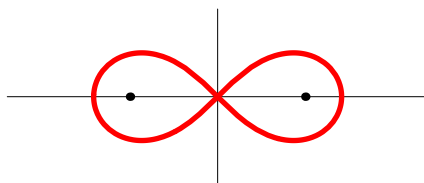


FIGURE 22. The lemniscate of Bernoulli and its foci.

4. **The catenary** is the curve assumed by a hanging flexible wire supported at its ends. Considerations from Physics show that the catenary can be described implicitly by¹⁰

$$y = a \cosh \frac{x}{a},$$

where a is a positive number. The obvious parametrization of the catenary is

$$\alpha(t) = (t, a \cosh \frac{t}{a}).$$

Exercise. Find a parametrization by arc length of the catenary, then find the curvature in terms of s .

5. **The tractrix** is the trace of the curve

$$\alpha(t) = (\sin t, \cos t + \log(\tan \frac{t}{2})),$$

where $0 < t < \pi$.

Exercise. Show that for any point P on the tractrix the length of the line segment between P and the y axis is constant (independent of t).

One can show that the evolute of the tractrix is a catenary (see for instance [Gr-Abb-Sa], section 4.2.).

REFERENCES

[Gr-Abb-Sa] A. Gray, E. Abbena, and S. Salamon, *Modern Differential Geometry with MATHEMATICA*, 3rd edition

⁸Because we want the midpoint of the segment determined by the foci to be on the curve.

⁹Check at least that x, y given by (8) satisfy (7).

¹⁰Recall that the hyperbolic cosine is $\cosh(u) := \frac{e^u + e^{-u}}{2}$.

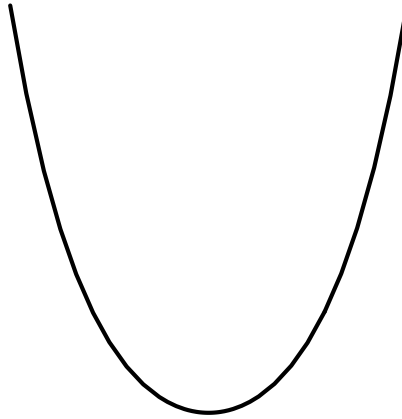


FIGURE 23. The catenary: think that you pick two arbitrary points on it; the piece of the curve between them is what a hanging wire attached at those two points looks like.

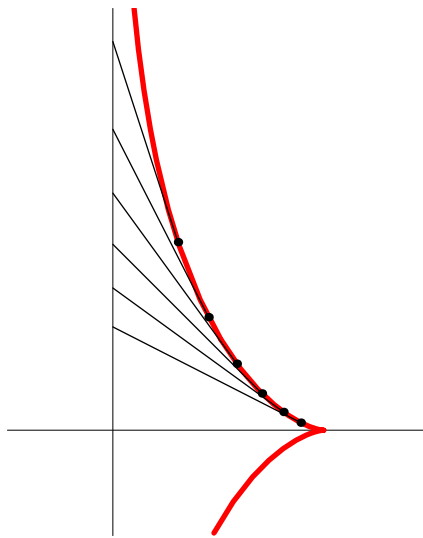


FIGURE 24. The tractrix: the trajectory of a point on the end of an inextensible rod as the other end of the rod is pulled vertically along the vertical line.