COHEN-MACAULAY MODULES WITH AN EYE TOWARD EQUIVARIANT COHOMOLOGY

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The equivariant cohomology module associated to an action of a compact Lie group G has a canonical structure of a $H^*(BG)$ module. (The coefficient field for cohomology is here $\mathbb{R}$.) One is especially pleased when this module is free: indeed, this happens for certain classes of group actions that have been extensively investigated over the past two to three decades[1]. However, this nice feature is very rare, as one can easily see in concrete situations. Instead of being free, one can require the above-mentioned equivariant cohomology module to satisfy a weaker condition, like for instance the so-called Cohen-Macaulay property. This appears to be a natural requirement: for instance, all transitive group actions fit into this class, even though only few of them satisfy the freenes condition above. The relevance of the Cohen-Macaulay condition in the context of equivariant cohomology was first noticed by Bredon in [Br]. More recently, this topic was investigated by Franz and Puppe [Fr-Pu], Goertsches and Töben [Go-Tö], Goertsches and Rollenske [Go-Ro], and Goertsches and Mare [Go-Ma]. The goal of this note is to provide the minimal background in commutative algebra needed in order to understand these papers. The treatment is sketchy: for instance, very few results are proved. However, the interested reader can find the details by following the bibliographical references which I have tried to make precise.

1. Modules over Noetherian local rings and their Krull dimension

1.1. **Rings.** All rings will be commutative with unit. If R is such a ring, an ideal $\mathfrak{p}$ is *prime* if $\mathfrak{p} \neq R$ and $xy \in \mathfrak{p} \Rightarrow x \in \mathfrak{p}$ or $y \in \mathfrak{p}$; equivalently, the ring $R/\mathfrak{p}$ is an integral domain. We set $\mathrm{Spec}(R) := \{\mathfrak{p} \subset R \, : \, \mathfrak{p} \text{ is a prime ideal}\}$. An ideal $\mathfrak{m}$ is *maximal* if there is no ideal $\mathfrak{a}$ such that $\mathfrak{m} \subset \mathfrak{a} \subset R$ (strict inclusion); equivalently, $R/\mathfrak{m}$ is a field. Hence a maximal ideal is automatically prime (see [At-Mc, p. 3]). Thus, any ring has at least one prime ideal (since it has a maximal one). A ring R is called *local* if it has a unique maximal ideal. Such rings can be constructed as follows: take $\mathfrak{p} \in \mathrm{Spec}(R)$, and denote by $R_\mathfrak{p}$ the ring of fractions of R with respect to $R \setminus \mathfrak{p}$ (note that the latter set is a multiplicative subset of R); then $R_\mathfrak{p}$ is a local ring, called the *localization of* R at $\mathfrak{p}$: the (only) maximal ideal consists of the "fractions" a/s with $a \in \mathfrak{p}$ and $s \in R \setminus \mathfrak{p}$, see [At-Mc, p. 38].

A ring R is *Noetherian* if it satisfies the ascending chain condition (a.c.c.) for ideals, i.e. any chain $\mathfrak{a}_1 \subseteq \mathfrak{a}_2 \subseteq \dots$ is eventually stationary; equivalently, every ideal in R is finitely

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[1]Most notably for Hamiltonian group actions on compact symplectic manifolds.

generated[2] see [At-Mc, p. 80]. For example, any PID is a Noetherian ring (recall that a *principal ideal domain* PID is an integral domain in which every ideal is principal, that is, of the form (r), with $r \in R$). In particular, one obtains the following concrete examples : any field K, the ring $K[x]$ of polynomials with coefficients in a field K, the ring $\mathbb{Z}$ etc. *Hilbert's Basis Theorem* says that if R is Noetherian, then so is $R[x_1, \ldots, x_n]$; in particular, if K is a field, then $K[x_1, \ldots, x_n]$ is a Noetherian ring.

The *(Krull) dimension* of an arbitrary ring R, denoted $\dim R$, is the supremum of the length of all strictly ascending chains of prime ideals in R. Recall that, by convention, if the chain is $\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \ldots \subset \mathfrak{p}_n$ (strict inclusions), then the length is n.

Examples.

• If K is a field, then $\dim K = 0$, since (0) is the only prime ideal.

• If K is a field, then $\dim K[x_1, \ldots, x_n] = n$. Proofs can be found in [Ma1, p. 83] and [Se, p. 56]. I just mention that a strictly ascending chain of length n is $(0) \subset (x_1) \subset (x_1, x_2) \subset \ldots \subset (x_1, \ldots, x_{n-1})$.

• $\dim \mathbb{Z} = 1$, because the only strictly ascending chains of prime ideals are $(0) \subset (p)$, where p is a prime integer. In general, any PID has dimension 1, because any prime ideal is maximal, see [At-Mc, p. 5].

It turns out that if R is also local, then $\dim R < \infty$, see [At-Mc, Corollary 11.11]. Note that, in general, the dimension of a Noetherian ring may be infinite, see [At-Mc, p. 126].

1.2. **Modules.** An R-module M is Noetherian if it satisfies the ascending chain condition (a.c.c.) for submodules, i.e. any chain $M_1 \subseteq M_2 \subseteq \ldots$ is eventually stationary; equivalently, every submodule of M is finitely generated, see [At-Mc, p. 75].

Proposition 1.1. ([At-Mc, p. 75]) *(a) Let $0 \to M' \to M \to M'' \to 0$ be a short exact sequence of R-modules. Then M is Noetherian if and only if M' and M'' are Noetherian.*

(b) Consequently, submodules and quotients of Noetherian modules are Noetherian modules as well.

The *(Krull) dimension* of M is by definition $\dim M := \dim R/\mathrm{Ann}(M)$, where $\mathrm{Ann}(M) = \{r \in R : rM = \{0\}\}$ is the annihilator of M (note that this is an ideal in R).

Some nice results about dimension can be obtained in the case when R is a Noetherian local ring and M a finitely generated R-module, see [Ma1, p. 73] and [At-Mc, p. 119]. I will only collect a few things from there.

Proposition 1.2. *If $(R, \mathfrak{m})$ is a Noetherian local ring and M a finitely generated R-module, then:*

(a) For any $n \geq 0$, the R-module $M/\mathfrak{m}^n M$ has finite length[3].

[2]Note that every ideal is also a Noetherian ring, because it is an R-submodule of R, see Proposition 1.1 below.

[3]Recall that the *length* of an A-module N is the length of a *composition series* of N, that is, a number $\ell(N) =: n$ for which there exists a sequence $(0) = N_0 \subset N_1 \subset \ldots \subset N_n = N$, where all inclusions are strict and no extra modules can be inserted (equivalently, any N_i/N_{i-1} is simple, i.e. has no non-trivial

(b) For n sufficiently large, $\chi(M;n) := \ell(M/\mathfrak{m}^n M)$ is a polynomial in n, of degree smaller than the least number of generators of $\mathfrak{m}$.

The polynomial $\chi(M;n)$ is called the *Hilbert polynomial* of M relative to[4] $\mathfrak{m}$. One is especially interested in its degree: set

$$d(M) := \deg \chi(M;n).$$

Proposition 1.3. ([Ma1, Theorem 17, p. 76]) *We have $d(M) = \dim(M)$. Consequently, $\dim(M) < \infty$.*

Proposition 1.4. ([Ma1, p. 74]) *If $0 \to M' \to M \to M'' \to 0$ is an exact sequence of finitely generated R-modules, then $d(M) = \max\{d(M'), d(M'')\}$. Consequently, $\dim(M) = \max\{\dim(M'), \dim(M'')\}$.*

2. Depth; Cohen-Macaulay modules

2.1. **Depth.** Let M be a module over a ring R. A sequence $r_1, \dots, r_m \in R$ is called *M-regular* if[5] r_i is not a zero-divisor on $M/(r_1, \dots, r_{i-1})M$, for all $1 \leq i \leq r$. If so, then all inclusions in the sequence $(r_1) \subset (r_1, r_2) \subset \dots \subset (r_1, \dots, r_m)$ are strict. Let $\mathfrak{a}$ be an ideal in R. The M-regular sequence $r_1, \dots, r_m$ with $r_i \in \mathfrak{a}$, $1 \leq i \leq m$, is said to be *maximal in* $\mathfrak{a}$ if there is no $r_{m+1} \in \mathfrak{a}$ such that $r_1, \dots, r_m, r_{m+1}$ is M-regular; in other words, all elements of $\mathfrak{a}$ are zero-divisors on $M/(r_1, \dots, r_m)M$.

Theorem 2.1. (see[6] [Ma1, p. 102]) *Let M be a finitely generated module over a Noetherian ring R and $\mathfrak{a} \subset R$ an ideal. Then all maximal M-regular sequences in $\mathfrak{a}$ have the same length n, which is given by*[7]

$$n = \min\{i \,:\, \mathrm{Ext}^i_R(R/\mathfrak{a}, M) \neq 0\}.$$

Definition 2.2. *For R, $\mathfrak{a}$, and M as in the theorem, the common length of all maximal M-regular sequences in $\mathfrak{a}$ is called the* $\mathfrak{a}$-depth *of M, denoted* $\mathrm{depth}_{\mathfrak{a}}(M)$.

Definition 2.3. *If $(R, \mathfrak{m})$ is a local Noetherian ring and M a finitely generated R-module, then the* depth *of M is*

$$\mathrm{depth}\,(M) := \mathrm{depth}_{\mathfrak{m}}(M).$$

By Theorem 2.1,

$$\mathrm{depth}(M) = \min\{i \,:\, \mathrm{Ext}^i_R(R/\mathfrak{m}, M) \neq 0\}. \tag{2.1}$$

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submodules). In fact, all composition series have the same number of terms, which can be finite or infinite. See [At-Mc, pp. 76-77]. See also [Ma1, p. 72] for characterizations of modules of finite length.

[4]The actual construction is slightly more general : it involves a choice of an ideal contained in $\mathfrak{m}$, called "ideal of definition" I in [Ma1], resp. "$\mathfrak{m}$-primary ideal" in [At-Mc]. Eventually it is shown that the number $d(M)$ obtained this way is independent on this choice. What I describe here is a correct definition of $d(M)$, which is sufficient for my goals.

[5]For $i = 1$, the condition is: r_1 is not a zero-divisor on M. For $i = m$, the condition is $M/(r_1, \dots, r_m)M \neq \{0\}$.

[6]See also [Br-He, Theorem 1.2.5].

[7]The functor Ext (along with Tor) is defined nicely in [Ma2, Appendix B, pp. 274-282].

The following inequalities are useful:

Proposition 2.4. (see [Br-He, Prop. 1.2.9]) *Let* $(R, \mathfrak{m})$ *be a local Noetherian ring and* $0 \to M' \to M \to M'' \to 0$ *an exact sequence of finitely generated* R*-modules. Then:*

(a) $\mathrm{depth}\,(M) \geq \min\{\mathrm{depth}\,(M'), \mathrm{depth}\,(M'')\}$,
(b) $\mathrm{depth}\,(M') \geq \min\{\mathrm{depth}\,(M), \mathrm{depth}\,(M'') + 1\}$,
(c) $\mathrm{depth}\,(M'') \geq \min\{\mathrm{depth}\,(M') - 1, \mathrm{depth}\,(M'')\}$

Proof's Main Idea. The short exact sequence induces the following long exact sequence (see [Ma2, pp. 279-280]):

$$\begin{aligned} \ldots \to \mathrm{Hom}(R/\mathfrak{m}, M') &\to \mathrm{Hom}(R/\mathfrak{m}, M) \to \mathrm{Hom}(R/\mathfrak{m}, M'') \to \\ \to \mathrm{Ext}^1(R/\mathfrak{m}, M') &\to \mathrm{Ext}^1(R/\mathfrak{m}, M) \to \mathrm{Ext}^1(R/\mathfrak{m}, M'') \to \\ \to \mathrm{Ext}^2(R/\mathfrak{m}, M') &\to \mathrm{Ext}^2(R/\mathfrak{m}, M) \to \mathrm{Ext}^2(R/\mathfrak{m}, M'') \to \ldots \end{aligned}$$

The inequalities follow readily from the exactness of this sequence and Equation (2.1). □

2.2. **Proof of** $\mathrm{depth}\, M \leq \dim M$**.** In what follows I would like to justify this inequality. To this end I first need the following obvious reformulation of the definition of the Krull dimension:

$$\dim R = \sup\{\mathrm{coht}(\mathfrak{p}) \, : \, \mathfrak{p} \in \mathrm{Spec}(R)\},$$

where[8] $\mathrm{coht}(\mathfrak{p})$ denotes the supremum of n for which there exists a sequence $\mathfrak{p} \subset \mathfrak{p}_1 \subset \ldots \subset \mathfrak{p}_n$ of strict inclusions of prime ideals. I get immediately that for any $\mathfrak{p} \in \mathrm{Spec}(R)$ one has

$$\dim R/\mathfrak{p} = \mathrm{coht}(\mathfrak{p}).$$

Moreover, if M is an R-module, then

$$\dim M = \sup\{\mathrm{coht}(\mathfrak{p}) \, : \, \mathfrak{p} \in V(\mathrm{Ann}(M))\},$$

where $V(\mathrm{Ann}(M))$ consists of all $\mathfrak{p} \in \mathrm{Spec}(R)$ with $\mathrm{Ann}(M) \subset \mathfrak{p}$. Note that in the above definition *it is sufficient to consider* $\mathfrak{p} \in V(\mathrm{Ann}(M))$ *that are minimal*, i.e., there is no $\mathfrak{p}' \in \mathrm{Spec}(R)$ with $\mathrm{Ann}(M) \subset \mathfrak{p}' \subset \mathfrak{p}$.

Proposition 2.5. *Assume that* R *is Noetherian and* M *is finitely generated.*

(a) One has

$$V(\mathrm{Ann}(M)) = \mathrm{Supp}(M) \stackrel{def.}{=} \{\mathfrak{p} \in \mathrm{Spec}(R) \, : \, M_{\mathfrak{p}} \neq 0\}.$$

(b) One has [9] $\mathrm{Supp}(M) \supseteq \mathrm{Ass}(M)$*. Any minimal element of* $\mathrm{Supp}(M)$ *is in* $\mathrm{Ass}(M)$*.*

(c) The set $\mathrm{Ass}(M)$ *is finite.*

(d) The union of all ideals that belong to Ass*(M) is the set of all zero-divisors for M.*

[8] Also recall that the height of $\mathfrak{p}$ is the supremum of n for which there exists a sequence $\mathfrak{p} \supset \mathfrak{p}_1 \supset \ldots \supset \mathfrak{p}_n$ of strict inclusions of prime ideals and $\dim R = \sup\{\mathrm{ht}(\mathfrak{p}) \, : \, \mathfrak{p} \in \mathrm{Spec}(R)\}$.

[9] Recall that $\mathrm{Ass}(M)$, or "the (set of) associated primes of M", consists of all $\mathfrak{p} \in \mathrm{Spec}(R)$ which are of the form $\mathfrak{p} = \mathrm{Ann}(x)$, $x \in M$. See [Ma1, p. 49].

Proof. (a) See [Ma1, Exercice 2, p. 16].

(b) See [Ma1, Theorem 9, p. 50].

(c) See [Ma1, Proposition 7.G, p. 52].

(d) See [Ma1, Corollary 2, p. 50]. □

I am getting closer to the actual goal of this section with the following lemma, which is the heart of the matter:

Lemma 2.6. *Let $(R, \mathfrak{m})$ be a Noetherian local ring and M a finitely generated R-module. If $r \in \mathfrak{m}$ is not a zero divisor, then*

$$\dim M/rM = \dim M - 1.$$

Proof of[10] $\dim M/rM < \dim M$. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_t$ be the elements of $\operatorname{Spec}(R)$ which contain $\operatorname{Ann}(M)$ and have the property that $\operatorname{coht}(\mathfrak{p}_1) = \ldots = \operatorname{coht}(\mathfrak{p}_t) = \dim(M)$; note that any other $\mathfrak{p} \in \operatorname{Spec}(R)$ which contains $\operatorname{Ann}(M)$ has $\operatorname{coht}(\mathfrak{p}) < \dim M$. The ideals $\mathfrak{p}_1, \ldots, \mathfrak{p}_t$ are minimal in $V(\operatorname{Ann}(M))$, hence, by Proposition 2.5, they lie in $\operatorname{Ass}(M)$, thus r is not contained in any of them. I obviously have $\operatorname{Ann}(M) \subset \operatorname{Ann}(M/rM)$. When calculating $\dim M/rM$, I consider the coht of $\mathfrak{p} \in \operatorname{Spec}(R)$ with $\operatorname{Ann}(M/rM) \subset \mathfrak{p}$, which implies $r \in \mathfrak{p}$; consequently, any such $\mathfrak{p}$ contains $\operatorname{Ann}(M)$, but is not among $\mathfrak{p}_1, \ldots, \mathfrak{p}_t$, hence $\operatorname{coht}(\mathfrak{p}) < \dim M$. This implies $\dim M/rM < \dim M$.

For a complete proof of the lemma I refer to [Ma1, Lemma 4, p. 105]. □

I deduce immediately:

Corollary 2.7. *Let $(R, \mathfrak{m})$ be a Noetherian local ring and M a finitely generated R-module.*

(a) If $r_1, \ldots, r_m \in \mathfrak{m}$ is an M-regular sequence, then

$$\dim M/(r_1, \ldots, r_m)M = \dim M - m.$$

(b) Consequently, $\operatorname{depth}(M) \leq \dim(M)$.

2.3. **Cohen-Macaulay modules.**

Definition 2.8. (see [Br-He, p. 57]) *Let $(R, \mathfrak{m})$ be a local Noetherian ring and M a finitely generated R-module. One says that M is* Cohen-Macaulay *(shortly* CM*) if $M = 0$ or* $\operatorname{depth}(M) = \dim(M)$.

To define CM-modules over arbitrary Noetherian rings, one needs localization of modules. For the definition of this notion I refer to [At-Mc, p. 38]: if M is an R-module and $\mathfrak{p} \subset R$ an ideal, one constructs $M_{\mathfrak{p}}$, which is a module over the localized ring $R_{\mathfrak{p}}$.

Relevant at this point is that if $(R, \mathfrak{m})$ is a local Noetherian ring and M an R-module then:

- $\dim M = \dim M_{\mathfrak{m}}$

[10] This is sufficient to achieve the actual goal stated in the title of this subsection: in the spirit of Corollary 2.7, one deduces $\dim M/(r_1, \ldots, r_m)M \leq \dim M - m$, which in turn implies $\operatorname{depth}(M) \leq \dim(M)$.

- depth M = depth $M_{\mathfrak{m}}$, where the depths are taken relative to $\mathfrak{m}$ and the ideal induced by $\mathfrak{m}$, respectively.

(Indeed, the first equality follows immediately from the definition of the dimension of a module and the exact description of the ideals in $R_{\mathfrak{m}}$, see, e.g., [At-Mc, Corollary 3.13 iv)]; for the second equality, see [Br-He, Proposition 1.5.15 (e)].) Consequently, M is CM iff $M_{\mathfrak{m}}$ is CM. Inspired by this property, one defines:

Definition 2.9. (see [Br-He, p. 57]) *Let R be an* arbitrary *Noetherian ring and M a finitely generated R-module. One says that M is* Cohen-Macaulay *if $M_{\mathfrak{m}}$ is Cohen-Macaulay over $R_{\mathfrak{m}}$ for any maximal ideal $\mathfrak{m}$ of R.*

Details and results concerning CM modules can be found in [Ma1, Section 16, p. 106], [Se, p. 83], and [Br-He, Ch. 2, p. 57]. The following result seems to be in none of these books.

Proposition 2.10. *Let $(R, \mathfrak{m})$ be a local ring and M', M'' two finitely generated R-modules. Then $M' \oplus M''$ is CM if and only if both M' and M'' are CM of the same dimension.*

Proof. [11] First assume that M' and M'' are CM of dimension n. Proposition 2.4 for the exact sequence $0 \to M' \to M' \oplus M'' \to M'' \to 0$ shows that depth $(M' \oplus M'') \geq n$. I use Proposition 1.3 for the same short exact sequence and obtain: $\dim(M' \oplus M'') = n$. Consequently,

$$n \leq \operatorname{depth}(M' \oplus M'') \leq \dim(M' \oplus M'') = n,$$

thus $M' \oplus M''$ is CM.

We now prove the converse implication. That is, assume that $M' \oplus M''$ is CM. Denote by n its dimension. The short exact sequence $0 \to M' \to M' \oplus M'' \to M'' \to 0$ induces the following long exact sequence:

$$\ldots \to \operatorname{Ext}^{i-1}(R/\mathfrak{m}, M') \to \operatorname{Ext}^{i-1}(R/\mathfrak{m}, M' \oplus M'') \to \operatorname{Ext}^{i-1}(R/\mathfrak{m}, M'') \to$$
$$\to \operatorname{Ext}^{i}(R/\mathfrak{m}, M') \to \operatorname{Ext}^{i}(R/\mathfrak{m}, M' \oplus M'') \to \operatorname{Ext}^{i}(R/\mathfrak{m}, M'') \to \ldots$$

Similarly, the short exact sequence $0 \to M'' \to M' \oplus M'' \to M' \to 0$ induces the following long exact sequence:

$$\ldots \to \operatorname{Ext}^{i-1}(R/\mathfrak{m}, M'') \to \operatorname{Ext}^{i-1}(R/\mathfrak{m}, M' \oplus M'') \to \operatorname{Ext}^{i-1}(R/\mathfrak{m}, M') \to$$
$$\to \operatorname{Ext}^{i}(R/\mathfrak{m}, M'') \to \operatorname{Ext}^{i}(R/\mathfrak{m}, M' \oplus M'') \to \operatorname{Ext}^{i}(R/\mathfrak{m}, M') \to \ldots$$

Along with Equation (2.1) for $M := M' \oplus M''$, whose depth is n, these sequences imply:

$$\operatorname{Ext}^{n-1}(R/\mathfrak{m}, M') = \operatorname{Ext}^{n-2}(R/\mathfrak{m}, M') = \ldots = \operatorname{Ext}^{0}(R/\mathfrak{m}, M') = 0$$
$$\operatorname{Ext}^{n-1}(R/\mathfrak{m}, M'') = \operatorname{Ext}^{n-2}(R/\mathfrak{m}, M'') = \ldots = \operatorname{Ext}^{0}(R/\mathfrak{m}, M'') = 0.$$

Thus depth $(M') \geq n$ and depth $(M'') \geq n$. On the other hand, from Proposition 1.4, I have $\dim(M') \leq n$ and $\dim(M'') \leq n$. Thus:

$$n \leq \operatorname{depth}(M') \leq \dim(M') \leq n,$$

which shows that M' is CM. Similarly, M'' is CM. □

[11] This proof was kindly provided to me by Oliver Goertsches.

3. Graded modules over graded rings

Let us now consider the following situation:

(i) $R = \bigoplus_{j\geq 0} R_j$, $R_i R_j \subset R_{i+j}$, for all $i, j \geq 0$, $R_0 = \mathbb{R}$, and R is a finitely generated R_0-algebra
(ii) $M = \bigoplus_{j\geq 0} M_j$, and $R_i M_j \subset M_{i+j}$ for all $i, j \geq 0$
(iii) M is finitely generated over R.

In the language of [Br-He, Section 1.5], M is a graded module over the graded ring R. Moreover, the ring R is Noetherian, see [Br-He, Theorem 1.5]. It is also a *local ring, i.e., it has a graded ideal[12], which is the unique maximal graded ideal: concretely, this is

$$\mathfrak{m}_0 := \bigoplus_{j\geq 1} R_j,$$

see [Br-He, Example 1.5.14 (b)]. Note that $\mathfrak{m}_0$ is also maximal among all ideals of R.

Example 3.1. The polynomial ring $R = \mathbb{R}[x, y]$ with the usual grading satisfies property (i) above. We have $\mathfrak{m}_0 = (x, y)$ (the ideal generated by x and y, which consists of all polynomials without free term). It's easy to see that $\mathfrak{m}_0$ is a maximal ideal: as a vector space, the quotient $\mathbb{R}[x, y]/(x, y)$ has dimension 1. The same argument shows that for any two numbers a and b, the ideal $(x + a, y + b)$ is also maximal (without being graded, unless $a = b = 0$).

By definition, the depth of a module of type (i)-(iii) is

$$\operatorname{depth} M := \operatorname{depth}_{\mathfrak{m}_0} M.$$

The following proposition is a useful tool when dealing with dimension and depth:

Proposition 3.2. *Let M be an R-module of type (i)-(iii), $M \neq 0$. Then*

(a) $M_{\mathfrak{m}_0} \neq \{0\}$

(b) $\dim M = \dim M_{\mathfrak{m}_0}$ *(in particular,* $\dim M$ *is finite)*

(c) $\operatorname{depth} M = \operatorname{depth} M_{\mathfrak{m}_0}$*, where the depth in the RHS is relative to the unique maximal ideal of* $R_{\mathfrak{m}_0}$ *(i.e., the one induced by* $\mathfrak{m}_0$*).*

Proof. (a) The following argument is from [Fr-Pu, Section 4]. By [Ma1, Corollary 1, p. 50], there exists at least one element $\mathfrak{p} \in \operatorname{Ass}(M)$. By [Br-He, Lemma 1.5.6], $\mathfrak{p}$ is a homogeneous ideal, hence it is contained in $\mathfrak{m}_0$. Finally, set $\mathfrak{p} = \operatorname{Ann}(x)$, $x \in M$, and take into account that the element $x/1$ in $M_{\mathfrak{m}_0}$ is different from 0 (because otherwise x would be annihilated by an element of $R \setminus \mathfrak{m}_0$, which contradicts $\operatorname{Ann}(x) \subset \mathfrak{m}_0$, see [At-Mc, Proof of Proposition 3.8]).

(b) The argument used here is taken from [Go-Tö, Proof of Proposition 5.1]. First, from the definition of dimension,

$$\dim M = \sup\{\dim M_{\mathfrak{m}} \ : \ \mathfrak{m} \subset R \text{ maximal ideal}\}.$$

[12] A *graded subring* (sometimes also called *homogeneous subring*) of R is a subring S with the property $S = \bigoplus_{j\geq 0}(S \cap R_j)$ which is maximal among all graded ideals. Any such subring is in turn a graded ring relative to the grading induced by the previous equation.

Thus, $\dim M_{\mathfrak{m}_0} \le \dim M$. Consider now an ideal $\mathfrak{m} \subset R$ which is maximal but not graded. Let $\mathfrak{m}^*$ be the ideal of R generated by the homogeneous elements of $\mathfrak{m}$. By [Br-He, Lemma 1.5.6], we have $\mathfrak{m}^* \in \mathrm{Supp}(M)$. By [Br-He, Theorem 1.5.8 (b)], we have $\dim M_{\mathfrak{m}} = \dim M_{\mathfrak{m}^*} + 1$. Since $\mathfrak{m}^*$ is homogeneous and $\mathfrak{m}^* \neq R$, we have $\mathfrak{m}^* \subseteq \mathfrak{m}_0$ (since $\mathfrak{m}^*$ must be contained in a maximal graded ideal of R, by the argument used in the proof of [At-Mc, Corollary 1.4]). In fact, the inclusion of $\mathfrak{m}^*$ in $\mathfrak{m}_0$ is strict, since otherwise we would have $\mathfrak{m} = \mathfrak{m}_0$, which is a contradiction. This implies $\dim M_{\mathfrak{m}^*} < \dim M_{\mathfrak{m}_0}$: indeed, if $d := \dim M_{\mathfrak{m}^*}$ then, by [Br-He, Theorem 1.5.8 (a)], there exists a chain of prime ideals $\mathfrak{p}_0 \subset \ldots \subset \mathfrak{p}_d = \mathfrak{m}^*$. We conclude that $\dim M_{\mathfrak{m}} \le \dim M_{\mathfrak{m}_0}$, hence $\dim M \le \dim M_{\mathfrak{m}_0}$.

(c) The claim follows readily from [Br-He, Theorem 1.5.15 (e)]. □

Consequently, several results involving the dimension and the depth of Noetherian modules over local rings become valid for modules of the type studied in this section: see for example Propositions 2.4 (with dim instead of d), 1.4, and 2.10 above, as well as [Go-Ma, Lemma 2.3] and [Go-Tö, Lemma 5.4] (which can be deduced from [Fr-Pu, Lemma 4.3]; the CM condition in the context of this section is discussed below). A somehow less obvious result is [Go-Ma, Lemma 2.5], see also [Go-Ro, Lemma 2.6]: For modules over *local* rings, the result is the content of [Se, Proposition 12, Section 4.b], see also [CR, Ch. 16, Proposition 1.9] or [Br-He, Exercise 1.2.26 (b)]. I would like in what follows to spell out the details of the proof.

Proposition 3.3. *Let S be a ring and M an S-module that satisfy the assumptions (i)-(iii) above. Let R be a graded ring which satisfies condition (i) above. Let $\varphi : R \to S$ be a homomorphism of graded rings. Assume that S is finitely generated relative to the R-module structure acquired via φ. Then*

$$\dim_R M = \dim_S M \quad \textit{and} \quad \mathrm{depth}\,_R M = \mathrm{depth}\,_S M.$$

Proof. The first of the two equations above is equivalent to

$$\dim R/(\mathrm{Ann}_R M) = \dim S/(\mathrm{Ann}_S M).$$

Observe that $\mathrm{Ann}_R M = \varphi^{-1}(\mathrm{Ann}_S M)$, hence $R/(\mathrm{Ann}_R M) = R/(\varphi^{-1}(\mathrm{Ann}_S M))$ can be viewed via φ as a subring of $S/(\mathrm{Ann}_S M)$. By hypothesis, the latter ring is finitely generated as a module over the former one. Hence they have the same dimension: see [Ta-Yu, 3.3.4 and 3.3.5] (and also [Se, Proposition 3, III A]). Note that this argument is quite general: the technical hypotheses that R and S are Noetherian and local, φ respects the grading, etc., are not used in the proof: we just need to know that M is finitely generated as module (over S or, equivalently, over R, see [Ta-Yu, Theorem 3.3.4]) and of course, that S is finitely generated over R.

I will now justify the formula concerning the depth. The method is inspired by the proof of [CR, Ch. 16, Proposition 1.9]. First note that both depths involved in the equation are finite numbers. Set $m := \mathrm{depth}\,_R M$. Let $r_1, \ldots, r_m \in \mathfrak{m}_0 = \oplus_{j\ge1} R_j$ be a maximal regular sequence relative to R. Since φ preserves the grading, the sequence $s_i := \varphi(r_i)$, $1 \le i \le m$, lies in $\oplus_{j\ge1} S_j$. It is clearly M-regular as well. I will now show it is maximal. Assume it isn't. Then $\oplus_{j\ge1} S_j$ contains elements that are not zero-divisors relative to $M/(r_1, \ldots, r_m)M$. Denote the latter space by N. It is also equal to $M/(s_1, \ldots, s_m)M$, hence its R-module structure is the one induced by the obvious S-module structure. From the maximality of $r_1, \ldots, r_m$, the R-depth of N is 0, hence

$$\mathrm{Ext}^0_R(R/\mathfrak{m}_0, N) = \mathrm{Hom}_R(R/\mathfrak{m}_0, N) \neq 0.$$

The space in the previous equation can be naturally embedded as an R-submodule into N: to the homomorphism $f : R/\mathfrak{m}_0 \to N$ one assigns $f(1 + \mathfrak{m}_0)$ (recall that the elements of $R/\mathfrak{m}_0$ are of the form $r + \mathfrak{m}_0$, with $r \in R$). The resulting subspace of N is obviously annihilated by $\mathfrak{m}_0$. Thus the space $N' := \{x \in N \ : \ rx = 0 \text{ for all } r \in \mathfrak{m}_0\}$ is a non-zero S-submodule of N. It is clearly a graded such submodule. Its annihilator is $\mathfrak{m}_0$, hence $\dim_R N' = 0$. Consequently, $\dim_S N' = 0$. In other words, N' is an Artinian S-module. The set $\mathrm{Ass}(N')$ is nonempty, see [Ma1, Corollary 1, p. 50]. If $\mathfrak{p} \in \mathrm{Ass}(N')$, then $\mathfrak{p}$ is a maximal ideal (use [At-Mc, Proposition 8.1] for the ring $R/\mathrm{Ann}(N')$; alternatively, see [La, Chapter 0, Proposition 0.40]). By [Br-He, Lemma 1.5.6], $\mathfrak{p}$ is also a graded ideal, hence $\mathfrak{p} = \oplus_{j\geq 1} S_j$. We conclude that $\oplus_{j\geq 1} S_j$ annihilates a certain element of N, which is a contradiction. $\square$

I would like now to get back to Proposition 3.2: along with the result proved in Section 2.2, it implies that for any module of type (i)-(iii) one has $\operatorname{depth} M \leq \dim M$. Furthermore, a such R-module M is CM according to Definition 2.9 above iff $M_{\mathfrak{m}_0}$ is CM over $R_{\mathfrak{m}_0}$, that is, iff $\operatorname{depth} M = \dim M$: for a proof, I refer to [Br-He, Exercise 2.1.27] or rather to the references indicated in [Br-He] on p. 86.

Finally, note that if G is a compact connected Lie group acting of a compact connected manifold X, then $H^*_G(X)$ equipped with the canonical structure of $H^*(BG)$-module satisfies hypotheses (i)-(iii) above. In fact, the only non-obvious fact is that this module is finitely generated: this has been proved by Quillen, see [Qu].

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