EQUIVARIANT K-THEORY OF QUATERNIONIC FLAG MANIFOLDS

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ABSTRACT. We consider the manifold $Fl_n(\mathbb{H}) = Sp(n)/Sp(1)^n$ of all complete flags in $\mathbb{H}^n$, where $\mathbb{H}$ is the skew-field of quaternions. We study its equivariant complex K-theory rings with respect to the action of two groups: $Sp(1)^n$ and a certain canonical subgroup $T = (S^1)^n$ (a maximal torus). For the first group action we obtain a Goresky-Kottwitz-MacPherson type description. For the second one, we describe the ring $K_T(Fl_n(\mathbb{H}))$ as a subring of $K_T(Sp(n)/T)$. This ring is well known, since $Sp(n)/T$ is a complex flag variety.

1. INTRODUCTION

The quaternionic flag manifold $Fl_n(\mathbb{H})$ is the space of all nested sequences

$$(V_\nu)_{1\le\nu\le n} = V_1 \subset \ldots \subset V_n$$

where V_ν is a ν-dimensional $\mathbb{H}$-vector subspace of $\mathbb{H}^n$, for all $1 \le \nu \le n$, and $\mathbb{H}$ is the skew-field of quaternions (by an $\mathbb{H}$-vector subspace we mean a left $\mathbb{H}$-submodule). Let $Sp(n)$ denote the group of all $\mathbb{H}$-linear transformations of $\mathbb{H}^n$ (that is, $n \times n$ matrices with coefficients in $\mathbb{H}$) which preserve the canonical inner product on $\mathbb{H}^n$. This group acts naturally on $Fl_n(\mathbb{H})$. In this paper we are particularly interested in the action on $Fl_n(\mathbb{H})$ of the following two subgroups of $Sp(n)$:

$$G = Sp(1)^n \text{ and } T = (S^1)^n.$$

To put matters otherwise, G is the group of all diagonal matrices in $Sp(n)$ and $T \subset G$ consists of all such matrices with entries in $\mathbb{C}$, where $\mathbb{C}$ is canonically embedded in $\mathbb{H}$ (as the set of all $a + bi$, where $a, b \in \mathbb{R}$). It is worth mentioning that G is actually the $Sp(n)$ stabilizer of the flag $(\mathbb{H}e_1 \oplus \ldots \oplus \mathbb{H}e_\nu)_{1\le\nu\le n}$, and since the action of $Sp(n)$ on $Fl_n(\mathbb{H})$ is transitive, we can identify

$$Fl_n(\mathbb{H}) = Sp(n)/G.$$

We investigate the (complex, topological) equivariant K-theory rings corresponding to the T and G actions. In general, the G-equivariant K-theory ring of any G-space $\mathcal{X}$ is denoted by $K_G(\mathcal{X})$ or $K^0_G(\mathcal{X})$ (in this paper the first notation will be used in most cases). By definition (see, for instance, [19]), it is the Grothendieck group of G-equivariant topological complex vector bundles over $\mathcal{X}$. It is a module over the ring $K_G(\text{pt.}) = R[G]$, which is the representation ring of G. We know that (see, for instance, [7, Chapter 14, Section 6])

$$R[T] = R[(S^1)^n] = \mathbb{Z}[x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}]$$

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and

$$R[G] = R[Sp(1)^n] = \mathbb{Z}[X_1, \ldots, X_n].$$

Here $X_\nu = x_\nu + x_\nu^{-1}$, $1 \le \nu \le n$, are copies of the (character of the) canonical representation of $Sp(1) = SU(2)$ on $\mathbb{H} = \mathbb{C}^2$.

The flag manifold $Fl_n(\mathbb{H})$ carries n canonical (complex) vector bundles $\mathcal{V}_1, \mathcal{V}_2, \ldots, \mathcal{V}_n$, of complex rank equal to 2, 4, $\ldots 2n$. The rank 2 quotient bundles $\mathcal{L}_\nu = \mathcal{V}_\nu/\mathcal{V}_{\nu-1}$, $1 \le \nu \le n$ play an important role (by convention, $\mathcal{V}_0$ is the rank 0 vector bundle). Namely, we take into account that T is a maximal torus in both $Sp(n)$ and G. Moreover, the Weyl group W_G of G is a normal subgroup of the Weyl group $W_{Sp(n)}$ and their quotient is

$$W_{Sp(n)}/W_G \simeq \mathcal{S}_n,$$

the symmetric group (see, for instance, [7, Chapter 14, Section 4] or Section 2, below). Results of [15, Section 4] (see also Proposition 5.2 of our paper), lead to:

$$\begin{aligned} K_G(Fl_n(\mathbb{H})) &\simeq K_{Sp(n)}(Fl_n(\mathbb{H})) \otimes_{R[Sp(n)]} R[G] \\ &\simeq R[G] \otimes_{R[Sp(n)]} R[G] \\ &\simeq \frac{\mathbb{Z}[[\mathcal{L}_1], \ldots, [\mathcal{L}_n], X_1, \ldots, X_n]}{\langle \sigma_k([\mathcal{L}_1], \ldots, [\mathcal{L}_n]) - \sigma_k(X_1, \ldots, X_n), 1 \le k \le n\rangle}. \end{aligned} \tag{1}$$

Here σ_k denotes the k-th symmetric polynomial in n variables. The ring $K_T(Fl_n(\mathbb{H}))$ is isomorphic to $K_G(Fl_n(\mathbb{H})) \otimes_{R[G]} R[T]$ (see [15, Section 4] or Proposition 4.2, below). Thus, we obtain

$$K_T(Fl_n(\mathbb{H})) \simeq \frac{\mathbb{Z}[[\mathcal{L}_1], \ldots, [\mathcal{L}_n], x_1^{\pm 1}, \ldots, x_n^{\pm 1}]}{\langle \sigma_k([\mathcal{L}_1], \ldots, [\mathcal{L}_n]) - \sigma_k(x_1 + x_1^{-1}, \ldots, x_n + x_n^{-1}), 1 \le k \le n\rangle}.$$

In this paper we give alternative descriptions of the two rings above. The approaches will be different for G and T, as follows.

The first main result describes $K_T(Fl_n(\mathbb{H}))$ as a subring of the T-equivariant K-theory ring of the principal adjoint orbit $Sp(n)/T$. The T-equivariant K-theory of principal adjoint orbits (that is, complete complex flag varieties) is well understood, see, for instance, [5], [10], [11], [15], [12], [17], [20], [21]. If we identify $Sp(n)/T$ with the quotient of $Sp(2n, \mathbb{C})$ by a Borel subgroup, we deduce from the general theory (see, e.g., [10, Lemma 4.9]) that $K_T(Sp(n)/T)$ has a natural basis over $R[T]$, namely the Schubert basis $\{[\mathcal{O}_w] \,:\, w \in W_{Sp(n)}\}$. Like for any flag variety, the Weyl group $W_{Sp(n)} = N_{Sp(n)}(T)/T$ acts on $Sp(n)/T$ via

$$(nT).(gT) = gn^{-1}T, \tag{2}$$

for any $n \in N_{Sp(n)}(T)$ and $g \in Sp(n)$. This action is T-equivariant. Therefore, by functoriality, it induces an action by ring homomorphisms on $K_T(Sp(n)/T)$. The canonical map

$$\pi : Sp(n)/T \to Sp(n)/G = Fl_n(\mathbb{H})$$

is T-equivariant too, hence it induces a homomorphism between the K_T-rings, which we denote by π_T^*. We can now state the theorem.

Theorem 1.1. *The map $\pi_T^* : K_T(Fl_n(\mathbb{H})) \to K_T(Sp(n)/T)$ is injective. Its image consists of all W_G-invariant elements of $K_T(Sp(n)/T)$. In this way, $K_T(Fl_n(\mathbb{H}))$ is the $R[T]$-subalgebra of $K_T(Sp(n)/T)$ generated by all $[\mathcal{O}_w]$, where $w \in W_{Sp(n)}$ is a maximal length representative of the quotient $W_{Sp(n)}/W_G$.*

Remark. A similar result holds for the general context of the $\mathcal{T}$-equivariant K-theories of $\mathcal{G}/\mathcal{P}$ and $\mathcal{G}/\mathcal{B}$, where $\mathcal{G}$ is a complex semisimple Lie group, $\mathcal{P}$ a parabolic subgroup which contains the Borel subgroup $\mathcal{B}$, and $\mathcal{T}$ a maximal torus of $\mathcal{G}$ such that $\mathcal{T} \subset \mathcal{G}$ (see, for instance, [10, Corollary 3.20]). However, Theorem 1.1 does not fit into this context, as $Fl_n(\mathbb{H})$ is not a complex flag variety.

For $K_G(Fl_n(\mathbb{H}))$ we will prove the following Goresky-Kottwitz-MacPherson (shortly GKM) type description. Before stating it, we just mention that the G fixed point set of $Fl_n(\mathbb{H})$ can be identified with the symmetric group $\mathcal{S}_n$, as follows (see, for instance, [13, Lemma 3.1]):

$$Fl_n(\mathbb{H})^G = \{(\mathbb{H}e_{\tau(1)} \oplus \ldots \oplus \mathbb{H}e_{\tau(\nu)})_{1\le \nu \le n} \ : \ \tau \in \mathcal{S}_n\} = \mathcal{S}_n. \tag{3}$$

And here is the theorem.

Theorem 1.2. *The ring homomorphism $K_G(Fl_n(\mathbb{H})) \to \prod_{\tau\in\mathcal{S}_n} R[G]$ induced by the inclusion map $Fl_n(\mathbb{H})^G \hookrightarrow Fl_n(\mathbb{H})$ is injective. Its image is*

$$\{(f_\tau) \in \prod_{\tau\in\mathcal{S}_n} \mathbb{Z}[X_1, \ldots, X_n] \ : \ f_\tau - f_{(\mu,\nu)\tau} \text{ is divisible by } X_\mu - X_\nu \text{ for all } 1 \le \mu < \nu \le n\}.$$

Here (μ, ν) denotes the transposition of μ and ν, that is, the element of $\mathcal{S}_n$ which interchanges μ and ν.

Remarks. 1. A description of the integral cohomology ring of $Fl_n(\mathbb{H})$ in terms of generators and relations has been obtained in [2] (see page 302). For the G-equivariant cohomology ring such a description has been obtained in [13]. In both cases one obtains the same formulas as for the complex flag manifold $Fl_n(\mathbb{C})$ (in the equivariant case the group acting on $Fl_n(\mathbb{C})$ is the standard maximal torus T of the unitary group $U(n)$). In the present paper we show that the same similarity can be noted for K-theory. Not only has the ring $K_G(Fl_n(\mathbb{H}))$ the same presentation as $K_T(Fl_n(\mathbb{C}))$ (see equation (1)), but also the same GKM description holds true (see Theorem 1.2 and compare with [15, Theorem 1.6] for $G = U(n)$). Another space for which we have the same analogy with the complex flag manifold at the level of equivariant cohomology and K-theory is the octonionic flag manifold $Fl(\mathbb{O})$ (see the recent paper [14]). It would be interesting to find more examples of spaces with group actions for which the equivariant K-theory has the same features as $Fl_n(\mathbb{H})$ and $Fl(\mathbb{O})$. What makes these two spaces special is as follows: First, they are homogeneous, of the form $\mathcal{G}/\mathcal{H}$ where $\mathcal{G}$ is a compact Lie group and $\mathcal{H}$ a closed subgroup of the same rank as $\mathcal{G}$; the group action is the one of $\mathcal{H}$, by multiplication from the left. Second, a maximal torus $\mathcal{T}$ of $\mathcal{H}$ has the same fixed point set as $\mathcal{H}$ itself. Third, $\mathcal{G}/\mathcal{H}$ admits a cell decomposition such that each cell is $\mathcal{T}$-invariant, homeomorphic to $\mathbb{C}^m$ for some $m \ge 0$, and the action of $\mathcal{T}$ on it is complex linear.

2. It is worth noticing that all the previously known GKM type descriptions of equivariant K-theory are for actions of tori (for instance, any subvariety of a complex projective space which is preserved by a linear torus action, see, for instance, [18, Appendix A]). Theorem 1.2 is a non-abelian version of this general result.

3. A GKM description exists for $K_T(Fl_n(\mathbb{H}))$ as well (see Proposition 4.5, below). It can also be deduced from the fact that $Fl_n(\mathbb{H})$ has a cell decomposition whose cells are complex vector spaces, the torus T leaving them invariant and acting on them complex linearly in a very explicit way (see Section 3, below). Thus, one can apply the main result of [8]: the main ingredient is the calculation of the Euler class in K_T for any cell and the observation that this is not a zero-divisor in $K_T(\text{pt.}) = R[T]$. We will not present the details. Our proof of Proposition 4.5 goes along different lines, using the GKM description of $K_T(Sp(n)/T)$. Concerning $K_G(Fl_n(\mathbb{H}))$, we do not know if the GKM description given in Theorem 1.2 is a direct consequence of [8]. Even though the cells mentioned above are G-invariant and the action of G on cells is also very explicit (see again Section 3), this action is $\mathbb{R}$-linear without being $\mathbb{C}$-linear. It seems difficult to find a way to compute the corresponding Euler classes in the ring $K_G(\text{pt.}) = R[G]$ of *complex* representations of G.

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2. The roots of $Sp(n)$

In this section we collect some background material concerning the roots of $Sp(n)$ and other related objects. The details can be found for instance in [4, Section 16.1].

Let $\mathfrak{sp}(2n, \mathbb{C})$ be the complexified Lie algebra of $Sp(n)$. It consists of all complex square matrices of the form

$$\begin{pmatrix} a & b \\ c & -a^t \end{pmatrix},$$

where a, b, c are $n \times n$ complex matrices with $b^t = b$, $c^t = c$. The elements of the complexified Lie algebra of T, call it $\mathfrak{h}$, are block matrices as above with a diagonal and $b = c = 0$. A linear basis of $\mathfrak{h}$ over $\mathbb{C}$ consists of the matrices $e_{\nu,\nu} - e_{\nu+n,\nu+n}$, $1 \le \nu \le n$, where $\{e_{\mu,\nu}\}_{1\le\mu,\nu\le 2n}$ is the canonical basis of the space of complex $2n \times 2n$ matrices. We denote by $\{L^\nu \ : \ 1 \le \nu \le n\}$ the corresponding dual basis of $\mathfrak{h}^*$. We have as follows:

- The set of roots is
$$\Delta = \{\pm L^\mu \pm L^\nu \ : \ 1 \le \mu < \nu \le n\} \cup \{\pm 2L^\nu, 1 \le \nu \le n\}.$$
- The set of positive roots is
$$\Delta^+ = \{L^\mu \pm L^\nu, 1 \le \mu < \nu \le n\} \cup \{2L^\nu, 1 \le \nu \le n\}.$$
- A simple root system is
$$\Pi = \{\alpha_1 = L^1 - L^2, \alpha_2 = L^2 - L^3, \dots, \alpha_{n-1} = L^{n-1} - L^n, \alpha_n = 2L^n\}.$$
- The weight lattice $\mathfrak{h}^*_{\mathbb{Z}}$ is the $\mathbb{Z}$-module generated by $L^1, L^2, \dots, L^n$.

- The weight lattice $\mathfrak{h}^*_{\mathbb{Z}}$ is canonically isomorphic to $R[T]$, and L^ν corresponds to x_ν in this isomorphism, $1 \le \nu \le n$. More generally, for any $\lambda \in \mathfrak{h}^*_{\mathbb{Z}}$, we will denote the corresponding character in $R[T]$ by e^λ.
- The Weyl group $W_{Sp(n)} = N_{Sp(n)}(T)/T$ is generated by the simple reflections s_ν corresponding to α_ν, for $1 \le \nu \le n$ (as usual, we denote by s_α the reflection corresponding to the positive root α). Concretely, $W_{Sp(n)}$ consists of all linear automorphisms η of $\mathfrak{h}^*$ such that for any $1 \le \nu \le n$, there exists $1 \le \mu \le n$ such that $\eta(L^\nu) = \pm L^\mu$. This means that $W_{Sp(n)}$ is the semi-direct product of the symmetric group $\mathcal{S}_n$ of permutations of the set $\{L^\nu \ : \ 1 \le \nu \le n\}$ and the group $\{-1,1\}^n$ of sign changes (here $\mathcal{S}_n$ acts on $\{-1,1\}^n$ by permuting the entries of an n-tuple). For $1 \le \nu \le n-1$, the reflection s_ν is the transposition $(\nu, \nu+1)$. The reflection s_n sends L^n to $-L^n$, and L^ν to itself, for $1 \le \nu < n$. More generally, for $1 \le \mu < \nu \le n$, the reflection corresponding to the root $L^\mu - L^\nu$ is the transposition (μ, ν), whereas the reflection corresponding to $L^\mu + L^\nu$ sends L^μ to $-L^\nu$, L^ν to $-L^\mu$, and leaves L^κ unchanged, for $\kappa \notin \{\mu, \nu\}$. The reflection corresponding to the root $2L^\nu$ sends L^ν to $-L^\nu$ and leaves L^μ unchanged for $\mu \neq \nu$.
- The Weyl group $W_G := N_G(T)/T$ is the subgroup of $W_{Sp(n)}$ generated by s_{2L^ν}, $1 \le \nu \le n$. It is isomorphic to $\{-1,1\}^n$.
- The quotient $W_{Sp(n)}/W_G$ is isomorphic to the group of permutations of the set $\{L^\nu \ : \ 1 \le \nu \le n\}$, which is the symmetric group $\mathcal{S}_n$. For any pair μ, ν such that $1 \le \mu < \nu \le n$, the cosets $s_{L^\mu - L^\nu} W_G$ and $s_{L^\mu + L^\nu} W_G$ are equal and are mapped by the isomorphism above to the transposition (μ, ν).

3. The cell decomposition

In this section we describe the Schubert cell decomposition of $Fl_n(\mathbb{H})$. We will be especially interested in the action of G on the cells.

The group $GL_n(\mathbb{H})$ of all invertible $n \times n$ matrices with entries in $\mathbb{H}$ acts linearly on $\mathbb{H}^n$. More precisely, $\mathbb{H}^n$ is regarded as a left $\mathbb{H}$-module and the action of $GL_n(\mathbb{H})$ is given by:

$$gh = h \cdot g^*$$

for any $g \in GL_n(\mathbb{H})$ and any $h \in \mathbb{H}^n$. Here $\cdot$ denotes the matrix multiplication and g^* the transposed conjugate of g. The group $GL_n(\mathbb{H})$ acts on $Fl_n(\mathbb{H})$ by

$$g(V_\nu)_{1 \le \nu \le n} = (gV_\nu)_{1 \le \nu \le n},$$

for any $g \in GL_n(\mathbb{H})$ and any $(V_\nu)_{1 \le \nu \le n} \in Fl_n(\mathbb{H})$. This group action is transitive and the stabilizer of the flag $(\mathbb{H}e_1 \oplus \ldots \oplus \mathbb{H}e_\nu)_{1 \le \nu \le n}$ is the group B consisting of all upper triangular matrices with entries in $\mathbb{H}$. In this way we obtain the identification

$$Fl_n(\mathbb{H}) = GL_n(\mathbb{H})/B.$$

The following result is a direct consequence of the Bruhat decomposition of $GL_n(\mathbb{H})$ (see [3, Section 19, Theorem 1]):

Proposition 3.1. *Any $g \in GL_n(\mathbb{H})$ can be written as $g = up_\tau b$, where:*

1. *$\tau \in \mathcal{S}_n$ and p_τ denotes the matrix $(\delta_{\mu,\tau(\nu)})_{1\le\nu,\mu\le n}$, where δ is the Kroenecker delta.*
2. *$b \in B$*
3. *both u and $(p_\tau u p_\tau^{-1})^t$ are upper triangular with all entries on the diagonal equal to 1 (the superscript t indicates the matrix transposed).*

Moreover, the matrices p_τ and u with properties 1 and 3 above are uniquely determined by g.

We deduce that

$$GL_n(\mathbb{H})/B = \bigsqcup_{\tau\in\mathcal{S}_n} \mathfrak{U}_\tau p_\tau B/B, \tag{4}$$

where $\mathfrak{U}_\tau$ denotes the set of all $n \times n$ matrices u with entries in $\mathbb{H}$ such that both u and $(p_\tau u p_\tau^{-1})^t$ are upper triangular with all entries on the diagonal equal to 1. The canonical map $\mathfrak{U}_\tau \to \mathfrak{U}_\tau p_\tau B/B$ is a homeomorphism. Indeed, this map is continuous and bijective, by Proposition 3.1. Its inverse is continuous too, because the map $\mathfrak{U}_\tau p_\tau B \to \mathfrak{U}_\tau$ which assigns to $g = up_\tau b$ the first factor u is continuous, as we can see from its explicit description in the proof of [3, Section 19, Theorem 1]. Now it is an easy exercise to see that an $n \times n$ matrix $u = (u_{\mu\nu})_{1\le\mu,\nu\le n}$ is in $\mathfrak{U}_\tau$ if and only if the diagonal entries are equal to 1 and the others are equal to 0, except for those $u_{\mu\nu}$ with $\mu < \nu$ and $\tau(\mu) > \tau(\nu)$ (the key point is the formula $p_\tau u p_\tau^{-1} = (u_{\tau(\mu)\tau(\nu)})_{1\le\mu,\nu\le n}$). This implies that $\mathfrak{U}_\tau$ can be identified with $\mathbb{H}^{\ell(\tau)}$, where $\ell(\tau)$ denotes the number of inversions of the permutation τ. Consequently, for any $\tau \in \mathcal{S}_n$ the element

$$C_\tau = \mathfrak{U}_\tau p_\tau B/B$$

of the decomposition (4) is homeomorphic to a cell of (real) dimension $4\ell(\tau)$. We call it a *Bruhat cell.*

The group G acts on $GL_n(\mathbb{H})/B$ by left multiplication. We claim that this action leaves any Bruhat cell $C_\tau = \mathfrak{U}_\tau p_\tau B/B$ invariant. To justify this, take $\gamma = \mathrm{Diag}(\gamma_1, \dots, \gamma_n)$ in G. We have

$$\gamma p_\tau = \mathrm{Diag}(\gamma_1, \dots, \gamma_n) p_\tau = p_\tau \mathrm{Diag}(\gamma_{\tau^{-1}(1)}, \dots, \gamma_{\tau^{-1}(n)}).$$

This implies that if $u \in \mathfrak{U}_\tau$, then

$$\gamma u p_\tau B = \gamma u \gamma^{-1} \gamma p_\tau B = \gamma u \gamma^{-1} p_\tau B.$$

We notice that if $u = (u_{\mu\nu})_{1\le\mu,\nu\le n}$, then $\gamma u \gamma^{-1} = (\gamma_\mu u_{\mu\nu} \gamma_\nu^{-1})_{1\le\mu,\nu\le n}$. Thus, if u is in $\mathfrak{U}_\tau$, then $\gamma u \gamma^{-1}$ is in $\mathfrak{U}_\tau$ as well.

We summarize our previous discussion as follows:

Proposition 3.2. *The Bruhat cell decomposition of the quaternionic flag manifold $Fl_n(\mathbb{H})$ is $Fl_n(\mathbb{H}) = \bigsqcup_{\tau\in\mathcal{S}_n} C_\tau$. The cell C_τ has real dimension $4\ell(\tau)$ and is G-invariant, being in fact G-equivariantly homeomorphic to $\bigoplus_{(\mu,\nu)} \mathbb{H}_{\mu\nu}$. Here the sum runs over all pairs (μ, ν)*

with $1 \le \mu < \nu \le n$ *such that* $\tau(\mu) > \tau(\nu)$, *and* $\mathbb{H}_{\mu\nu}$ *is a copy of* $\mathbb{H}$. *The action of* G *on* $\mathbb{H}_{\mu\nu}$ *is*

$$(\gamma_1, \ldots, \gamma_n).h = \gamma_\mu h \gamma_\nu^{-1}, \tag{5}$$

for all $(\gamma_1, \ldots, \gamma_n) \in G$ *and* $h \in \mathbb{H}$.

Remark. Let us identify

$$\mathbb{H} = \mathbb{C} \oplus j\mathbb{C} = \mathbb{C}^2.$$

It is an easy exercise to see that if in equation (5) we take $(\gamma_1, \ldots, \gamma_n) \in T$, the resulting transformation of $\mathbb{H}$ is $\mathbb{C}$-linear. In other words, T acts complex linearly on each cell C_τ. However, if $(\gamma_1, \ldots, \gamma_n)$ is in G but not in T, the transformation is in general not $\mathbb{C}$-linear.

There are two alternative presentations of the cell C_τ, which are given in what follows.

Lemma 3.3. *We have*

$$C_\tau = Bp_\tau B/B.$$

Proof. We have

$$GL_n(\mathbb{H}) = \bigsqcup_{\tau \in \mathcal{S}_n} \mathfrak{U}_\tau p_\tau B$$

and $\mathfrak{U}_\tau \subset B$ for all $\tau \in \mathcal{S}_n$. Thus, it is sufficient to prove that if $\tau_1, \tau_2 \in \mathcal{S}_n$, $\tau_1 \neq \tau_2$, then $(Bp_{\tau_1}B) \cap (Bp_{\tau_2}B) = \emptyset$. This can be proved by using the same arguments as in the proof of [6, Chapter III, Proposition 4.6]. □

We can also describe C_τ by using the actual definition of $Fl_n(\mathbb{H})$, as the set of all flags in $\mathbb{H}^n$. Our model is the presentation of the Bruhat cells in $Fl_n(\mathbb{C})$, as given, for instance, in [6, Chapter III, Section 4]. First, to each r-dimensional linear subspace V of $\mathbb{H}^n$ we assign the set $s(V) = \{m_1, \ldots, m_r\}$ where $m_1 < \ldots < m_r$ are determined by $V \cap \mathbb{H}^{m_t - 1} \neq V \cap \mathbb{H}^{m_t}$, $1 \le t \le r$ (here $\mathbb{H}^m$ denotes $\mathbb{H}e_1 \oplus \ldots \oplus \mathbb{H}e_m$, for all $1 \le m \le n$, and $\mathbb{H}^0 = \{0\}$). Note that if V and V' are subspaces such that $V \subset V'$, then $s(V) \subset s(V')$. To the flag $V_\bullet = (V_\nu)_{1 \le \nu \le n}$ we assign the permutation $\tau = \tau^{V_\bullet}$ which is defined recursively as follows:

- $\{\tau(1)\} = s(V_1)$.
- if $\tau(1), \ldots, \tau(k)$ are known, we set $\{\tau(k+1)\} = s(V_{k+1}) \setminus s(V_k)$

Lemma 3.4. *We have*

$$C_\tau = \{V_\bullet \in Fl_n(\mathbb{H}) \ : \ \tau^{V_\bullet} = \tau\}.$$

Proof. We are using the identification $GL_n(\mathbb{H})/B = Fl_n(\mathbb{H})$ given by

$$gB = g\mathbb{H}_\bullet,$$

for any $g \in GL_n(\mathbb{H})$. Here $\mathbb{H}_\bullet$ denotes the flag $(\mathbb{H}e_1 \oplus \ldots \oplus \mathbb{H}e_\nu)_{1 \le \nu \le n}$. In this way, $p_\tau B$ is the same as the flag $(\mathbb{H}e_{\tau(1)} \oplus \ldots \oplus \mathbb{H}e_{\tau(\nu)})_{1 \le \nu \le n}$, which we denote by $\tau\mathbb{H}_\bullet$. The B-orbit of this flag is just C_τ (see Lemma 3.3). The lemma is a straightforward consequence of the following two facts:

$$\tau^{\tau\mathbb{H}_\bullet} = \tau$$
$$\tau^{bV_\bullet} = \tau^{V_\bullet} \text{ for all } b \in B \text{ and all } V_\bullet \in Fl_n(\mathbb{H}).$$

□

In the same way as in [6, p. 122], we see that the closure of C_τ in $Fl_n(\mathbb{H})$ consists of all flags $V_\bullet$ such that $\tau^{V_\bullet} \preceq \tau$. Here $\preceq$ denotes the Bruhat ordering on the symmetric group $\mathcal{S}_n$ (see [6, Chapter I, Section 6]). Note that if $\tau_1 \preceq \tau_2$ then $\ell(\tau_1) \leq \ell(\tau_2)$.

Proposition 3.5. *The closure of C_τ in $Fl_n(\mathbb{H})$ can be expressed as follows:*

$$\overline{C_\tau} = \bigsqcup_{\tau' \preceq \tau} C_{\tau'}.$$

Any of the cells $C_{\tau'}$ above, with $\tau' \neq \tau$, has dimension strictly less than the dimension of C_τ.

4. T-equivariant K-theory

Throughout this section we will use the notations

$$\mathcal{X} = Fl_n(\mathbb{H}) = Sp(n)/G \text{ and } \mathcal{Y} = Sp(n)/T.$$

Our main goal here is to prove Theorem 1.1. We will use the injectivity of the restriction to fixed points, which is the content of the next proposition. We first note that the T and G fixed points of $Fl_n(\mathbb{H})$ are the same (see [13, Lemma 3.1]). By equation (3) we have

$$\mathcal{X}^G = \mathcal{X}^T = \mathcal{S}_n.$$

Let $i : \mathcal{S}_n \to \mathcal{X}$ be the inclusion map and $i_T^* : K_T(\mathcal{X}) \to \prod_{\tau \in \mathcal{S}_n} R[T]$ the corresponding ring homomorphism.

Proposition 4.1. *(a) The T-equivariant K-theory of $\mathcal{X}$ is a free $R[T]$-module of rank $n!$.*

(b) The restriction to fixed points $i_T^* : K_T(\mathcal{X}) \to \prod_{\tau \in \mathcal{S}_n} R[T]$ is injective.

Proof. (a) Let $\mathcal{X} = \bigsqcup_{\tau \in \mathcal{S}_n} C_\tau$ be the cell decomposition of $\mathcal{X}$ (see Proposition 3.2). Each cell C_τ is T-equivariantly homeomorphic to $\mathbb{C}^{2\ell(\tau)}$. Notice that

$$\ell(\tau) \leq \frac{n(n-1)}{2},$$

for all $\tau \in \mathcal{S}_n$. For any integer number ν with $0 \leq \nu \leq \frac{n(n-1)}{2}$, we set

$$\mathcal{X}_\nu = \bigsqcup_{\tau \in \mathcal{S}_n, \ell(\tau) \leq \nu} C_\tau.$$

By Proposition 3.5, this is a closed subspace of $\mathcal{X}$. We prove by induction that for all $0 \leq \nu \leq \frac{n(n-1)}{2}$ we have:

- $K_T(\mathcal{X}_\nu) = K_T^0(\mathcal{X}_\nu)$ is a free $R[T]$-module of rank equal to the number of cells in $\mathcal{X}_\nu$
- $K_T^{-1}(\mathcal{X}_\nu) = \{0\}$.

The definition of the functors K_T^{-1} and K_T^1 can be found for instance in [19]. We have

$$K_T^{-1}(\mathrm{pt.}) = K_T^1(\mathrm{pt.}) = \{0\}. \tag{6}$$

This implies our claim for $\nu = 0$, since $\mathcal{X}_0$ consists of only one point (of course we have $K_T(\mathrm{pt.}) = R[T]$).

Let us assume that the claim is true for ν. We will prove it for $\nu + 1$. We consider the space

$$\mathcal{X}_{\nu+1} \setminus \mathcal{X}_\nu = \bigsqcup_{\tau \in \mathcal{S}_n, \ell(\tau) = \nu+1} C_\tau. \tag{7}$$

We consider now the exact sequence of the pair $(\mathcal{X}_{\nu+1}, \mathcal{X}_\nu)$. Since $\mathcal{X}_\nu$ is closed in $\mathcal{X}_{\nu+1}$, we deduce that $K_T^*(\mathcal{X}_{\nu+1}, \mathcal{X}_\nu) = K_T^*(\mathcal{X}_{\nu+1} \setminus \mathcal{X}_\nu)$ and obtain the following sequence (see [19, Section 2]):

$$\begin{aligned} &K_T^{-1}(\mathcal{X}_{\nu+1} \setminus \mathcal{X}_\nu) \to K_T^{-1}(\mathcal{X}_{\nu+1}) \to K_T^{-1}(\mathcal{X}_\nu) \to \\ &\to K_T^{0}(\mathcal{X}_{\nu+1} \setminus \mathcal{X}_\nu) \to K_T^{0}(\mathcal{X}_{\nu+1}) \to K_T^{0}(\mathcal{X}_\nu) \to K_T^{1}(\mathcal{X}_{\nu+1} \setminus \mathcal{X}_\nu). \end{aligned}$$

By the induction hypothesis, $K_T^{-1}(\mathcal{X}_\nu) = 0$, and $K_T^0(\mathcal{X}_\nu)$ is a free $R[T]$-module of rank equal to the number of cells in $\mathcal{X}_\nu$. Each of the cells C_τ in the union given by equation (7) is a connected component of $\mathcal{X}_{\nu+1} \setminus \mathcal{X}_\nu$: indeed, by Proposition 3.5, any such C_τ is a closed subspace of $\mathcal{X}_{\nu+1} \setminus \mathcal{X}_\nu$, hence it is open as well (as its complement is closed). We obtain

$$K_T^1(\mathcal{X}_{\nu+1} \setminus \mathcal{X}_\nu) = K_T^{-1}(\mathcal{X}_{\nu+1} \setminus \mathcal{X}_\nu) = \{0\},$$

where we have used the Thom isomorphism for each of the cells C_τ in the union given by (7) (recall that C_τ is T-equivariantly homeomorphic to $\mathbb{C}^{2\ell(\tau)}$, the action of T being complex linear, as mentioned in the remark following Proposition 3.2). We also have

$$K_T^0(\mathcal{X}_{\nu+1} \setminus \mathcal{X}_\nu) \simeq \bigoplus_{\tau \in \mathcal{S}_n, \ell(\tau) = \nu+1} K_T^0(C_\tau) = \bigoplus_{\tau \in \mathcal{S}_n, \ell(\tau) = \nu+1} R[T].$$

In other words, $K_T^0(\mathcal{X}_{\nu+1} \setminus \mathcal{X}_\nu)$ is a free $R[T]$-module of rank equal to the number of cells in $\mathcal{X}_{\nu+1} \setminus \mathcal{X}_\nu$. The desired conclusion follows.

Finally, for $\nu = \frac{n(n-1)}{2}$, we obtain point (a) of the proposition.

Point (b) is a consequence of point (a). Indeed, let $Q[T]$ denote the fraction field of $R[T]$. According to the localization theorem (see [19]), the homomorphism $K_T(\mathcal{X}) \otimes_{R[T]} Q[T] \to \prod_{\tau \in \mathcal{S}_n} Q[T]$ induced by i_T^* is an isomorphism. Moreover, since $K_T(\mathcal{X})$ is a free $R[T]$-module, the canonical map $K_T(\mathcal{X}) \to K_T(\mathcal{X}) \otimes_{R[T]} Q[T]$ is an embedding. Since the following diagram is commutative, we deduce that i_T^* is injective.

$$\begin{tikzcd} K_T(\mathcal{X}) \arrow[r, hook] \arrow[d, "i_T^*"] & K_T(\mathcal{X}) \otimes_{R[T]} Q[T] \arrow[d, "\simeq"] \\ \prod_{\mathcal{S}_n} R[T] \arrow[r] & \prod_{\mathcal{S}_n} Q[T] \end{tikzcd}$$

□

Remark. The arguments used in the proof are standard: see [15, proof of Lemma 2.2] or [10, proof of Theorem 3.13].

Let $\rho : G \times \mathbb{C}^m \to \mathbb{C}^m$ be a complex representation of G. The quotient space

$$V_\rho = Sp(n) \times \mathbb{C}^m/((k, v) \sim (kg, \rho(g^{-1})v) \text{ for all } k \in Sp(n), g \in G, v \in \mathbb{C}^m) \tag{8}$$

has a natural structure of a vector bundle over $Sp(n)/G = Fl_n(\mathbb{H})$. Moreover, there is a natural action of G (consequently, also of T) on V_ρ given by

$$g(k, v) = (gk, v), \text{ for all } g \in G, k \in Sp(n), \text{and } v \in \mathbb{C}^m. \tag{9}$$

The $R[T]$-linear extension of the assignment $\rho \mapsto V_\rho$ gives the homomorphism $\kappa_T : R[T] \otimes R[G] \to K_T(\mathcal{X})$. It is easy to check that if ρ is a representation of $Sp(n)$, then V_ρ is isomorphic to the trivial bundle $(Sp(n)/G) \times \mathbb{C}^m$. In other words, if $\chi \in R[Sp(n)]$, then we have $\kappa_T(\chi \otimes 1 - 1 \otimes \chi) = 0$. Consequently, we obtain a homomorphism

$$\overline{\kappa}_T : R[T] \otimes_{R[Sp(n)]} R[G] \to K_T(\mathcal{X}).$$

We are now ready to prove the first part of Theorem 1.1, concerning the image of the map

$$\pi_T^* : K_T(\mathcal{X}) \to K_T(\mathcal{Y}).$$

Proposition 4.2. *The homomorphism π_T^* in Theorem 1.1 is injective and its image is equal to $K_T(\mathcal{Y})^{W_G}$. Moreover, $\overline{\kappa}_T$ is a ring isomorphism.*

Proof. We first note that

$$\pi_T^*(K_T(\mathcal{X})) \subset K_T(\mathcal{Y})^{W_G}. \tag{10}$$

This is because for any $w \in W_G = N_G(T)/T$ we have $\pi \circ w = \pi$ (here w is regarded as an automorphism of $\mathcal{Y} = Sp(n)/T$, see equation (2)).

As mentioned in the introduction, the space $\mathcal{Y} = Sp(n)/T$ is a complete complex flag variety. Thus, the fixed point set of the T action is

$$(Sp(n)/T)^T = W_{Sp(n)}.$$

The image of this set under π is $Fl_n(\mathbb{H})^T = \mathcal{S}_n$. In fact, the restriction of π to the fixed point set is the canonical projection $W_{Sp(n)} \to W_{Sp(n)}/W_G = \mathcal{S}_n$ (see Section 2). The homomorphism between the T-equivariant K-theories induced by this map is the obvious map

$$p : \prod_{\mathcal{S}_n} R[T] \to \prod_{W_{Sp(n)}} R[T],$$

which is injective. This homomorphism is the bottom arrow in the following commutative diagram (where i_T^* is the restriction homomorphism).

$$\begin{tikzcd} K_T(\mathcal{X}) \arrow[r, "\pi_T^*"] \arrow[d, "i_T^*"] & K_T(\mathcal{Y}) \arrow[d, "i_T^*"] \\ \prod_{\mathcal{S}_n} R[T] \arrow[r, "p"] & \prod_{W_{Sp(n)}} R[T] \end{tikzcd}$$

The maps i_T^* and i_T^* are also injective, by Proposition 4.1, respectively the fact that $\mathcal{Y}$ is a complex flag variety (for such spaces, injectivity is proved for instance in [15, Lemma 2.2]). We deduce that π_T^* is injective.

We now prove that the image of π_T^* is the whole $K_T(\mathcal{Y})^{W_G}$ and that $\overline{\kappa}_T$ is an isomorphism. Since $\mathcal{Y} = Sp(n)/T$ we deduce that

$$K_T(\mathcal{Y}) = R[T] \otimes_{R[Sp(n)]} R[T]. \tag{11}$$

More precisely, by [15, The Main Theorem], we have an isomorphism

$$R[T] \otimes_{R[Sp(n)]} R[T] \to K_T(\mathcal{Y}) \tag{12}$$

which is the $R[T]$-linear extension of the map $\sigma \mapsto V'_\sigma$, for any complex representation $\sigma : T \times \mathbb{C}^m \to \mathbb{C}^m$. Here we define

$$V'_\sigma = Sp(n) \times \mathbb{C}^m/((k,v) \sim (kg, \sigma(g^{-1})v), \text{ for all } k \in Sp(n), g \in T, v \in \mathbb{C}^m), \tag{13}$$

which is a complex vector bundle over $\mathcal{Y} = Sp(n)/T$, with an obvious T action. We claim that the composed homomorphism

$$\pi_T^* \circ \overline{\kappa}_T : R[T] \otimes_{R[Sp(n)]} R[G] \to R[T] \otimes_{R[Sp(n)]} R[T]$$

is induced by the canonical embedding $R[G] \to R[T]$: this follows from equations (9) and (13) and the fact that π_T^* maps a vector bundle to its pull-back via π_T. Consequently, $\pi_T^* \circ \overline{\kappa}_T$ is an isomorphism between its domain and $R[T] \otimes_{R[Sp(n)]} R[G]$. This space is just $K_T(\mathcal{Y})^{W_G}$ (as we will see below). Thus, the image of π_T^* is $K_T(\mathcal{Y})^{W_G}$ (see also equation (10)). We also deduce that $\overline{\kappa}_T$ is an isomorphism.

It only remains to show that, via the identification (11), we have

$$K_T(\mathcal{Y})^{W_G} = R[T] \otimes_{R[Sp(n)]} R[G].$$

Indeed, this is a consequence of the fact that the isomorphism given by equation (12) is $W_{Sp(n)}$-equivariant with respect to the action given by

$$w.(\chi_1 \otimes \chi_2) = \chi_1 \otimes (w\chi_2),$$

for all $w \in W_{Sp(n)}$, $\chi_1, \chi_2 \in R[T]$ (see [15, Section 1.4] for a proof); we deduce that

$$K_T(\mathcal{Y})^{W_G} = R[T] \otimes_{R[Sp(n)]} R[T]^{W_G} = R[T] \otimes_{R[Sp(n)]} R[G],$$

as required. The proposition is now proved. □

We now focus on the last statement in Theorem 1.1. Since $\mathcal{Y}$ is a complex flag variety, $K_T(\mathcal{Y})$ can be identified with the Grothendieck group of the T-equivariant coherent sheaves on $\mathcal{Y}$, and the Bruhat decomposition gives an $R[T]$-basis of $K_T(\mathcal{Y})$. More precisely, this is the set of all classes $[\mathcal{O}_w]$ of the structure sheaves of the Schubert varieties $\mathcal{Y}_w$, for $w \in W_{Sp(n)}$ (cf. e.g. [10, Section 4]).

The following lemma will be needed later.

Lemma 4.3. *Let $w \in W_{Sp(n)}$ and let s_ν be a simple reflection of $W_{Sp(n)}$. If $ws_\nu \leq w$ in the Bruhat order, then $s_\nu[\mathcal{O}_w] = [\mathcal{O}_w]$.*

Proof. The multiplication by s_ν from the right (see equation (2)) induces a smooth map from $\mathcal{Y}$ to $\mathcal{Y}$. Consequently, since $\mathcal{Y}$ is a smooth variety, for any (not necessarily smooth) subvariety Z of $Sp(n)/T$, we have $s_\nu[\mathcal{O}_Z] = [\mathcal{O}_{s_\nu(Z)}]$ (note that $s_\nu^{-1} = s_\nu$). In particular, for $w \in W_{Sp(n)}$ we have $s_\nu[\mathcal{O}_w] = [\mathcal{O}_{s_\nu(\mathcal{Y}_w)}]$. Since $ws_\nu \leq w$, we have $s_\nu(\mathcal{Y}_w) = \mathcal{Y}_w$ (as we will see below), hence $s_\nu[\mathcal{O}_w] = [\mathcal{O}_w]$, as needed.

We still need to prove that $s_\nu(\mathcal{Y}_w) = \mathcal{Y}_w$. Since $s_\nu^{-1} = s_\nu$, it is enough to show that $s_\nu(\mathcal{Y}_w) \subset \mathcal{Y}_w$. We have

$$\mathcal{Y} = Sp(n)/T = Sp(2n, \mathbb{C})/\mathcal{B},$$

where $\mathcal{B}$ is a Borel subgroup of $Sp(2n, \mathbb{C})$ with $T \subset \mathcal{B}$. By definition, $\mathcal{Y}_w = \overline{\mathcal{B}w\mathcal{B}/\mathcal{B}}$. Hence we have

$$s_\nu(\mathcal{Y}_w) = s_\nu(\overline{\mathcal{B}w\mathcal{B}/\mathcal{B}}) \subset \overline{s_\nu(\mathcal{B}w\mathcal{B}/\mathcal{B})} \subset \overline{(\mathcal{B}w\mathcal{B}\mathcal{B}s_\nu\mathcal{B})/\mathcal{B}}.$$

From the general theory of Tits systems (see for example [2]), we know that $\mathcal{B}w\mathcal{B}\mathcal{B}s_\nu\mathcal{B} \subset \mathcal{B}w\mathcal{B} \cup \mathcal{B}ws_\nu\mathcal{B}$. This union is contained in $\mathcal{Y}_w$, since $ws_\nu \leq w$ in the Bruhat order. The conclusion follows. □

From here we deduce the desired result (see again Theorem 1.1), namely:

Proposition 4.4. *The set*

$$\{[\mathcal{O}_w] \ : \ w \in W_{Sp(n)} \text{ is a maximal length representative in } W_{Sp(n)}/W_G\}$$

is a $R[T]$-basis of $\pi_T^*(K_T(\mathcal{X}))$.

Proof. Let us denote by W the subset of $W_{Sp(n)}$ consisting of all maximal length representatives of $W_{Sp(n)}/W_G$. The previous lemma implies that if $v \in W_G$ and $w \in W$, then

$$v[\mathcal{O}_w] = [\mathcal{O}_w].$$

More precisely, we write $v = s_1 \dots s_k$, where $s_1, \dots, s_k \in \{s_{2L^1}, \dots, s_{2L^n}\}$, which is the generating set of W_G (see Section 2); by Lemma 4.3 we have $s_\nu[\mathcal{O}_w] = [\mathcal{O}_w]$, for all $\nu \in \{1, 2, \dots, k\}$. We deduce from Proposition 4.2 that $\pi_T^*(K_T(\mathcal{X}))$ contains all $[\mathcal{O}_w]$ with $w \in W$.

To prove the converse inclusion, let us suppose that there exists $\xi \in K_T(\mathcal{X})$ such that $\xi \notin \bigoplus_{w \in W} R[T][\mathcal{O}_w]$. Since $K_T(\mathcal{Y}) = \bigoplus_{w \in W_{Sp(n)}} R[T][\mathcal{O}_w]$, we can write

$$\xi = \sum_{w \in W_{Sp(n)}} a_w[\mathcal{O}_w],$$

where $a_w \in R[T]$. We deduce

$$\xi' := \xi - \sum_{w \in W} a_w[\mathcal{O}_w] \in K_T(\mathcal{X}) \setminus \{0\}.$$

Consequently, the set $\{\xi'\} \cup \{[\mathcal{O}_w] \ : \ w \in W\}$ is an $R[T]$-free family of $n! + 1$ elements in $K_T(\mathcal{X})$. This is not possible, since, by Proposition 4.1, $K_T(\mathcal{X})$ is a free $R[T]$-module of rank $n!$. The contradiction finishes the proof. □

Remark. We would like to point out that a result similar to Proposition 4.4 holds true for the usual (that is, non-equivariant) K-theory group of $Fl_n(\mathbb{H})$. Namely, from Pittie's theorem [16], we have $K(\mathcal{X}) = R[G]/R[Sp(n)]$ and $K(\mathcal{Y}) = R[T]/R[Sp(n)]$. In terms of these identifications, the homomorphism $\pi^* : K(\mathcal{X}) \to K(\mathcal{Y})$ induced by $\pi : \mathcal{Y} \to \mathcal{X}$ is the inclusion induced by the (injective) map $R[G] \to R[T]$ which assigns to a representation of G its restriction to T. We deduce that

$$K(\mathcal{X}) = K(\mathcal{Y})^{W_G}.$$

Consequently, $K(\mathcal{X})$ is the subring of $K(\mathcal{Y})$ generated by the elements of the Schubert basis induced by $w \in W_{Sp(n)}$ which are maximal length representatives of $W_{Sp(n)}/W_G$: this can be proved by using the non-equivariant analogue of Proposition 4.4.

We conclude the section with a GKM description of $K_T(\mathcal{X})$. We will deduce it from the GKM description of $K_T(\mathcal{Y})$ (recall that $\mathcal{Y}$ is a complete complex flag variety) and the fact that $K_T(\mathcal{X}) = K_T(\mathcal{Y})^{W_G}$ (see Proposition 4.2). The notations established in Section 2 are used in what follows.

First, we identify $\mathcal{Y}^T = W_{Sp(n)}$. By [15, Theorem 1.6], the ring homomorphism $K_T(\mathcal{Y}) \to K_T(W_{Sp(n)})$ induced by the inclusion map is injective and its image is

$$\{(f_w) \in \prod_{w \in W_{Sp(n)}/W_G} R[T] \ : \ e^\alpha - 1 \text{ divides } f_w - f_{s_\alpha w} \text{ for all } \alpha \in \Delta^+\}.$$

The isomorphism is $W_{Sp(n)}$-equivariant if we let the Weyl group $W_{Sp(n)}$ act on the space in the right-hand side of the previous equation by

$$v.(f_w) = (f_{wv^{-1}})$$

for all $v \in W_{Sp(n)}$. This can be proved as follows: take $v, w \in W_{Sp(n)}$ and consider the ring automorphism v^* of $K_T(\mathcal{Y})$ induced by v (see equation (2)), as well as the map $i_w^* : K_T(\mathcal{Y}) \to K_T(\{w\})$ induced by the inclusion $\{w\} \hookrightarrow \mathcal{Y}$; for any $x \in K_T(\mathcal{Y})$ we have

$$i_w^*(v^*(x)) = (v \circ i_w)^*(x) = i_{v(w)}^*(x) = i_{wv^{-1}}^*(x).$$

Consequently, $K_T(\mathcal{Y})^{W_G}$ can be identified with

$$\{(f_{\overline{w}}) \in \prod_{\overline{w} \in W_{Sp(n)}/W_G} R[T] \ : \ e^\alpha - 1 \text{ divides } f_{\overline{w}} - f_{\overline{s_\alpha w}} \text{ for all } \alpha \in \Delta^+ \text{ such that } s_\alpha \notin W_G\}.$$

Here $W_{Sp(n)}/W_G$ denotes the set of right cosets $\overline{w} = wW_G$ with $w \in W_{Sp(n)}$. The reason why in the previous equation we only need to consider roots $\alpha \in \Delta^+$ such that $s_\alpha \notin W_G$ is that if s_α does belong to W_G then

$$\overline{s_\alpha w} = \overline{ww^{-1}s_\alpha w} = \overline{w},$$

for any $w \in W_{Sp(n)}$ (because W_G is a normal subgroup of $W_{Sp(n)}$). The roots $\alpha \in \Delta^+$ such that $s_\alpha \notin W_G$ are $L^\mu - L^\nu$ and $L^\mu + L^\nu$, where $1 \le \mu < \nu \le n$. We saw in Section 2 that via the identification $W_{Sp(n)}/W_G = \mathcal{S}_n$, we have $s_{L^\mu - L^\nu} W_G = s_{L^\mu + L^\nu} W_G = (\mu, \nu)$. Let us denote

$$e^{L_\nu} = x_\nu, \ 1 \le \nu \le n.$$

This implies that $e^{L^\mu - L^\nu} - 1 = x_\mu x_\nu^{-1} - 1$ and $e^{L^\mu + L^\nu} - 1 = x_\mu x_\nu - 1$. Since these two polynomials are relatively prime in $\mathbb{Z}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$, we deduce the following proposition.

Proposition 4.5. *The homomorphism $K_T(Fl_n(\mathbb{H})) \to K_T(Fl_n(\mathbb{H})^T)$ induced by the inclusion $Fl_n(\mathbb{H})^T \to Fl_n(\mathbb{H})$ is injective. Its image is equal to*

$$\{(f_\tau) \in \prod_{\tau \in \mathcal{S}_n} \mathbb{Z}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}] \ : \ (x_\mu x_\nu^{-1} - 1)(x_\mu x_\nu - 1) \textit{ divides } f_\tau - f_{(\mu,\nu)\tau} \textit{ for all } 1 \le \mu < \nu \le n\}.$$

5. G-equivariant K-theory

In this section we will prove Theorem 1.2. Like in the previous section, we denote $\mathcal{X} = Fl_n(\mathbb{H})$, which is the same as the homogeneous space $Sp(n)/G$. We consider the following commutative diagram:

$$\begin{tikzcd}
K_G(\mathcal{X}) \arrow[r, "j^*"] \arrow[d, "i_G^*"] & K_T(\mathcal{X}) \arrow[d, "i_T^*"] \\
\prod_{\mathcal{S}_n} R[G] \arrow[r, "\tilde{p}"] & \prod_{\mathcal{S}_n} R[T]
\end{tikzcd}$$

Here j^* and $\tilde{p}$ are induced by the restriction of the G action to T. The maps i_T^*, $\tilde{p}$, and j^* are injective: the first by Proposition 4.5, the second by e.g. [7, Chapter 13, Section 8], and the last by [15, Theorem 4.4]. We deduce that i_G^* is injective too.

We are interested in the image of i_G^*. Let us consider the action of W_G on $R[T]\otimes_{R[Sp(n)]}R[G]$ given by

$$w.(\chi_1 \otimes \chi_2) = (w\chi_1) \otimes \chi_2,$$

for any $\chi_1 \in R[T], \chi_2 \in R[G], w \in W_G$, as well as the diagonal action of W_G on $\prod_{\mathcal{S}_n} R[T]$. We need the following lemma.

Lemma 5.1. *The map*

$$i_T^* \circ \overline{\kappa}_T : R[T] \otimes_{R[Sp(n)]} R[G] \to \prod_{\mathcal{S}_n} R[T]$$

is a W_G-equivariant homomorphism.

Proof. We take $w \in \mathcal{X}^T$ and show that the map $R[T] \times R[G] \to R[T]$ given by

$$(\chi_1, \chi_2) \mapsto i_T^*(\kappa_T(\chi_1 \otimes \chi_2))_w \tag{14}$$

is W_G-equivariant (the map κ_T was defined in Section 4). We identify $\mathcal{X} = Sp(n)/G$ and write $w = kG$, where $k \in N_{Sp(n)}(G)$. Assume that χ_1, χ_2 in equation (14) are the characters of the representations (V_1, ρ_1), respectively (V_2, ρ_2). An easy exercise shows that $\kappa_T(\chi_1 \otimes \chi_2)_w$ is the T-representation on $V_1 \otimes V_2$ given by

$$t.(v_1 \otimes v_2) = v_1 \otimes \rho_2(k^{-1}tk)(v_2),$$

for all $v_1 \in V_1, v_2 \in V_2$ and $t \in T$. This is the tensor product of two T-representations, the first one being trivial and the second one lying in $R[G]$, which is the same as $R[T]^{W_G}$. The W_G-equivariance of our map is now clear. □

From now on we will identify

$$K_T(\mathcal{X}) = R[T] \otimes_{R[Sp(n)]} R[G],$$

by using Proposition 4.2. In this way, W_G acts on $K_T(\mathcal{X})$. From the previous lemma we deduce that the map i_T^* is a W_G-equivariant homomorphism. Since $R[G] = R[T]^{W_G}$, the image of the map

$$\tilde{p} \circ i_G^* = i_T^* \circ j^*$$

is included in $(i_T^*(K_T(\mathcal{X})))^{W_G}$. Since i_T^* is injective, we deduce that the image of j^* is contained in $K_T(\mathcal{X})^{W_G}$.

Like in the T-equivariant case (see the previous section), we consider the homomorphism

$$\overline{\kappa}_G : R[G] \otimes_{R[Sp(n)]} R[G] \to K_G(\mathcal{X}),$$

which is $R[G]$-linear and satisfies

$$\overline{\kappa}_G(1 \otimes \chi) = [V_\rho]$$

for any representation $\rho : G \times V \to V$ of character χ. For the definition of V_ρ, see equation (8). The composition

$$j^* \circ \overline{\kappa}_G : R[G] \otimes_{R[Sp(n)]} R[G] \to K_T(\mathcal{X}) = R[T] \otimes_{R[Sp(n)]} R[G]$$

is induced by the restriction map from G to T. Since $R[G] = R[T]^{W_G}$, the map $j^* \circ \overline{\kappa}_G$ is an isomorphism between $R[G] \otimes_{R[Sp(n)]} R[G]$ and its image in $K_T(\mathcal{X})$. The image is actually equal to $K_T(\mathcal{X})^{W_G}$. Consequently, the image of j^* is $K_T(\mathcal{X})^{W_G}$. Since j^* is injective (as we saw above), we deduce that it is an isomorphism between $K_G(\mathcal{X})$ and $K_T(\mathcal{X})^{W_G}$. Incidentally, we have also proved the following result:

Proposition 5.2. *The map $\overline{\kappa}_G$ defined above is a ring isomorphism.*

From the commutative diagram at the beginning of the section we deduce that the image of i_G^* consists of all W_G-invariants in the image of i_T^*. By Proposition 4.5, the image of i_T^* is

$$\{(f_\tau) \in \prod_{\tau \in \mathcal{S}_n} \mathbb{Z}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]^{W_G} \ : \ (x_\mu x_\nu^{-1} - 1)(x_\mu x_\nu - 1) \text{ divides } f_\tau - f_{(\mu,\nu)\tau} \text{ for all } 1 \le \mu < \nu \le n\}.$$

By Lemma 5.3, below, this is the same as

$$\{(f_\tau) \in \prod_{\tau \in \mathcal{S}_n} \mathbb{Z}[X_1, \dots, X_n] \ : \ f_\tau - f_{(\mu,\nu)\tau} \text{ is divisible by } X_\mu - X_\nu \text{ for all } 1 \le \mu < \nu \le n\}.$$

Theorem 1.2 is now completely proved. The following lemma has been used above.

Lemma 5.3. *Let $x_1, \dots, x_n$ be some variables and set*

$$X_\nu = x_\nu + x_\nu^{-1},$$

$1 \le \nu \le n$. An element of $\mathbb{Z}[X_1, \dots, X_n]$ is divisible by $(x_\mu x_\nu^{-1} - 1)(x_\mu x_\nu - 1)$ in the ring $\mathbb{Z}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ if and only if it is divisible by $X_\mu - X_\nu$ in $\mathbb{Z}[X_1, \dots, X_n]$.

Proof. We first prove the following claim.

Claim. The ring $\mathbb{Z}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ is a free module over $\mathbb{Z}[X_1, \dots, X_n]$ of basis $x_1^{\epsilon_1} \dots x_n^{\epsilon_n}$, where $\epsilon_\nu \in \{-1, 0\}$, $1 \le \nu \le n$.

We first notice that if x is a variable, $X = x + x^{-1}$, and R is an arbitrary unit ring, then $R[x, x^{-1}]$ is a free module over $R[X]$ of basis $1, x^{-1}$. This implies the claim by a recursive argument. Namely, we take successively $R = \mathbb{Z}[x_\mu^{\pm 1}, \dots, x_n^{\pm 1}]$ for $\mu = 2, \dots, n$, which gives $R[x_{\mu-1}^{\pm 1}] = \mathbb{Z}[x_{\mu-1}^{\pm 1}, \dots, x_n^{\pm 1}]$.

We now turn to the proof of the lemma. For sake of simplicity let us make $\mu = 1$ and $\nu = 2$. We note that

$$X_1 - X_2 = x_1^{-1}(x_1 x_2^{-1} - 1)(x_1 x_2 - 1).$$

Thus, if f is divisible by $X_1 - X_2$ then it is divisible by $(x_1 x_2^{-1} - 1)(x_1 x_2 - 1)$. We will now prove the converse. Assume that $f \in \mathbb{Z}[X_1, \dots, X_n]$ is divisible by $(x_1 x_2^{-1} - 1)(x_1 x_2 - 1)$. We deduce that

$$f = (X_1 - X_2)g, \tag{15}$$

where $g \in \mathbb{Z}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$. We consider the expansion of g with respect to the basis indicated in the claim and denote by $g_0 \in \mathbb{Z}[X_1, \dots, X_n]$ the coefficient of 1. Equation (15) implies

$$f = (X_1 - X_2)g_0.$$

This finishes the proof. □

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