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REAL LOCI OF BASED LOOP GROUPS

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Abstract. Let (G, K) be a Riemannian symmetric pair of maximal rank, where G is a compact simply connected Lie group and K is the fixed point set of an involutive automorphism σ. This induces an involutive automorphism τ of the based loop space $\Omega(G)$. There exists a maximal torus $T \subset G$ such that the canonical action of $T \times S^1$ on $\Omega(G)$ is compatible with τ (in the sense of Duistermaat). This allows us to formulate and prove a version of Duistermaat's convexity theorem. Namely, the images of $\Omega(G)$ and $\Omega(G)^\tau$ (fixed point set of τ) under the $T \times S^1$ moment map on $\Omega(G)$ are equal. The space $\Omega(G)^\tau$ is homotopy equivalent to the loop space $\Omega(G/K)$ of the Riemannian symmetric space G/K. We prove a stronger form of a result of Bott and Samelson which relates the cohomology rings with coefficients in $\mathbb{Z}_2$ of $\Omega(G)$ and $\Omega(G/K)$. Namely, the two cohomology rings are isomorphic, by a degree-halving isomorphism (Bott and Samelson [BS] had proved that the Betti numbers are equal). A version of this theorem involving equivariant cohomology is also proved. The proof uses the notion of conjugation space in the sense of Hausmann, Holm, and Puppe [HHP].

Introduction

Let G be a compact connected simply connected Lie group. Consider the space

$$\Omega(G) := \{\gamma : S^1 \to G \ : \ \gamma \text{ of Sobolev class } H^1, \gamma(1) = e\}$$

of all based loops in G, where S^1 is the unit circle. It is known that $\Omega(G)$ is an infinite-dimensional symplectic manifold which behaves in many respects like a *compact* symplectic manifold. For example, let us consider the canonical action of the product $T \times S^1$ on $\Omega(G)$, where $T \subset G$ is a maximal torus. This action is described in detail in Section 1.1 below. One can show that it is Hamiltonian. Moreover, by the convexity theorem of Atiyah and Pressley [AP], the image of the corresponding moment map is a convex unbounded polyhedron. By *convex polyhedron* we always mean in this paper the convex hull of an infinite but discrete collection of points. Another instance of the same phenomenon is that the $T \times S^1$-equivariant cohomology of $\Omega(G)$ can be computed by Goresky–Kottwitz–MacPherson-type formulas: this has been obtained by Harada, Henriques, and Holm in [HHH].

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Let σ be a Lie group automorphism of G with the following properties:

- $\sigma \circ \sigma = \mathrm{id}_G$, that is, σ is an involution; and
- there exists a maximal torus $T \subset G$ such that $\sigma(t) = t^{-1}$ for all $t \in T$.

It is known (cf., e.g. [L, Chap. VI, Theorem 4.2]) that any simply connected compact Lie group G admits such an an automorphism σ. This σ is unique up to an inner automorphism of G. For example, if $G = \mathrm{SU}(n)$, σ is given by

$$\sigma((a_{k\ell})_{1\le k,\ell\le n}) = (\bar{a}_{k\ell})_{1\le k,\ell\le n}$$

for any special unitary $n \times n$ matrix $(a_{k\ell})_{1\le k,\ell\le n}$. Here the bar indicates the complex conjugate. Examples of such involutions for other Lie groups are presented in Section 3.

The automorphism σ gives rise to the involution τ of $\Omega(G)$ given by

$$\tau(\gamma)(z) = \sigma(\gamma(\bar{z})) \tag{1}$$

for all $\gamma \in \Omega(G)$ and $z \in S^1$. One can see that τ is an antisymplectic automorphism of $\Omega(G)$, that is, it satisfies $\tau^*(\omega) = -\omega$, where ω is the symplectic form of $\Omega(G)$ (cf. [K]). The automorphism τ of $\Omega(G)$ is compatible with the $T \times S^1$ action mentioned above: that is, we have

$$\tau((t,z).\gamma) = (t^{-1}, z^{-1}).\tau(\gamma) \tag{2}$$

for all $(t,z) \in T \times S^1$ and all $\gamma \in \Omega(G)$ (see Proposition 2.2.4 below). Real loci of compact (finite-dimensional) symplectic manifolds with compatible torus actions have been investigated by several authors, like Duistermaat [D], O'Shea and Sjamaar [OSS], Biss, Guillemin, and Holm [BGH], and Hausmann, Holm, and Puppe [HHP]. The loop space $\Omega(G)$ is infinite-dimensional, thus we cannot directly apply the results in the papers above. The goal of our paper is to show that the following two results can be extended to $\Omega(G)$: the Duistermaat convexity theorem (cf. [D], see also Theorem 1.1.1 below) and a more recent result of Hausmann, Holm, and Puppe which relates the (equivariant) cohomology rings of the manifold and of the fixed point set of the involutive automorphism (cf. [HHP]). More precisely, we prove Theorems 1 and 2 below. The first theorem concerns the moment map of the $T \times S^1$ action on $\Omega(G)$, which is a map from $\Omega(G)$ to $(\mathrm{Lie}(T) \oplus \mathbb{R})^*$. The explicit description of this map is given in Section 1. It turns out that it is more convenient to describe it by endowing $\mathrm{Lie}(G)$ with an $\mathrm{Ad}(G)$-invariant inner product and restricting it to $\mathrm{Lie}(T)$, and then endowing $\mathbb{R}$ with the canonical inner product: we identify in this way $(\mathrm{Lie}(T) \oplus \mathbb{R})^* = \mathrm{Lie}(T) \oplus \mathbb{R}$.

Theorem 1. *If $\Phi : \Omega(G) \to \mathrm{Lie}(T) \oplus \mathbb{R}$ is the moment map of the $T \times S^1$ action, then we have*

$$\Phi(\Omega(G)) = \Phi(\Omega(G)^\tau).$$

Here $\Omega(G)^\tau$ denotes the fixed point set of τ.

Remarks. 1. Let us consider the more general situation when σ is an *arbitrary* involutive Lie group automorphism of G. The differential map $d\sigma_e$ is an involutive

Lie algebra automorphism of $\mathfrak{g} := \mathrm{Lie}(G)$. Let $\mathfrak{k}, \mathfrak{p} \subset \mathfrak{g}$ denote the corresponding $+1$ (resp. -1) eigenspaces. We have $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{g}$ with $\mathfrak{a} \subset \mathfrak{p}$. Then $A := \exp(\mathfrak{a})$ is a torus in G (cf., e.g. [H, Chap. VII]). Let $T \subset G$ be a maximal torus such that $A \subset T$. Consider again the involution τ of $\Omega(G)$ given by equation (1). Again, τ is an antisymplectic automorphism of $\Omega(G)$. The subgroup $A \times S^1$ of $T \times S^1$ acts on $\Omega(G)$ and this action is compatible with τ. Let $\Phi_A : \Omega(G) \to \mathfrak{a} \oplus \mathbb{R}$ be the moment map of the $A \times S^1$ action. Here we have used the identification $(\mathfrak{a} \oplus \mathbb{R})^* = \mathfrak{a} \oplus \mathbb{R}$. Both $\Phi_A(\Omega(G))$ and $\Phi_A(\Omega(G)^\tau)$ are convex polyhedra in $\mathfrak{a} \oplus \mathbb{R}$: the first by Atiyah and Pressley's theorem mentioned above, the second by the convexity theorem of Terng [T3, Theorem 1.6] for infinite-dimensional isoparametric submanifolds (for more details, see Section 1.2 below). It is not known whether $\Phi_A(\Omega(G)^\tau) = \Phi_A(\Omega(G))$. In the special case when A is a maximal torus T in G, this equality is the content of Theorem 1 above. The proof of this theorem relies essentially on the fact that any based loop in G which is fixed by the $T \times S^1$ action is automatically fixed by τ, too (see Section 1.1). Since this is not true for arbitrary σ, the proof breaks down in the general situation.

2. It is also worth investigating whether the result in Theorem 1 remains true if instead of loops of Sobolev class H^1 we consider other classes of loops. For example, let us consider the space $\Omega_{\mathrm{alg}}(G)$ of algebraic loops in G (see Section 2.1 for the exact definition of this notion). Note that $\Omega_{\mathrm{alg}}(G)$ is a $T \times S^1$ invariant subspace of $\Omega(G)$. Atiyah and Pressley [AP] showed that we have $\Phi(\Omega(G)) = \Phi(\Omega_{\mathrm{alg}}(G))$. Consequently, the latter set is also an unbounded convex polyhedron. The automorphism τ leaves $\Omega_{\mathrm{alg}}(G)$ invariant. We do not know whether $\Phi(\Omega_{\mathrm{alg}}(G)^\tau) = \Phi(\Omega_{\mathrm{alg}}(G))$. An important step towards the proof of this conjecture would be made by taking the Bruhat cells in $\Omega_{\mathrm{alg}}(G)$ (see Section 2.1 below) and their closures, which are finite-dimensional projective varieties. They are both $T \times S^1$ and τ invariant. One should first verify whether for any such variety X we have $\Phi(X) = \Phi(X^\tau)$.

3. The proof of Theorem 1 will be given in Section 1. The main ingredients of the proof are as follows. First, the set $\Phi(\Omega(G)^\tau)$ is a convex subset of $\mathfrak{t} \oplus \mathbb{R}$, which is a fact proved in Section 1.2; second, by [AP, Sect. 1, Remark 2], the vertices of Atiyah and Pressley's polyhedron $\Phi(\Omega(G))$ are of the form $\Phi(\lambda)$, where $\lambda : S^1 \to T$ is a group homomorphism.

The following is the second main result of the paper.

Theorem 2. *One has the following two ring isomorphisms:*

$$\text{(a)}\ \ H^{2*}(\Omega(G); \mathbb{Z}_2) \simeq H^*(\Omega(G)^\tau; \mathbb{Z}_2),$$
$$\text{(b)}\ \ H^{2*}_{T \times S^1}(\Omega(G); \mathbb{Z}_2) \simeq H^*_{T_2 \times \mathbb{Z}_2}(\Omega(G)^\tau; \mathbb{Z}_2),$$

where $T_2 \times \mathbb{Z}_2 := \{(t, z) \in T \times S^1 \ : \ t^2 = 1$ *and* $z = \pm 1\}$.

Note that the right-hand side of equation (b) above is well defined: by the compatibility condition (2), the group $T_2 \times \mathbb{Z}_2$ leaves $\Omega(G)^\tau$ invariant.

This theorem is related to a result of Bott and Samelson [BS] concerning the space of loops in a symmetric space. To be more precise, let G be, as before, a compact simply connected Lie group and σ a group automorphism of G with the property that $\sigma \circ \sigma = \mathrm{id}_G$. The assumption that $\sigma(t) = t^{-1}$ for all $t \in T$

is temporarily dropped. Then the fixed point set $K = G^\sigma$ is a connected closed subgroup of G and the homogeneous space G/K has a natural structure of a Riemannian symmetric space. Explicit formulas for the $\mathbb{Z}_2$ Betti numbers of the loop space $\Omega(G/K)$ are given in [BS, Cor. 3.10]. This result also gives the $\mathbb{Z}_2$ Betti numbers of $\Omega(G)^\tau$, since the latter space is homotopy equivalent to $\Omega(G/K)$ (see, for instance, Proposition 2.2.6 below).

We now recall the assumption that $\sigma(t) = t^{-1}$ for all $t \in T$. This implies that G/K is a symmetric space of maximal rank (i.e., rank $G/K =$ rank G). Under this assumption, Bott and Samelson proved that

$$\dim H^{2q}(\Omega(G); \mathbb{Z}_2) = \dim H^q(\Omega(G/K); \mathbb{Z}_2) \tag{3}$$

for all $q \geq 0$ (see [BS, Prop. 4.1]). The homotopy equivalence between $\Omega(G)^\tau$ and $\Omega(G/K)$ mentioned above is $(T_2 \times \mathbb{Z}_2)$-equivariant with respect to a certain natural action of $T_2 \times \mathbb{Z}_2$ on $\Omega(G/K)$ (see Section 2.2 below, especially Proposition 2.2.6). Consequently, $\Omega(G)^\tau$ and $\Omega(G/K)$ have the same cohomology rings, both equivariant and nonequivariant. In this way, the following result can be deduced from Theorem 2. Before stating it, we note that it is a stronger form of the result given by equation (3).

Corollary 3. *If G/K is a symmetric space of maximal rank, then one has the following two ring isomorphisms:*

(a) $H^{2*}(\Omega(G); \mathbb{Z}_2) \simeq H^*(\Omega(G/K); \mathbb{Z}_2)$,
(b) $H^{2*}_{T\times S^1}(\Omega(G); \mathbb{Z}_2) \simeq H^*_{T_2\times\mathbb{Z}_2}(\Omega(G/K); \mathbb{Z}_2)$.

Remarks. 1. The following result was also proved by Bott and Samelson. As usual, G/K is a Riemannian symmetric space of maximal rank. Take $x \in \mathfrak{t}$ and the orbits $\mathrm{Ad}_G(G)x = G/G_x$ and $\mathrm{Ad}_G(K)x = K/K_x$. We have

$$\dim H^{2q}(G/G_x; \mathbb{Z}_2) = \dim H^q(K/K_x; \mathbb{Z}_2)$$

for all $q \geq 0$ (see [BS, Prop. 4.3]). Stronger forms of this result have been obtained by Hausmann, Holm, and Puppe in [HHP]. Namely, they proved the following ring isomorphisms:

$$H^{2*}(G/G_x; \mathbb{Z}_2) \simeq H^*(K/K_x; \mathbb{Z}_2), \tag{4}$$

$$H^{2*}_T(G/G_x; \mathbb{Z}_2) \simeq H^*_{T_2}(K/K_x; \mathbb{Z}_2), \tag{5}$$

where $T_2 := \{t \in T \ : \ t^2 = 1\}$. The main idea of their proof is that σ induces an antisymplectic involutive automorphism of G/G_x, which is compatible with the T action and whose fixed point set is K/K_x; the upshot is that this automorphism together with the Schubert cell decomposition makes G/G_x into a *spherical conjugation complex*, and this automatically implies the isomorphisms (4) and (5). Our proof of Theorem 2 uses a similar argument. Namely, we use the Bruhat cell decomposition of the space $\Omega_{\mathrm{alg}}(G)$ in order to show that this, together with the involution τ, is a spherical conjugation complex. These arguments also show that Theorem 2 remains valid if $\Omega(G)$ is replaced by $\Omega_{\mathrm{alg}}(G)$. Finally, we use a theorem which says that the inclusion $\Omega_{\rm alg}(G) \hookrightarrow \Omega(G)$ is a homotopy equivalence. The details can be found in Section 2.

2. The following result can also be proved by using the methods of our paper, combined with a theorem of Franz and Puppe (see [FP, Theorem 1.3]). Let κ denote any of the two isomorphisms given at points (a) and (b) of Corollary 3, which are maps from the (equivariant) cohomology of $\Omega(G)$ to the (equivariant) cohomology of $\Omega(G/K)$. Then we have

$$\kappa \circ {\rm Sq}^{2q} = {\rm Sq}^q \circ \kappa$$

for all $q \geq 0$. Here ${\rm Sq}^{2q}$ and ${\rm Sq}^q$ denote the Steenrod squaring operations on the (equivariant) cohomology rings of $\Omega(G)$ (resp. $\Omega(G/K)$).

Note. By S^1 we will interchangeably denote the unit circle in the complex plane and the quotient space $\mathbb{R}/2\pi\mathbb{Z}$. It will be clear from the context which of these two presentations is used.

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1. The image of $\Omega(G)^\tau$ under the moment map

1.1. Duistermaat type convexity for $(\Omega(G), \tau, T \times S^1)$

Duistermaat proved the following theorem.

Theorem 1.1.1 ([D]). *Let M be a compact symplectic manifold equipped with a Hamiltonian action of a torus $\mathcal{T}$ and an antisymplectic involution ρ which are compatible, in the sense that*

$$\rho(tx) = t^{-1}\rho(x) \tag{6}$$

for all $t \in \mathcal{T}$ and all $x \in M$. If $\mu : M \to {\rm Lie}(\mathcal{T})^*$ is the moment map of the $\mathcal{T}$ action, then we have

$$\mu(M) = \mu(M^\rho),$$

where M^ρ is the fixed point set of ρ.

Our Theorem 1 is an extension of this result to the case where M, ρ are $\Omega(G)$, respectively τ. We note that $\Omega(G)$ is an infinite-dimensional manifold, so Theorem 1.1.1 does not apply directly to this situation. In this section we prove Theorem 1. The considerations made in the Introduction right before stating this theorem are in force here. We denote by $\mathfrak{g}$ the Lie algebra of G and choose an ${\rm Ad}(G)$ invariant inner product on $\mathfrak{g}$ (e.g. the negative of the Killing form): if $X \in \mathfrak{g}$ then $|X|$ denotes the length of X.

We consider the action of T on $\Omega(G)$ given by pointwise conjugation of loops, that is,

$$(t.\gamma)(\theta) = t\gamma(\theta)t^{-1}, \tag{7}$$

for all $\gamma \in \Omega)G)$, $t \in T$, and $\theta \in S^1$. There is also an action of S^1 on $\Omega(G)$, given by the rotation of loops. Concretely, if $e^{i\varphi} \in S^1$ and $\gamma \in \Omega(G)$, then

$$(e^{i\varphi}.\gamma)(\theta) := \gamma(\theta + \varphi)\gamma(\varphi)^{-1} \tag{8}$$

for all $\theta \in S^1$.

The details concerning the following results can be found for instance in [AP]. First, the moment map of the T action on $\Omega(G)$ is $p : \Omega(G) \to \mathfrak{t}$ given by

$$p(\gamma) = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{pr}_{\mathfrak{t}}(\gamma(\theta)^{-1}\gamma'(\theta))\, d\theta = \mathrm{pr}_{\mathfrak{t}}\left(\frac{1}{2\pi}\int_0^{2\pi} \gamma(\theta)^{-1}\gamma'(\theta)\, d\theta\right),$$

where $\mathfrak{t}$ is the Lie algebra of T and $\mathrm{pr}_{\mathfrak{t}} : \mathfrak{g} \to \mathfrak{t}$ is the orthogonal projection. The factors $1/4\pi$ in equation (9) and $1/2\pi$ in equation (10) are due to a canonical choice of the symplectic form on $\Omega(G)$, cf., e.g. [AP]. Second, the moment map of the S^1 action on $\Omega(G)$ is the energy functional $E : \Omega(G) \to \mathbb{R}$,

$$E(\gamma) = \frac{1}{4\pi}\int_0^{2\pi} |\gamma(\theta)^{-1}\gamma'(\theta)|^2 d\theta. \tag{9}$$

The actions of T and S^1 commute with each other. The moment map of the $T \times S^1$ action is

$$\Phi = p \times E : \Omega(G) \to \mathfrak{t} \oplus \mathbb{R}.$$

The following theorem was proved by Atiyah and Pressley in [AP].

Theorem 1.1.2 ([AP]). *We have*

$$\Phi(\Omega(G)) = \mathrm{cvx}\{\Phi(\lambda) \ : \ \lambda : S^1 \to T \textit{ is a group homomorphism}\},$$

where cvx *stands for convex hull.*

We note that the group homomorphisms $S^1 \to T$ are precisely the elements of $\Omega(G)$ which are fixed by the $T \times S^1$ action.

An important ingredient of this section is the following result, which is a consequence of the convexity theorem of Terng (see [T3]). This is an immediate corollary of Theorem 1.2.1, proved in Section 1.2 below.

Theorem 1.1.3. *The space $\Phi(\Omega(G)^\tau)$ is a convex subset of $\mathfrak{t} \oplus \mathbb{R}$.*

We are now ready to prove Theorem 1.

Proof of Theorem 1. First note that

$$\Phi(\Omega(G)^\tau) \subset \Phi(\Omega(G)).$$

To prove the opposite inclusion, we note that if $\lambda : S^1 \to T$ is a group homomorphism, then $\lambda \in \Omega(G)^\tau$. Indeed, for any $z \in S^1$, we have

$$\tau(\lambda)(z) = \sigma(\lambda(\bar{z})) = \sigma(\lambda(z^{-1})) = \sigma(\lambda(z)^{-1}) = \lambda(z).$$

From Theorem 1.1.2 we deduce that $\Phi(\Omega(G))$ is in the convex hull of points which are all in $\Phi(\Omega(G)^\tau)$. Since the latter set is convex (by Theorem 1.1.3), we deduce that $\Phi(\Omega(G)) \subset \Phi(\Omega(G)^\tau)$. This finishes the proof. □

1.2. Convexity for $(\Omega(G)^\tau, A \times S^1)$

The goal of this subsection is to prove Theorem 1.1.3. In fact we will prove a stronger form of it. Namely, we consider the situation described in Remark 1 following Theorem 1 and we show as follows.

Theorem 1.2.1. *The space $\Phi_A(\Omega(G)^\tau)$ is a convex subset of $\mathfrak{a} \oplus \mathbb{R}$.*

The following expression of the moment map $\Phi_A : \Omega(G) \to \mathfrak{a} \oplus \mathbb{R}$ will be needed in the proof (it can be deduced immediately from the description of $\Phi : \Omega(G) \to \mathfrak{t} \oplus \mathbb{R}$ given in the previous subsection): we have $\Phi_A = p_A \times E$, where

$$p_A(\gamma) = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{pr}_{\mathfrak{a}}(\gamma(\theta)^{-1}\gamma'(\theta))\, d\theta = \mathrm{pr}_{\mathfrak{a}}\left(\frac{1}{2\pi}\int_0^{2\pi} \gamma(\theta)^{-1}\gamma'(\theta)\, d\theta\right). \tag{10}$$

We also need the following considerations, which can be found in [T4]. We consider the loop group

$$L(G) = \{\gamma : S^1 \to G \ : \ \gamma \text{ of Sobolev class } H^1\}.$$

It acts by "gauge transformations" on the Hilbert space $H^0(S^1, \mathfrak{g})$, by

$$\gamma \star u = \gamma u \gamma^{-1} - \gamma'\gamma^{-1} \tag{11}$$

for all $\gamma \in L(G)$ and $u \in H^0(S^1, \mathfrak{g})$. The stabilizer of the constant loop $0 \in H^0(S^1, \mathfrak{g})$ consists of all $\gamma \in L(G)$ with $\gamma'\gamma^{-1} = 0$, which means that γ is a constant loop in G. We deduce that the $L(G)$ orbit of 0 can be identified with $L(G)/G$, which is the same as $\Omega(G)$. Henceforth, we will make the identification

$$\Omega(G) = L(G) \star 0 = \{\gamma'\gamma^{-1} \ : \ \gamma \in \Omega(G)\}, \tag{12}$$

which is a subspace of $H^0(S^1, \mathfrak{g})$: more precisely, any based loop $\gamma : S^1 \to G$ of Sobolev class H^1 is identified with $\gamma \star 0 = \gamma^{-1}\gamma'$, which is an element of $H^0(S^1, \mathfrak{g})$. In this way, the moment map corresponding to the $T \times S^1$ action on $\Omega(G)$ is $\Phi : \Omega(G) \to \mathfrak{t} \oplus \mathbb{R}$,

$$\Phi(u) = (P_{\mathfrak{t}}(u), \tfrac{1}{2}\|u\|^2) \tag{13}$$

for all $u \in \Omega(G)$. Here we regard $\mathfrak{t}$ as a subspace of $H^0(S^1, \mathfrak{g})$ (consisting of constant loops) and we denote by $P_{\mathfrak{t}} : H^0(S^1, \mathfrak{g}) \to \mathfrak{t}$ the orthogonal projection with respect to the canonical inner product on $H^0(S^1, \mathfrak{g})$. We recall that this is given by

$$(u, v) = \frac{1}{2\pi}\int_0^{2\pi} \langle u(\theta), v(\theta)\rangle\, d\theta \tag{14}$$

for all $u, v \in H^0(S^1, \mathfrak{g})$ (here $\langle\ ,\ \rangle$ is the $\mathrm{Ad}(G)$ invariant inner product on $\mathfrak{g}$ we chose at the beginning of this section). By $\|\cdot\|$ we denote the corresponding norm on $H^0(S^1, \mathfrak{g})$. To justify equation (13), we show that

$$P_{\mathfrak{t}}(u) = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{pr}_{\mathfrak{t}}(u(\theta))\, d\theta, \tag{15}$$

$$\|u\|^2 = \frac{1}{2\pi}\int_0^{2\pi} |u(\theta)|^2\, d\theta, \tag{16}$$

for all $u \in H^0(S^1, \mathfrak{g})$ (see also equations (9) and (10)). equation (16) follows immediately from (14). To prove (15), we consider an orthonormal basis $e_1, \ldots, e_r$ of $\mathfrak{t}$, in the sense that $\langle e_i, e_j \rangle = \delta_{ij}$ for all $1 \leq i, j \leq r$ (here δ_{ij} is the Kronecker delta). By using equation (14), we deduce that $(e_i, e_j) = \delta_{ij}$ for all $1 \leq i, j \leq r$. Thus

$$\begin{aligned} P_{\mathfrak{t}}(u) &= \sum_{i=1}^{r} (u, e_i) e_i = \sum_{i=1}^{r} \left(\frac{1}{2\pi} \int_0^{2\pi} \langle u(\theta), e_i \rangle \, d\theta \right) e_i \\ &= \frac{1}{2\pi} \int_0^{2\pi} \sum_{i=1}^{r} \langle u(\theta), e_i \rangle e_i \, d\theta = \frac{1}{2\pi} \int_0^{2\pi} \mathrm{pr}_{\mathfrak{t}}(u(\theta)) \, d\theta. \end{aligned}$$

Equation (13) is now completely justified.

We recall now that σ is an involution of G whose fixed point set is K. We denote

$$\hat{K} := \{\gamma \in L(G) \ : \ \gamma(-\theta) = \sigma(\gamma(\theta)) \text{ for all } \theta \in S^1\}.$$

This is a subgroup of $L(G)$ which leaves invariant the closed vector subspace

$$\hat{\mathfrak{p}}(\mathfrak{g}, \sigma) := \{u \in H^0(S^1, \mathfrak{g}) \ : \ u(-\theta) = -d\sigma_e(u(\theta)) \text{ for all } \theta \in S^1\}$$

of $H^0(S^1, \mathfrak{g})$. As before, $\mathfrak{a}$ is a maximal abelian subspace of $\mathfrak{p}$. It can be made into a subspace of $\hat{\mathfrak{p}}(\mathfrak{g}, \sigma)$ by regarding every element of $\mathfrak{a}$ as a constant loop. In what follows we will need the notion of *isoparametric submanifold* in Hilbert space. By definition, this is a finite-codimensional Riemannian submanifold for which the normal vector bundle is flat relative to the normal connection and satisfies some other assumptions: for instance, if v is a parallel normal vector field on the manifold, then the shape operators $A_{v(p)}$ and $A_{v(q)}$ corresponding to any two points p and q on the manifold are orthogonally conjugate. For the exact definition we refer the reader to [T1, Sect. 6] (see also Chapter 7 of the monograph [PT]). We note that any isoparametric submanifold induces a foliation of Hilbert space by parallel submanifolds, which are not necessarily isoparametric submanifolds. We will refer to this as an isoparametric foliation.

Proposition 1.2.2.

(a) *The orbits of the $\hat{K}$ action on $\hat{\mathfrak{p}}(\mathfrak{g}, \sigma)$ are elements of an isoparametric foliation of the Hilbert space $\hat{\mathfrak{p}}(\mathfrak{g}, \sigma)$.*

(b) *There exists $a \in \mathfrak{a}$ such that the orbit $\hat{K} \star a$ is an isoparametric submanifold of $\hat{\mathfrak{p}}(\mathfrak{g}, \sigma)$. The normal space at a to this submanifold is $\mathfrak{a}$.*

Proof. We use the following identifications (see also [T2, Remark 3.4]):

$$\hat{K} = \{\gamma : [0, \pi] \to G \ : \ \gamma(0), \gamma(\pi) \in K\} =: P(G, K \times K),$$
$$\hat{\mathfrak{p}}(\mathfrak{g}, \sigma) = H^0([0, \pi], \mathfrak{g}).$$

By [T4, Theorem 1.2] the action of $P(G, K \times K)$ on $H^0([0, \pi], \mathfrak{g})$ given by (11) is polar (by definition, which can be found in full detail in [T4], this means essentially that there exists a section of this action, that is, a submanifold of $H^0([0, \pi], \mathfrak{g})$

which meets all orbits of the action and meets them orthogonally). By [T1, Theorem 8.10] the orbits of this action are an isoparametric foliation. In particular, the principal orbits are isoparametric submanifolds. We are looking for such orbits. To find them, we recall that the action of $K \times K$ on G given by

$$(k_1, k_2).g = k_1 g k_2^{-1}$$

for all $k_1, k_2 \in K$ and $g \in G$ is polar; a section of this action is $A = \exp(\mathfrak{a})$ (cf., e.g. [C]). From [T4, Theorem 1.2] we deduce that $\mathfrak{a}$ (the space of constant maps from $[0, \pi]$ to $\mathfrak{a}$) is a section of the $P(G, K \times K)$ action on $H^0([0, \pi], \mathfrak{g})$. To prove our proposition, we only need to pick $a \in \mathfrak{a}$ a regular point (i.e., one whose orbit is principal). Such a point exists due to the following criterion (see [T4, Theorem 1.2, (6)]): a point $a \in \mathfrak{a}$ is regular for the $P(G, K \times K)$ action on $H^0([0, \pi], \mathfrak{g})$ if and only if $\exp(a)$ is regular for the $K \times K$ action on G. Moreover, a general result says that any section of a polar action of a compact Lie group on a simply connected compact manifold contains regular points (see, e.g. [T4, Theorem 1.6]). This finishes the proof. $\square$

In order to prove Theorem 1.1.3 we will show that, via the identification (12), $\Omega(G)^\tau$ is the same as the element $\hat{K}\star 0$ of the isoparametric foliation in the previous proposition. Then we use the convexity theorem for isoparametric foliations of Terng [T3]. For the moment, we will prove the following lemma.

Lemma 1.2.3. *Take $\gamma \in \Omega(G)$ and denote $\gamma_0 = \tau(\gamma)$. Then we have*

$$\gamma_0{}'(\theta)\gamma_0{}^{-1}(\theta) = -d\sigma_e(\gamma'(-\theta)\gamma^{-1}(-\theta))$$

for all $\theta \in S^1$.

Proof. If $g \in G$, then the tangent space to G at g consists of vectors of the form $Xg = dR(g)_e(X)$, where $X \in T_eG$. Here $R(g) : G \to G$ is the right multiplication by g. Moreover, we have

$$d\sigma_g(Xg) = d\sigma_e(X)\sigma(g).$$

Indeed,

$$d\sigma_g(Xg) = d\sigma_g(dR(g)_e(X)) = d(\sigma\circ R(g))_e(X) = d(R(\sigma(g))\circ\sigma)_e(X) = d\sigma_e(X)\sigma(g).$$

We deduce that

$$\begin{aligned}\gamma_0{}'(\theta)\gamma_0{}^{-1}(\theta) &= d\sigma_{\gamma(-\theta)}(-\gamma'(-\theta))\sigma(\gamma(-\theta)^{-1})\\ &= -d\sigma_{\gamma(-\theta)}(\gamma'(-\theta)\gamma(-\theta)^{-1}\gamma(-\theta))\sigma(\gamma(-\theta)^{-1})\\ &= -d\sigma_e(\gamma'(-\theta)\gamma(-\theta)^{-1}). \qquad \square\end{aligned}$$

From this lemma we deduce

$$\Omega(G)^\tau = \{u \in \Omega(G) \ : \ -d\sigma_e(u(\theta)) = u(-\theta) \text{ for all } \theta \in S^1\} = \Omega(G)\cap\hat{\mathfrak{p}}(\mathfrak{g}, \sigma). \quad (17)$$

This space is the same as the orbit $\hat{K} \star 0$, as the following lemma shows.

Lemma 1.2.4. *We have*

$$\Omega(G)^\tau = \hat{K} \star 0.$$

Proof. The inclusion $\hat{K} \star 0 \subset \Omega(G)^\tau$ is clear, because $\hat{K} \star 0$ is a subset of both $\hat{\mathfrak{p}}(\mathfrak{g}, \sigma)$ and $L(G) \star 0$. We now prove the reverse inclusion. Take $\gamma \in \Omega(G)^\tau$: by identifying it with the element $\gamma \star 0 = \gamma'\gamma^{-1}$ of $H^0(S^1, \mathfrak{g})$ and taking into account equation (17), we have

$$d\sigma_e(\gamma'(\theta)\gamma^{-1}(\theta)) = -\gamma'(-\theta)\gamma^{-1}(-\theta)$$

for all $\theta \in S^1$. We show that $\gamma \in \hat{K}$, as follows. We have

$$\begin{aligned}\frac{d}{d\theta}\sigma(\gamma(\theta)) &= d\sigma_{\gamma(\theta)}(\gamma'(\theta))\\ &= d\sigma_e(\gamma'(\theta)\gamma^{-1}(\theta))\sigma(\gamma(\theta))\\ &= -\gamma'(-\theta)\gamma^{-1}(-\theta)\sigma(\gamma(\theta)),\end{aligned}$$

which implies

$$\frac{d}{d\theta}(\sigma(\gamma(\theta)))\sigma(\gamma(\theta))^{-1} = \frac{d}{d\theta}(\gamma(-\theta))\gamma(-\theta)^{-1}.$$

We deduce that the loops $\theta \mapsto \sigma(\gamma(\theta))$ and $\theta \mapsto \gamma(-\theta)$ are equal. Thus $\tau(\gamma) = \gamma$, in other words, $\gamma \in \hat{K}$. $\square$

We are now ready to prove our main result.

Proof of Theorem 1.2.1. By the convexity theorem of Terng (see [T3, Theorem 1.6]), the image of the map $\Psi_A : \hat{K} \star 0 \to \mathfrak{a} \oplus \mathbb{R}$ given by

$$\Psi_A(u) = (P_{\mathfrak{a}}(u), \|u\|^2)$$

is a convex polyhedron in $\mathfrak{a} \oplus \mathbb{R}$ (we are also using Proposition 1.2.2). Here $P_{\mathfrak{a}} : \hat{\mathfrak{p}}(\mathfrak{g}, \sigma) \to \mathfrak{a}$ is the orthogonal projection with respect to the Hilbert space metric. By Lemma 1.2.4, $\Psi_A(\Omega(G)^\tau)$ is a convex polyhedron. If we now compare the map Ψ_A with the moment map $\Phi_A = p_A \times E$ (see equations (9) and (10)), we note that the two maps are essentially the same. More specifically, by taking into account the identification given by (12), we have

$$\Phi_A(u) = \left(P_{\mathfrak{a}}(u), \tfrac{1}{2}\|u\|^2\right)$$

for all $u \in \Omega(G)^\tau$ (this can be proved in the same way as equation (13)). We deduce that the set $\Phi_A(\Omega(G)^\tau)$ is obtained from $\Psi_A(\Omega(G)^\tau)$ by the automorphism of $\mathfrak{a} \oplus \mathbb{R}$ given by

$$(a, r) \mapsto \left(a, \tfrac{1}{2}r\right)$$

for all $(a, r) \in \mathfrak{a} \oplus \mathbb{R}$. Thus $\Phi_A(\Omega(G)^\tau)$ is a convex polyhedron as well. This finishes the proof. $\square$

2. (Equivariant) cohomology ring of $\Omega(G/K)$

2.1. (Equivariant) cohomology of $\Omega(G)^\tau$

In this subsection we will prove Theorem 2. An important ingredient of the proof will be the space $\Omega_{\rm alg}(G)$ of algebraic loops in G. By definition, this is

$$\Omega_{\rm alg}(G) = L_{\rm alg}(G^{\mathbb{C}}) \cap \Omega(G).$$

Here $G^{\mathbb{C}}$ denotes the complexification of the Lie group G and $L_{\rm alg}(G^{\mathbb{C}})$ is the set of all (free) loops $\gamma : S^1 \to G^{\mathbb{C}}$ which are restrictions of algebraic maps from $\mathbb{C}^*$ to $G^{\mathbb{C}}$. In the case where $G^{\mathbb{C}}$ is a subgroup of some general linear group $GL_n(\mathbb{C})$, elements of $L_{\rm alg}(G^{\mathbb{C}})$ are Laurent series of the form

$$\gamma(z) = \sum_{p=-k}^{k} z^p A_p \tag{18}$$

for some $k \geq 0$, where A_p are elements of ${\rm Mat}^{n\times n}(\mathbb{C})$. For a fixed k, the space of all maps γ of the form (18) is equipped with the standard metric topology which comes from its identification with $({\rm Mat}^{n\times n}(\mathbb{C}))^{2k+1}$; we denote by $\Omega^k_{\rm alg}(G)$ the space of all γ of type (18) which map S^1 to G, and equip it with the subspace topology. We endow $\Omega_{\rm alg}(G)$ with the direct limit topology coming from the filtration $\{\Omega^k_{\rm alg}(G)\}_{k\geq 0}$. The following theorem has been proved by Mitchell in [M] (see Theorem 4.1 and the theorem in the Introduction of his paper, where the result is attributed to Quillen). Another proof can be found in [K] (see Theorem 3.1.4 of that paper).

Theorem 2.1.1 ([M], [K]).

(a) *The inclusion map $\Omega_{\rm alg}(G) \to \Omega(G)$ is a homotopy equivalence.*
(b) *The automorphism τ of $\Omega(G)$ leaves $\Omega_{\rm alg}(G)$ invariant and the inclusion map $\Omega_{\rm alg}(G)^\tau \to \Omega(G)^\tau$ is a homotopy equivalence.*

The advantage of dealing with $\Omega_{\rm alg}(G)$ instead of $\Omega(G)$ is that the former space has a natural CW-decomposition. Its elements are the Bruhat cells, which are described in what follows (the details of this construction can be found in [M, Sects. 2 and 3]). First, we make the identification

$$\Omega_{\rm alg}(G) = L_{\rm alg}(G^{\mathbb{C}})/L^+_{\rm alg}(G^{\mathbb{C}}), \tag{19}$$

where $L^+_{\rm alg}(G^{\mathbb{C}})$ is the subgroup of $L_{\rm alg}(G^{\mathbb{C}})$ consisting of loops of the form (18) for some $k \geq 0$, where $A_p = 0$ for all $p < 0$. We consider the roots of G with respect to T, which are linear functions $\mathfrak{t} \to \mathbb{R}$. The root space decomposition of $\mathfrak{g}^{\mathbb{C}} := \mathfrak{g} \otimes \mathbb{C}$ is

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{t}^{\mathbb{C}} \oplus \sum_{\alpha} \mathfrak{g}^{\mathbb{C}}_{\alpha},$$

where the sum runs over all the roots of G with respect to T. We fix a simple root system $\alpha_1, \ldots, \alpha_\ell$ and denote by B^- the (Borel) connected subgroup of $G^{\mathbb{C}}$

whose Lie algebra is $\mathfrak{t}^{\mathbb{C}} \oplus \sum_\alpha \mathfrak{g}^{\mathbb{C}}_\alpha$, where the sum runs over all negative roots α. The Bruhat decomposition of $\Omega_{\rm alg}(G)$ is

$$\Omega_{\rm alg}(G) = \bigsqcup_\lambda \mathcal{B}\lambda, \tag{20}$$

where the union runs over all group homomorphisms $\lambda : S^1 \to T$ such that $\lambda'(0)$ is in the closure of the fundamental Weyl chamber of $\mathfrak{t}$. Here $\mathcal{B}$ is the subgroup of $L^+_{\rm alg}(G^{\mathbb{C}})$ consisting of all loops γ of the form (18) for some $k \geq 0$, where $A_p = 0$ for all $p < 0$ and $A_0 \in B^-$. The decomposition described by (20) is a CW decomposition (cf., e.g. [M, Sect. 3]). The orbits $\mathcal{B}\lambda$ are the Bruhat cells.

Any Bruhat cell is homeomorphic to some complex vector space. Proposition 2.1.1 below gives a more precise description of this homeomorphism. In order to state it, we need to make some more considerations. First, we note that the set of group homomorphisms $\lambda : S^1 \to T$ can be identified with the integer lattice $I = \ker(\exp : \mathfrak{t} \to T)$. Let W be the Weyl group of G. We recall that this is the group of linear transformations of $\mathfrak{t}$ generated by the reflections about the hyperplanes $\ker \alpha_1, \ker \alpha_2, \ldots, \ker \alpha_\ell$; let us denote these reflections by $s_1, s_2, \ldots, s_\ell$. The affine Weyl group $\tilde{W}$ is the semidirect product $W \ltimes I$. It is the same as the group of affine transformations of $\mathfrak{t}$ generated by $s_1, s_2, \ldots, s_\ell$, and s_0. Here s_0 is the reflection about the affine hyperplane $\{x \in \mathfrak{t} \ : \ \alpha_0(x) = 1\}$, where α_0 is the highest root of G. To any $s \in \{s_0, s_1, \ldots, s_\ell\}$ we assign the subgroup U_s of $L_{\rm alg}(G^{\mathbb{C}})$, as follows:

- For $j \in \{1, \ldots, \ell\}$ we have $U_{s_j} := \exp(\mathfrak{g}^{\mathbb{C}}_{\alpha_j})$ (its elements are constant loops in $G^{\mathbb{C}}$). Since U_{s_j} is a unipotent group, the exponential map is an isomorphism between U_{s_j} and its Lie algebra $\mathfrak{g}^{\mathbb{C}}_{\alpha_j}$. More precisely, by fixing E_{α_j} a nonzero vector in $\mathfrak{g}^{\mathbb{C}}_{\alpha_j}$, the map $\mathbb{C} \to U_{\alpha_j}$ given by $x \mapsto \exp(xE_{\alpha_j})$ is a homeomorphism.
- U_{s_0} consists of loops of the form $z \mapsto \exp(z^{-1}X)$, $z \in S^1$, where $X \in \mathfrak{g}^{\mathbb{C}}_{-\alpha_0}$. Again, since U_{s_0} is a unipotent group, the exponential map is an isomorphism between U_{s_0} and $\mathfrak{g}^{\mathbb{C}}_{-\alpha_0}$. By fixing again $E_{-\alpha_0}$ a nonzero vector in $\mathfrak{g}^{\mathbb{C}}_{-\alpha_0}$, the map $\mathbb{C} \to U_{\alpha_0}$ which assigns to $x \in \mathbb{C}$ the loop $z \mapsto \exp(z^{-1}xE_{-\alpha_0})$ is a homeomorphism.

We mention without any further explanation that the groups U_s are the root subgroups of $L_{\rm alg}(G^{\mathbb{C}})$ corresponding to a certain canonical simple affine root system of G (note that the Lie algebra of $L_{\rm alg}(G^{\mathbb{C}})$ has a root decomposition labeled by the affine roots).

Take $\lambda \in I = \tilde{W}/W$ and consider the element $\tilde{w}$ of $\tilde{W}$ which has minimal length (with respect to the generating set $s_0, s_1, \ldots, s_\ell$) and satisfies $\lambda = \tilde{w}W$. Let $\tilde{w} = s_{i_1} \ldots s_{i_k}$ be any reduced decomposition of $\tilde{w}$, where $i_1, \ldots, i_k \in \{0, 1, \ldots, \ell\}$. The following result has been proved by Mitchell in [M].

Proposition 2.1.2 ([M]). *The map*

$$\mathbb{C}^k = U_{s_{i_1}} \times \ldots \times U_{s_{i_k}} \to L_{\rm alg}(G^{\mathbb{C}})/L^+_{\rm alg}(G^{\mathbb{C}}) = \Omega_{\rm alg}(G), \tag{21}$$
$$(u_1, \ldots, u_k) \mapsto u_1 \ldots u_k L^+_{\rm alg}(G^{\mathbb{C}}),$$

is a homeomorphism onto the Bruhat cell $\mathcal{B}\lambda$.

Let σ be the automorphism of G defined in the Introduction. We note that the involutive automorphism τ of $\Omega(G)$ given by (1) leaves $\Omega_{\rm alg}(G)$ invariant. To understand this, we first extend σ to a group automorphism of $G^{\mathbb{C}}$, namely the one whose differential at the identity element is the anticomplex linear extension of the differential of the original σ. That is, we have

$$d\sigma_e(X+iY) = d\sigma_e(X) - i\, d\sigma_e(Y) \tag{22}$$

for all $X, Y \in \mathfrak{g}$. Then we extend τ to a group automorphism of $L_{\rm alg}(G^{\mathbb{C}})$, namely the one described by equation (1) with γ in $L_{\rm alg}(G^{\mathbb{C}})$. This map leaves $L^+_{\rm alg}(G^{\mathbb{C}})$ invariant and induces the original automorphism τ of $\Omega_{\rm alg}(G)$ via the identification (19).

We now consider the decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, where $\mathfrak{k} = \{X \in \mathfrak{g} \,:\, d\sigma_e(X) = X\}$ and $\mathfrak{p} = \{X \in \mathfrak{g} \,:\, d\sigma_e(X) = -X\}$. Note that $\mathfrak{t}$ is a subset of $\mathfrak{p}$. The automorphism $d\sigma_e$ of $\mathfrak{g}^{\mathbb{C}}$ has fixed point set equal to $\mathfrak{g}_0 := \mathfrak{k} + i\mathfrak{p}$. The latter space is a real form of $\mathfrak{g}^{\mathbb{C}}$. Any root α of $\mathfrak{g}^{\mathbb{C}}$ with respect to $\mathfrak{t} \otimes \mathbb{C}$ takes real values on the subspace $i\mathfrak{t}$ of $\mathfrak{g}_0$. This means that $\mathfrak{g}_0$ is a split real form of $\mathfrak{g}^{\mathbb{C}}$ (cf., e.g [FH, Sect. 26.1]). We deduce that we have the splitting

$$\mathfrak{g}_0 = i\mathfrak{t} \oplus \sum \mathbb{R}E_\alpha,$$

where the sum runs over all the roots α of G with respect to T and E_α is a (nonzero) root vector for any such root α. In constructing the groups $U_{s_0}, U_{s_1}, \dots, U_{s_\ell}$ (see above) we use the vectors $E_{-\alpha_0}, E_{\alpha_1}, \dots, E_{\alpha_\ell}$ in the previous equation.

We will prove the following result (see also [M, Proof of Theorem 5.9]).

Proposition 2.1.3. *Any Bruhat cell* $\mathcal{B}\lambda$ *in* $\Omega_{\rm alg}(G)$ *remains invariant under* τ. *Moreover, via the homeomorphism* $\mathbb{C}^k \simeq \mathcal{B}\lambda$ *described by equation* (21), τ *acts on* $\mathcal{B}\lambda$ *by complex conjugation.*

Proof. We have already seen that if $\lambda : S^1 \to T$ is a group homomorphism then $\tau(\lambda) = \lambda$ (see the proof of Theorem 1 at the end of Section 1.1). The automorphism τ leaves $\mathcal{B}$ invariant: this follows from the definition of $\mathcal{B}$ and the fact that the Borel subgroup B^- is σ-invariant. Consequently, τ leaves the orbit $\mathcal{B}\lambda$ invariant. The homeomorphism $U_{s_{i_1}} \times \cdots \times U_{s_{i_k}} \to \mathcal{B}\lambda$ described by equation (21) is τ-equivariant, where τ acts diagonally on the domain of the map. The reason is that τ is a group automorphism of $L_{\rm alg}(G^{\mathbb{C}})$. The last statement in the proposition follows from the fact that τ leaves U_{s_j} invariant, for any $j \in \{0, 1, \dots, \ell\}$; moreover, via the identification $U_{s_j} = \mathfrak{g}^{\mathbb{C}}_{\alpha_j} = \mathbb{C}$ (see above), τ acts as the complex conjugation. Indeed, if $j \neq 0$, then $\mathfrak{g}^{\mathbb{C}}_{\alpha_j} = \mathbb{C}E_{\alpha_j}$ and by equation (22), for any $x \in \mathbb{C}$, we have

$$\tau(\exp(xE_{\alpha_j})) = \sigma(\exp(xE_{\alpha_j})) = \exp(d\sigma_e(xE_{\alpha_j})) = \exp(\overline{x}E_{\alpha_j});$$

for $j = 0$, we use that for any complex number x, the loop $z \mapsto \exp(z^{-1}xE_{-\alpha_0})$ is mapped by τ to

$$z \mapsto \sigma(\exp(zxE_{-\alpha_0})) = \exp(\overline{zx}E_{-\alpha_0})) = \exp(z^{-1}\overline{x}E_{-\alpha_0}). \qquad \square$$

Our proof of Theorem 2 uses the notion of a spherical conjugation complex, defined in [HHP]. By definition, a spherical conjugation complex is a (finite or infinite) cell complex X equipped with an involutive automorphism ρ with the following properties:

- each cell in X is a complex cell, that is, it is homeomorphic to $\mathbb{C}^k$, for some $k \in \mathbb{Z}$, $k \geq 0$.
- ρ leaves each cell $\mathbb{C}^k$ invariant, acting on it as the complex conjugation. That is, we have
$$\rho(z_1, \ldots, z_k) = (\bar{z}_1, \ldots, \bar{z}_k)$$
for all $(z_1, \ldots, z_k) \in \mathbb{C}^k$.

The following theorem has been proved in [HHP, Sects. 5 and 7].

Theorem 2.1.4 ([HHP]). *Let (X, ρ) be a spherical conjugation complex and denote by X^ρ the fixed point set of ρ. Then we have as follows:*

(a) *There exists a degree-halving ring isomorphism $H^{2*}(X; \mathbb{Z}_2) \simeq H^*(X^\rho; \mathbb{Z}_2)$.*
(b) *Let $\mathcal{T}$ be a compact torus acting on X such that the action is compatible with ρ, in the sense that*
$$\rho(tx) = t^{-1}\rho(x)$$
for all $t \in \mathcal{T}$ and all $x \in X$. Then there exists a degree-halving ring isomorphism $H^{2*}_{\mathcal{T}}(X; \mathbb{Z}_2) \simeq H^*_{\mathcal{T}_2}(X^\rho; \mathbb{Z}_2)$. Here $\mathcal{T}_2$ denotes the set of all $t \in \mathcal{T}$ with $t^2 = 1$.

Without any further comments we mention that the key point of this theorem is that a spherical conjugation complex is a conjugation space (for the definition of this notion, see [HHP]).

We are now ready to give the desired proof.

Proof of Theorem 2. By Proposition 2.1.3, $\Omega_{\mathrm{alg}}(G)$ together with the involution τ is a spherical conjugation complex. Theorem 2.1.4(a) implies that we have a ring isomorphism
$$H^{2*}(\Omega_{\mathrm{alg}}(G)) \simeq H^*(\Omega_{\mathrm{alg}}(G)^\tau).$$
Combined with Theorem 2.1.1, this implies point (a) of Theorem 2. Point (b) follows from the fact that the actions of $T \times S^1$ and τ on $\Omega_{\mathrm{alg}}(G)$ are compatible, see Proposition 2.2.4 below. We use Theorem 2.1.4(b) and again Theorem 2.1.1. □

2.2. The Bott-Samelson theorem for $\Omega(G/K)$

Throughout this subsection G will be a compact connected simply connected Lie group and σ an arbitrary involutive automorphism of G. The notations established in Remark 1 following Theorem 1 are in force. We consider again the group $K = G^\sigma$ and the homogeneous space G/K, which has a canonical structure of a Riemannian symmetric space. Let us also consider the loop space
$$\Omega(G/K) := \{\mu : [0, \pi] \to G/K \ : \ \mu \text{ is of Sobolev class } H^1 \text{ and } \mu(0) = \mu(\pi) = eK\}$$

where eK denotes the coset of e in G/K. The reason why the loops in the previous definition are defined on $[0,\pi]$, and not on $[0,2\pi]$ or $S^1=\mathbb{R}/2\pi\mathbb{Z}$, as usual, will be understood later (see the proof of Proposition 2.2.6). Consider the group

$$A_2 := \{a \in A \ : \ a^2 = e\}.$$

In this subsection we will define $A_2 \times \mathbb{Z}_2$ actions on $\Omega(G/K)$ and $\Omega(G)^\tau$, show that these two spaces are equivariantly homotopy equivalent, and finally prove Corollary 3.

We first note that $A_2 = A\cap K$. This can be justified as follows: if $a\in A$ then $a=\exp(X)$ where $X\in\mathfrak{a}$, so $\sigma(a)=a^{-1}$; consequently,

$$a\in K \quad\Leftrightarrow\quad \sigma(a)=a \quad\Leftrightarrow\quad a^{-1}=a \quad\Leftrightarrow\quad a^2=e.$$

The group A_2 acts on $\Omega(G/K)$ by pointwise multiplication of the loops from the left:

$$(a.\mu)(\theta) = a\mu(\theta)$$

for all $a\in A_2$, $\mu\in\Omega(G/K)$ and $\theta\in[0,\pi]$. There is also an action of $\mathbb{Z}_2$ on $\Omega(G/K)$, which is more subtle. It is determined by the involutive automorphism $\mu\mapsto\tilde{\mu}$ of $\Omega(G/K)$, defined below. We first prove a lemma.

Lemma 2.2.1. *Any loop $\mu\in\Omega(G/K)$ can be written as*

$$\mu(\theta)=\gamma(\theta)K,$$

where $\gamma:[0,\pi]\to G$ is an H^1 map such that $\gamma(0)=e$ and $\gamma(\pi)\in K$.

Proof. We use the Path Lifting Theorem (cf., e.g. [AMR, Theorem 3.4.30]) for the locally trivial bundle $G\to G/K$. □

Definition 2.2.2. Let $\mu\in\Omega(G/K)$ be of the form $\mu(\theta)=\gamma(\theta)K$, $\theta\in[0,\pi]$, as in the previous lemma. We define $\tilde{\mu}$ by

$$\tilde{\mu}(\theta) := \sigma(\gamma(\pi-\theta))K,$$

$\theta\in[0,\pi]$.

We first verify that the map $\mu\mapsto\tilde{\mu}$ is independent of the choice of γ: if γ_1 is another representative of μ, that is, if $\gamma_1(\theta)=\gamma(\theta)k$ for some $k\in K$, then

$$\sigma(\gamma_1(\pi-\theta))K = \sigma(\gamma(\pi-\theta)k)K = \sigma(\gamma(\pi-\theta))\sigma(k)K = \sigma(\gamma(\pi-\theta))kK = \sigma(\gamma(\pi-\theta))K.$$

Next we verify that the map $\mu\mapsto\tilde{\mu}$ is involutive, that is $\tilde{\tilde{\mu}}=\mu$. To do this, we write

$$\tilde{\mu}(\theta) := \sigma(\gamma(\pi-\theta))\gamma(\pi)^{-1}K,$$

and deduce that

$$\tilde{\tilde{\mu}}(\theta) := \sigma(\sigma(\gamma(\pi-(\pi-\theta))\gamma(\pi)^{-1}))K = \gamma(\theta)K = \mu(\theta).$$

In this way we have defined our $\mathbb{Z}_2$ action on $\Omega(G/K)$.

Lemma 2.2.3. *The A_2 and $\mathbb{Z}_2$ actions on $\Omega(G/K)$ defined above commute with each other and thus define an action of $A_2 \times \mathbb{Z}_2$.*

Proof. Take $a \in A_2$ and $\mu \in \Omega(G/K)$ of the form $\mu(\theta) = \gamma(\theta)K$, as in Lemma 2.2.1. Since $a \in K$, we can write

$$(a.\mu)(\theta) = a\mu(\theta) = a\gamma(\theta)K = a\gamma(\theta)a^{-1}K.$$

Then

$$\begin{aligned}\widetilde{(a\mu)}(\theta) &= \sigma(a\gamma(\pi-\theta)a^{-1})K = \sigma(a)\sigma(\gamma(\pi-\theta))\sigma(a^{-1})K \\ &= a\sigma(\gamma(\pi-\theta))a^{-1}K = a\sigma(\gamma(\pi-\theta))K = a\tilde{\mu}(\theta). \qquad \square\end{aligned}$$

We consider again the action of $A \times S^1$ on $\Omega(G)$ given by equations (7) and (8). We also recall (see equation (1)) that τ is the involutive automorphism of $\Omega(G)$ given by

$$\tau(\gamma)(\theta) = \sigma(\gamma(-\theta)),$$

$\theta \in S^1$ (see equation (1)). The following proposition shows that the $A \times S^1$ action and the involution τ are compatible in the sense of Duistermaat [D].

Proposition 2.2.4. *We have*

$$\tau((a,z).\gamma) = (a^{-1}, z^{-1}).\tau(\gamma)$$

for any $\gamma \in \Omega(G)$ and any $(a,z) \in A \times S^1$.

Proof. We take the A and S^1 actions separately. First, if $a \in A$ then we have $\sigma(a) = a^{-1}$, thus

$$\tau(a.\gamma)(\theta) = \sigma(a\gamma(-\theta)a^{-1}) = \sigma(a)\sigma(\gamma(-\theta))\sigma(a^{-1}) = a^{-1}\sigma(\gamma(-\theta))a = (a^{-1}.\tau(\gamma))(\theta).$$

Second, if $z = e^{i\varphi}$, then

$$\begin{aligned}\tau(z.\gamma)(\theta) = \sigma(\gamma(-\theta+\varphi)\gamma(\varphi)^{-1}) &= \sigma(\gamma(-\theta+\varphi))\sigma(\gamma(\varphi)^{-1}) \\ &= \tau(\gamma)(\theta-\varphi)\tau(\gamma)(-\varphi)^{-1} = (z^{-1}.\tau(\gamma))(\theta). \qquad \square\end{aligned}$$

We deduce immediately as follows.

Corollary 2.2.5. *The fixed point set*

$$\Omega(G)^\tau := \{\gamma \in \Omega(G) \ : \ \sigma(\gamma(\theta)) = \gamma(-\theta), \theta \in S^1\}$$

is invariant under the action of the group $A_2 \times \mathbb{Z}_2$ which consists of all pairs $(a,z) \in A \times S^1$ with $a^2 = 1$ and $z = \pm 1$.

The following proposition makes the connection between the spaces $\Omega(G)^\tau$ and $\Omega(G/K)$. It is an equivariant version of a result whose origins go back to Bott and Samelson [BS] (see also [M], [K]).

Proposition 2.2.6. *There is a homotopy equivalence between* $\Omega(G/K)$ *and* $\Omega(G)^\tau$ *which is equivariant with respect to the* $A_2 \times \mathbb{Z}_2$ *actions defined in Lemma* 2.2.3 *and Corollary* 2.2.5.

Proof. We use the idea of [K, Prop. 3.1.3] (see also [M, Sect. 5]). The homotopy equivalence is the map $F : \Omega(G)^\tau \to \Omega(G/K)$ given by

$$F(\gamma) := \gamma|_{[0,\pi]}K.$$

This map is well defined since if γ is in $\Omega(G)^\tau$ then $\gamma(\pi) = \sigma(\gamma(\pi))$, thus $\gamma(\pi) \in K$ and, consequently, $\gamma(\pi)K = \gamma(0)K = eK$. To prove that F is a homotopy equivalence, we note that we can identify $\Omega(G)^\tau$ with the space of all paths $\beta : [0,\pi] \to G$ with $\beta(0) = e$ and $\beta(\pi) \in K$. The map F is given by $\beta \mapsto \beta K$ for all paths β as above. This is a principal bundle whose fibre is the group $\{\beta : [0,\pi] \to K \ : \ \beta(0) = e\}$. Since the latter space is contractible, F is a homotopy equivalence, as desired.

It remains to show that F is $A_2 \times \mathbb{Z}_2$ equivariant. Only the $\mathbb{Z}_2$-equivariance is nontrivial. Let us consider $\gamma \in \Omega(G)^\tau$ and verify that

$$F((-1).\gamma) = \widetilde{F(\gamma)}.$$

Here the loop $(-1).\gamma$ is given by

$$((-1).\gamma)(\theta) = \gamma(\theta + \pi)\gamma(\pi)^{-1}$$

for all $\theta \in S^1$, see equation (8). Thus we have

$$F((-1).\gamma)(\theta) = \gamma(\theta + \pi)\gamma(\pi)^{-1}K = \gamma(\theta + \pi)K,$$

since $\gamma(\pi) \in K$ (see above). On the other hand, for any $\theta \in S^1$, we have

$$\widetilde{F(\gamma)}(\theta) = \sigma(\gamma(\pi - \theta))K = \gamma(\theta - \pi)K = \gamma(\theta + \pi)K.$$

Here we have used that $\tau(\gamma) = \gamma$, which implies that $\sigma(\gamma(\pi - \theta)) = \gamma(\theta - \pi)$. □

Finally, we can spell out the details of the proof of Corollary 3: this follows from Theorem 2 by using Proposition 2.2.6 above (in the particular situation when $A = T$).

3. Examples and counterexamples

3.1. Examples

The basic assumption of this paper is that the involutive automorphism σ of the simply connected and compact Lie group G satisfies $\sigma(t) = t^{-1}$ for all t in a maximal torus $T \subset G$. In other words, if K denotes the fixed point set of σ, the Riemannian symmetric pair (G, K) is of maximal rank: by this we mean that the rank of the symmetric space G/K is equal to the rank of G (not to be confused with the situation when the homogeneous space G/K has maximal rank, which means rank G = rank K). Each Lie group G as above has essentially one such involution σ. In the following table we describe σ when G is one of the classical simply connected compact Lie groups: in each case it is sufficient to describe the automorphism $\theta := d\sigma_e$ of $\mathfrak{g}$. We are using [H, Chap. X, Sect. 2, Subsect. 3].

G	$\mathfrak{g}$	$\theta := d\sigma_e$
$\mathrm{SU}(n)$	$\mathfrak{su}(n)$: $n \times n$ complex skew-Hermitean matrices X	$\theta(X) = \overline{X}$ (complex conjugation)
$\mathrm{Spin}(2n)$	$\mathfrak{so}(2n)$: $2n \times 2n$ real skew-symmetric matrices X	$\theta(X) = I_{n,n} X I_{n,n}$, where $I_{n,n} := \begin{pmatrix} -I_n & 0 \\ 0 & I_n \end{pmatrix}$
$\mathrm{Spin}(2n+1)$	$\mathfrak{so}(2n+1)$: $(2n+1) \times (2n+1)$ real skew-symmetric matrices X	$\theta(X) = I_{n+1,n} X I_{n+1,n}$, where $I_{n+1,n} := \begin{pmatrix} -I_{n+1} & 0 \\ 0 & I_n \end{pmatrix}$
$\mathrm{Sp}(n)$	$\mathfrak{sp}(n)$: $X = \begin{pmatrix} Z_{11} & Z_{12} \\ -\overline{Z}_{12} & Z_{22} \end{pmatrix}$, where Z_{ij} are $n \times n$ complex matrices, Z_{11} and Z_{22} skew-Hermitean, and Z_{12} symmetric	$\theta(X) = J_n X J_n^{-1}$, where $J_n := \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$

The pairs (G, σ) in the table above correspond to the symmetric spaces G/K of type AI, BDI (with $p = q$ or $p = q+1$), and CI: for the meaning of these types, that is, for the classification of the irreducible Riemannian symmetric spaces, we refer the reader to [H, Table V, p. 518] or [B, Table 2, pp. 312–313]. For the exceptional Lie groups, one can also consult the last two tables: the maximal rank types are $EI, EV, EVIII, FI$, and G.

3.2. Counterexamples

In the remaining part of this section we will show that the hypothesis which says that the pair (G, K) is of maximal rank is essential for the two main results of the paper. The notations established in Remark 1 following Theorem 1 are in force here.

Let us start with Theorem 1. We show that there exist simply connected compact Lie groups G with an involution σ and a maximal torus $T \subset G$ such that $\Phi(\Omega(G)^\tau)$ is strictly contained in $\Phi(\Omega(G))$. We first recall that, in general, the vertices of the polyhedron $\Phi(\Omega(G))$ in $\mathfrak{t} \oplus \mathbb{R}$ are $\Phi(\gamma_\xi) = (\xi, \frac{1}{2}|\xi|^2)$, where ξ is in the integral lattice I of T and $\gamma_\xi : S^1 \to T$, $\gamma_\xi(\theta) = \exp(\theta\xi)$, for all $\theta \in S^1$, is the corresponding group homomorphism (see [AP, Sect. 1, Remark 2]). Pick ξ_0 in the integer lattice I such that $d\sigma_e(\xi_0) \neq -\xi_0$ (we will comment below on the existence of such ξ_0). Let $\gamma_0 : S^1 \to T$, $\gamma_0(\theta) = \exp(\theta\xi_0)$ be the corresponding group homomorphism and consider $\Phi(\gamma_0) = (\xi_0, \frac{1}{2}|\xi_0|^2)$. Assume that there exists $\gamma \in \Omega(G)^\tau$ such that $\Phi(\gamma) = \Phi(\gamma_0)$. Then $\Phi(\gamma)$ is on the paraboloid of equation $E = \frac{1}{2}|p|^2$ in $\mathfrak{t} \oplus \mathbb{R}$, hence γ must be a group homomorphism $S^1 \to T$ (by [AP, Sect. 1, Remark

3]). Thus γ is of the form $\gamma(\theta) = \exp(\theta\xi)$, for all $\theta \in S^1$, where $\xi \in I$. A simple calculation shows that the condition $\tau(\gamma) = \gamma$ implies $d\sigma_e(\xi) = -\xi$, thus $\xi \in \mathfrak{a}$. From $\Phi(\gamma) = \Phi(\gamma_0)$ we deduce

$$(\xi, \tfrac{1}{2}|\xi|^2) = (\xi_0, \tfrac{1}{2}|\xi_0|^2),$$

thus $\xi = \xi_0$, which contradicts $d\sigma_e(\xi_0) \neq -\xi_0$.

One can easily find examples of symmetric spaces G/K for which there exists $\xi_0 \in I$ with $d\sigma_e(\xi_0) \neq -\xi_0$. For example, one can take

$$\mathbb{C}P^{n-1} = \mathrm{SU}(n)/\mathrm{S}(\mathrm{U}(1) \times \mathrm{U}(n-1)).$$

This is a rank 1 symmetric space (cf., e.g. [H, Chap. X, Sect. 6, Table V] or [M, Example 6.6]). Recall that the rank of a general symmetric space G/K is equal to the dimension of $\mathfrak{a}$ (cf., e.g. [H, Chap. V, Sect. 6], see also Remark 1 following Theorem 1). Thus, in the case at hand, we have $\dim \mathfrak{a} = 1$. We can extend $\mathfrak{a}$ to a maximal abelian subspace of $\mathrm{Lie}(\mathrm{SU}(n))$, call it $\mathfrak{t}$, which is $d\sigma_e$ invariant and such that

$$\mathfrak{a} = \{x \in \mathfrak{t} \ : \ d\sigma_e(x) = -x\}.$$

Put $T = \exp(\mathfrak{t})$, which is a maximal torus in $\mathrm{SU}(n)$. It is clear that if $n \geq 3$, then $\dim \mathfrak{t} = n-1$ is at least 2, and so not all integral elements of T are in $\mathfrak{a}$. We note that the pair $(\mathrm{SU}(n), \mathrm{S}(\mathrm{U}(1) \times \mathrm{U}(n-1))$ is far from being of maximal rank, as $\mathrm{rank}\,\mathrm{SU}(n) = n-1$, whereas rank $\mathbb{C}P^{n-1} = 1$. An even more extreme example is given by the pair $(\mathrm{Sp}(n), \mathrm{U}(n))$ (see [H, Chap. X, Sect. 2, Subsect. 3]). This pair is of maximal rank: indeed, rank $\mathrm{Sp}(n)/\mathrm{U}(n)$ = rank $\mathrm{Sp}(n) = n$. Hence there exists a torus $T \subset \mathrm{Sp}(n)$ with $\sigma(t) = t^{-1}$ for all $t \in T$: Theorem 1 applies in this situation. However, we also have rank $\mathrm{Sp}(n)$ = rank $\mathrm{U}(n) = n$, thus there exists another maximal torus in $\mathrm{Sp}(n)$, call it T', such that $T' \subset \mathrm{U}(n)$. This implies that $d\sigma_e(\xi) = \xi$ for all $\xi \in \mathrm{Lie}(T')$; thus $d\sigma_e(\xi) \neq -\xi$, unless $\xi = 0$. As a general comment, the natural thing to expect in the case where rank $G/K <$ rank G is that $\Phi_A(\Omega(G)) = \Phi_A(\Omega(G)^\tau)$ (see Remark 1 following Theorem 1).

Let us now turn to Theorem 2. This time we show that there exists a simply connected compact Lie group G with an involution σ such that for some $q \geq 0$ we have $\dim H^{2q}(\Omega(G); \mathbb{Z}_2) \neq \dim H^q(\Omega(G)^\tau; \mathbb{Z}_2)$. Indeed, let us consider again the Riemannian symmetric pair $(\mathrm{SU}(n), \mathrm{S}(\mathrm{U}(1) \times \mathrm{U}(n-1)))$: the corresponding symmetric space is $\mathrm{SU}(n)/(\mathrm{S}(\mathrm{U}(1) \times \mathrm{U}(n-1)) = \mathbb{C}P^{n-1}$ (see above). The $\mathbb{Z}_2$ Poincaré series of $\Omega(\mathrm{SU}(n))^\tau$ and $\Omega(\mathbb{C}P^{n-1})$ are the same, being equal to $(1+t)(1-t^{2n-2})^{-1}$ (see [M, Sect. 6, Example 6.6]). The $\mathbb{Z}_2$ Poincaré series of $\Omega(\mathrm{SU}(n))$ is $[(1-t^2)(1-t^4)\dots(1-t^{2n-2})]^{-1}$ (cf., e.g. [BS, equation (13.2)]). Thus if $n \geq 3$, then we have $\dim H^4(\Omega(\mathrm{SU}(n)); \mathbb{Z}_2) = 2$, whereas $\dim H^2(\Omega(\mathrm{SU}(n))^\tau; \mathbb{Z}_2) = 0$.

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