Steepest descent on complex flag manifolds

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The adjoint orbits of compact semisimple Lie groups K are called *complex* flag manifolds. Any such manifold admits a standard embedding in $\mathrm{Lie}(K)$; if the latter is equipped with an $\mathrm{Ad}(K)$-invariant inner product, the embedding is taut, i.e. all height functions are perfect over $\mathbb{Z}$. At the same time, an adjoint orbit is of the type K^c/P, where P is a parabolic subgroup of K^c; hence it is a complex manifold. In fact, the natural KKS-symplectic form makes it into a Kähler manifold. Now there is a nice observation which says that the gradient flows with respect to the Kähler metric of height functions are just 1-parameter subgroups of G^c. A proof of this fact has been sketched by M. Guest and Y. Ohnita in the Appendix of [2]. The main goal of these notes is to give all details of their proof. At the end I will also make a reference to [1], where J. Eschenburg and myself dealt with flow lines on *real* flag manifolds: in the particular case of complex flag manifolds, I will show how to recover the theorem of Guest and Ohnita.

Let K be a compact semisimple Lie group of Lie algebra $\mathfrak{k}$, and $T \subset K$ a maximal torus of Lie algebra $\mathfrak{t}$. Consider the adjoint orbit $M = \mathrm{Ad}(K)(x_o)$ for $x_o \in \mathfrak{k}$. If $G = K^{\mathbb{C}}$ is the complexification of K, then G/K is a non-compact symmetric space and

$$\mathfrak{g} = \mathfrak{k} + \mathrm{i}\mathfrak{k}$$

is a Cartan decomposition of $\mathfrak{g} = \mathrm{Lie}(G) = \mathfrak{k} \otimes \mathbb{C}$ (the involution σ is just the complex conjugation). Since M is — up to a multiple of i — an isotropy orbit of G/K, the results of the previous section can be applied here, too. The goal of this section is to point out that there exists already a natural metric on M with the property that the lines of steepest descent of the height functions with respect to it are orbits of one-parameter subgroups of G — namely the Kähler metric (cf. [2]).

The complex structure J on M is an important ingredient. It can be defined as follows: Start by fixing $\mathfrak{t}_o \subset \mathfrak{k}$ a maximal abelian subspace with $x_o \in \mathfrak{t}_o$. For any $x \in M$, take $\mathfrak{t}$ a maximal abelian subspace of $\mathfrak{k}$ such that x_o and x, respectively $\mathfrak{t}_o$ and $\mathfrak{t}$ are Ad-conjugate by the same element of K. The root decomposition of $\mathfrak{g}$ corresponding to $\mathfrak{t}$ is $\mathfrak{g} = \mathfrak{t} \otimes \mathbb{C} + \sum_{\alpha \in R} \mathfrak{g}_\alpha$ where the roots α are linear functions on $\mathfrak{t}$ with the property that

$$\mathfrak{g}_\alpha = \{z \in \mathfrak{g}; [\xi, z] = \mathrm{i}\alpha(\xi)z, \ \forall \xi \in \mathfrak{t}\}$$

is nonzero. Let $\mathfrak{n}_- = \sum_{\alpha(x)>0} \mathfrak{g}_\alpha$ and consider the complex subgroup

$$H = \{g \in G; \mathrm{Ad}(g)(x + \mathfrak{n}_-) = x + \mathfrak{n}_-\}$$

of G, where $C = \{k \in K; \mathrm{Ad}(k)x = x\}$. As in section 3 of [1], H is independent on the choice of $\mathfrak{t}$. Like in Lemma 3.1 of [1], the exists a natural diffeomeorphism $M \simeq G/H$ which maps x to the coset of e and induces in this way a complex structure on T_xM. In order to describe it more precisely, take $v \in T_x(M) = [x, \mathfrak{k}]$ and its decomposition $v = \sum_+ (z_\alpha + \bar{z}_\alpha)$, where $z_\alpha \in \mathfrak{g}_\alpha$ and $\sum_+$ denotes $\sum_{\alpha(x)>0}$. We must have

$$J(v) = J_x(v) = J_x \sum_+ (z_\alpha + \bar{z}_\alpha) = \sum_+ \mathrm{i}(z_\alpha - \bar{z}_\alpha).$$

Let us note the following property of J:

Lemma 0.1 *The infinitesimal action of G on M satisfies*

$$J(q.x) = (iq).x$$

for any $x \in M$ and $q \in \mathfrak{k}$.

Proof. We can write $q = \sum_+ (z_\alpha + \bar{z}_\alpha)$, with $z_\alpha \in \mathfrak{g}_\alpha$. We have $(\mathrm{i}q).x = r.x$, where $r \in \mathfrak{k}$ has the property $\mathrm{i}q - r \in \mathfrak{h}$. But one can easily see that $r = \sum_+ \mathrm{i}(z_\alpha - \bar{z}_\alpha)$ satisfies this property. Hence $(\mathrm{i}q).x = [x, r] = \sum_+ -\alpha(x)(z_\alpha + \bar{z}_\alpha)$. The last expression is obviously the same as

$$J(q.x) = J_x[x, q] = J_x \sum_+ \alpha(x)\mathrm{i}(z_\alpha - \bar{z}_\alpha).$$

□

Let us fix a K-invariant inner product $\langle\ ,\ \rangle$ on $\mathfrak{k}$ (e.g. the negative of the Killing form). There exists a natural symplectic form ω on M, which is given by

$$\omega_x([x,u],[x,v]) = \langle x, [u,v]\rangle = \langle [x,u], v\rangle$$

for any $x \in M$ and any two tangent vectors $[x,u], [x,v] \in T_xM$, where $u, v \in \mathfrak{k}$. The symplectic form ω and the complex form J make M into a Kähler manifold. The corresponding Kähler metric $(\ ,\ )$ is defined by

$$(X,Y) = \omega_x(X, JY), \tag{1}$$

for any two vectors $X, Y \in T_x(M)$. Let us consider the action of K on M and the corresponding momentum map $\mu : M \to \mathfrak{k}^*$. One can see that for any $q \in \mathfrak{k}$, the map $\mu(\cdot)(q) := \mu^q : M \to \mathbb{R}$ is just the height function $h_q = \langle q, \cdot\rangle$. This means that we must have

$$d(h_q)_x = \omega_x([x,q], \cdot).$$

From (1) we deduce that the gradient of h_q with respect to the Kähler metric is

$$\nabla(h_q)_x = -J[q,x],$$

$x \in M$. We would like to find the corresponding gradient lines $x(t)$, i.e. solutions of the equation

$$x'(t) = -J[q, x(t)].$$

By Lemma 0.1, we can express this differential equation in terms of the infinitesimal action of G on M, as follows:

$$x'(t) = (iq).x(t).$$

The solution of this equation is obviously

$$x(t) = \exp(\mathrm{i}tq).x(0),$$

where the right hand side one uses the action of G on M.

In fact we can obtain the same result if we use Theorem 4.1 of [1] and the following result:

Proposition 0.2 *If M is an adjoint orbit, then the metric s on M defined by Theorem 4.1 of [1] is the same as the Kähler metric.*

Proof. Recall that for any $x \in M$ we have

$$T_x(M) = \sum_+ (\mathfrak{g}_\alpha + \mathfrak{g}_{-\alpha}) \cap \mathfrak{k}$$

and the metric s is defined by

$$\langle v, w \rangle_s = \sum_+ \frac{1}{\alpha(x)} \langle v_\alpha, w_\alpha \rangle.$$

If α is an arbitrary root with $\alpha(x) > 0$, take $z_\alpha, \zeta_\alpha \in \mathfrak{g}_\alpha$, then $z_\alpha + \bar{z}_\alpha$ and $\zeta_\alpha + \bar{\zeta}_\alpha$ the corresponding tangent vectors. Their product with respect to the Kähler metric (see (1)) is

$$\begin{aligned}
(z_\alpha + \bar{z}_\alpha, \zeta_\alpha + \bar{\zeta}_\alpha) &= \omega_x(z_\alpha + \bar{z}_\alpha, \mathrm{i}(\zeta_\alpha - \bar{\zeta}_\alpha)) \\
&= -\frac{1}{\alpha(x)^2} \omega_x([x, \mathrm{i}(z_\alpha - \bar{z}_\alpha)], [x, \zeta_\alpha + \bar{\zeta}_\alpha]) \\
&= -\frac{1}{\alpha(x)^2} \langle [x, \mathrm{i}(z_\alpha - \bar{z}_\alpha)], \zeta_\alpha + \bar{\zeta}_\alpha \rangle \\
&= \frac{1}{\alpha(x)} \langle z_\alpha + \bar{z}_\alpha, \zeta_\alpha + \bar{\zeta}_\alpha \rangle \\
&= \langle z_\alpha + \bar{z}_\alpha, \zeta_\alpha + \bar{\zeta}_\alpha \rangle_s
\end{aligned}$$

□

By Theorem 4.1 of [1], the gradient lines of the function $f(x) = \langle -\mathrm{i}q, \mathrm{i}x \rangle = \langle q, x \rangle$, $x \in M$, are

$$x(t) = \exp(\mathrm{i}tq)x(0),$$

as expected.

References

[1] J.-H. Eschenburg, A.-L. Mare, *Steepest descent on real flag manifolds*, preprint

[2] M. A. Guest, Y. Ohnita, *Group actions and deformations for harmonic maps*, J. Math. Soc. Japan, **45** 1993, 671-704

[3] M. A. Guest, *Harmonic Maps, Loop Groups and Integrable Systems*, LMS Student Texts 38, Cambridge Univ. Press, 1997

[4] S. Helgason: *Differential Geometry, Lie groups and Symmetric Spaces*, Academic Press 1978