

NON-ABELIAN GKM THEORY

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ABSTRACT. We describe a generalization of GKM theory for actions of arbitrary compact connected Lie groups. To an action satisfying the non-abelian GKM conditions we attach a graph encoding the structure of the non-abelian 1-skeleton, i.e., the subspace of points with isotopy rank at most one less than the rank of the acting group. We show that the algebra structure of the equivariant cohomology can be read off from this graph. In comparison with ordinary abelian GKM theory, there are some special features due to the more complicated structure of the non-abelian 1-skeleton.

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1. INTRODUCTION

The action of a compact torus $T = (S^1)^r$ on a compact connected manifold M is particularly nice when the corresponding equivariant cohomology group $H_T^*(M)$, regarded as a module over $H^*(BT) = S(\mathfrak{t}^*)$, is free: such actions are called *equivariantly formal*. (In this paper the coefficient ring for cohomology will always be \mathbb{R} .) In fact, the equivariant cohomology group has a canonical structure of algebra over $H^*(BT)$: the by now classical Chang-Skjelbred Lemma [3, Lemma 2.3] states that, under the equivariant formality assumption, this structure is encoded into the 1-skeleton of the T -action, i.e., the subspace $M_{r-1,T}$ of points in M whose isotropy group has codimension at most one in T . Important progress was made by Goresky, Kottwitz, and MacPherson, who, originally in the context of algebraic actions of tori on complex projective varieties, found an explicit combinatorial description of $H_T^*(M)$ in terms of the T -action on $M_{r-1,T}$, see [8, Theorem 7.2]. As it turns out, their result holds for any T -action on M , under the additional assumptions that the fixed point set of the T -action is finite and $M_{r-1,T}$ is a union of 2-spheres. (In the case of an orientable M , the last condition is equivalent to the pairwise independence of the weights of the isotropy representations at the fixed points). The relevant data can be encoded in a graph with vertices corresponding to the fixed points and edges corresponding to the 2-spheres in $M_{r-1,T}$.

This approach to equivariant cohomology, nowadays referred to as GKM theory, has received much attention in the last years. For example, the interplay between the combinatorics of the GKM graph and the geometry of M has been investigated in the case when M is symplectic and T acts in Hamiltonian fashion, see, e.g., [13], [14], and also when M is a quotient G/H of two compact connected Lie groups G and H of equal rank and T is a maximal torus in H , see [11]. Various versions and generalizations of GKM theory were proposed: Guillemin and Holm [10] allowed the group action to have non-isolated fixed points (in the setting of Hamiltonian torus actions on symplectic manifolds). In [6], the first named author, together with Nozawa and Töben, considered GKM theory for Cohen-Macaulay torus actions,

which are similar to equivariantly formal actions, but for which the fixed point set is replaced by the orbits of lowest dimension. Actions of arbitrary, possibly non-abelian, topological groups on certain stratified spaces have been considered by Harada, Henriques, and Holm in [15].

Our aim is to present a generalization of the classical GKM theory to the action of a non-abelian compact connected Lie group G on a compact connected manifold M . As already motivated by the Borel localization theorem, see, e.g., [9, Theorem C.70], the natural replacement for the fixed point set in this context is the space M_{\max} of points whose isotropy group has rank equal to the rank of G . In Section 2 we will show that the natural replacement of $M_{r-1,T}$ is the space M_{r-1} of points whose isotropy rank is at most one smaller than the rank of G , in the sense that there is a non-abelian version of the Chang-Skjelbred Lemma involving this space, see Theorem 2.2.

Having thus laid the theoretical foundation for our theory, we will, after reviewing (a slight generalization of) the classical GKM theory in Section 3, develop non-abelian GKM theory in Sections 4-6, under the so-called *non-abelian GKM conditions*:

- (i) The G -action is equivariantly formal.
- (ii) The space M_{\max} consists of finitely many G -orbits.
- (iii) For every $p \in M_{\max}$, the weights of the isotropy representation of G_p on T_pM are pairwise linearly independent.

We remark that these conditions are equivalent to the ordinary (abelian) GKM conditions for the induced action of a maximal torus of G : see Lemma 4.2 below.

Our main task will be to understand the structure of the space M_{r-1} . In the torus case, the GKM conditions imply that $M_{r-1,T}$ is always the union of two-dimensional manifolds which connect the fixed points, and on which the torus acts with one-dimensional maximal orbits (i.e., very simple cohomogeneity-one manifolds). Abstractly, these can be defined as the closures of the components of $M_{r-1,T} \setminus M^T$, where M^T is the fixed point set. In the non-abelian setting, we will still consider the closures N of the components of $M_{r-1} \setminus M_{\max}$. Although these subspaces of M are not necessarily smooth, see Remark 4.12, the group G acts on them with cohomogeneity-one in the sense that their G -orbit space is homeomorphic to either a closed interval or a circle. Any of the spaces N above is of one of the following three types:

- (I) The case which is most similar to that of a 2-sphere in the 1-skeleton of a torus action is that of a space N whose G -orbit space is a closed interval, with both endpoints of the interval corresponding to orbits in M_{\max} .
- (II) It can happen that the G -orbit space of N is an interval, but only one of the endpoints corresponds to an orbit in M_{\max} . In the torus case this phenomenon occurs only if one allows nonorientable manifolds; then, there can appear real projective two-planes in $M_{r-1,T}$, see Section 3. In the non-abelian setting this phenomenon is not connected to orientability, see Example 6.5.
- (III) The case when the G -orbit space of N is a circle can only appear if N is not smooth, see Example 5.1. This case has no analogue for torus actions on manifolds.

The GKM graph associated to a torus action contains a vertex for each fixed point and an edge for each two-sphere in the 1-skeleton connecting the two vertices which

correspond to the two fixed points in the sphere. By analogy, the vertices of the nonabelian GKM graph to be constructed in Section 6 should be related to the orbits of maximal rank, and the edges to the spaces N above. However, this simple analogy is not sufficient: for example, we see from case (II) above that we need a second type of vertices representing those endpoints which correspond to orbits not contained in M_{\max} . For the detailed construction of the non-abelian GKM graph, we refer to Section 6.

A final comment concerns the effectiveness of our method for computing the $S(\mathfrak{g}^*)^G$ -algebra $H_G^*(M)$. This can also be done by using the well-known formula

$$H_G^*(M) = H_T^*(M)^{W(G)},$$

where $T \subset G$ is a maximal torus and $W(G)$ the corresponding Weyl group. As already pointed out, in our context the algebra $H_T^*(M)$ can be expressed by means of the classical GKM theory. Since the action of $W(G)$ on this algebra can also be made precise, the formula above leads to concrete descriptions of $H_G^*(M)$. Although such a description has in general a more complicated appearance than the one which results from our approach, it can, in principle, be reduced to that one. However, a considerable amount of work is sometimes needed to get this reduction. For a discussion of concrete examples we refer to Remark 6.8. Perhaps even more important than the simplicity of our presentations of the equivariant cohomology algebra is their “naturalness” in the sense that they reflect closely the geometry of the action. In other words, the focus here is not that much on combinatorics, but rather on geometry.

2. A NON-ABELIAN CHANG-SKJELBRED LEMMA

Let G be a compact connected Lie group of rank r acting on a compact connected manifold M . For each $i = 0, \dots, r$ we define $M_i := \{p \in M \mid \text{rank } G_p \geq i\}$. (As usual, G_p is the isotropy group at p .) We also write $M_{\max} := M_r$.

Consider first the case when the Lie group G is Abelian, i.e., a torus T of rank r . Then, we have $M_{\max} = M^T$, the fixed point set of the action. Assuming that the action is equivariantly formal, it is well-known that the restriction map $H_T^*(M) \rightarrow H_T^*(M^T)$ is injective. The Chang-Skjelbred Lemma [3, Lemma 2.3] then states that its image is the same as the image of $H_T^*(M_{r-1}) \rightarrow H_T^*(M^T)$. Put together, we have a short exact sequence

$$(2.1) \quad 0 \longrightarrow H_T^*(M) \longrightarrow H_T^*(M^T) \longrightarrow H_T^{*+1}(M_{r-1}, M^T).$$

In this section we derive a non-abelian version of this short exact sequence, see Theorem 2.2 below.

From now on, G is again an arbitrary compact connected Lie group of rank r . We need some considerations concerning the Krull dimension of certain (G -equivariant cohomology) $H^*(BG)$ -modules: throughout this section, dimension means always *Krull* dimension. The following result is a slight generalization of [4, Lemma 4.4]. It can be proven with exactly the same arguments as that result (the only new ingredient needed in the proof is the general result saying that if K is a compact, possibly non-connected, closed subgroup of G , then the dimension of $H^*(BK)$ over $H^*(BG)$ is equal to the rank of K , see, e.g., [5, Proposition 2.6]).

Lemma 2.1. *The dimension of $H_G^*(M, M_i)$ is at most equal to $i - 1$.*

We are now in a position to state and prove the main result of this section.

Theorem 2.2. *If an action of a compact connected Lie group G on a compact connected manifold M is equivariantly formal, then*

$$0 \longrightarrow H_G^*(M) \longrightarrow H_G^*(M_{\max}) \longrightarrow H_G^{*+1}(M_{r-1}, M_{\max})$$

is exact.

Proof. The injectivity of $H_G^*(M) \rightarrow H_G^*(M_{\max})$ was proven in [7, Proposition 4.8]. The exactness at $H_G^*(M_{\max})$ can be proven with the same method as the exactness of the Atiyah-Bredon sequence for torus actions [2, Main Lemma], see also [4]. We decided to include the proof here for the sake of completeness. Because

$$0 \longrightarrow H_G^*(M) \longrightarrow H_G^*(M_{\max}) \longrightarrow H_G^{*+1}(M, M_{\max}) \longrightarrow 0$$

is exact, we have $\text{depth } H_G^*(M, M_{\max}) \geq r - 1$. On the other hand, the dimension of $H_G^*(M, M_{\max})$ is at most $r - 1$, by Lemma 2.1, so it follows that $H_G^*(M, M_{\max})$ has dimension equal to $r - 1$ (recall that, in general, the depth cannot exceed the Krull dimension).

Since the dimension of $H_G^*(M, M_{r-1})$ is at most equal to $r - 2$, the same can be said about the dimension of the image of the natural map $H_G^*(M, M_{r-1}) \rightarrow H_G^*(M, M_{\max})$. Thus, this image is equal to 0 (since otherwise, by [4, Lemma 4.3], it would have dimension equal to $r - 1$). It follows that

$$0 \longrightarrow H_G^*(M, M_{\max}) \longrightarrow H_G^*(M_{r-1}, M_{\max}) \longrightarrow H_G^{*+1}(M, M_{r-1}) \longrightarrow 0$$

is exact. Thus, an element in the kernel of $H_G^*(M_{\max}) \rightarrow H_G^{*+1}(M_{r-1}, M_{\max})$ must already lie in the kernel of $H_G^*(M_{\max}) \rightarrow H_G^{*+1}(M, M_{\max})$ and hence is in the image of $H_G^*(M) \rightarrow H_G^*(M_{\max})$. \square

Remark 2.3. The theorem suggests that there might exist a non-abelian version of the Atiyah-Bredon sequence as in [2]. However this is unclear, as to copy the proof given there one needs to know that the modules $H_G^*(M_i, M_{i+1})$ are Cohen-Macaulay of dimension i . This is so far only proven for $i = r$: [7, Corollary 4.5] shows that $H_G^*(M_r, M_{r+1}) = H_G^*(M_{\max})$ is a free module over $H^*(BG)$.

3. ABELIAN GKM THEORY

In this section we review the theorem of Goresky, Kottwitz, and MacPherson [8, Theorem 7.2] (see also [12, Section 11.8]) concerning the description of the equivariant cohomology of a torus action in terms of the 1-skeleton. A compact, connected manifold M with an action of a torus T is a *GKM space* if:

- (i) The T -action on M is equivariantly formal.
- (ii) The fixed point set M^T is finite.
- (iii) For every $p \in M^T$, the weights of the isotropy representation of T on (the tangent space) $T_p M$ are pairwise linearly independent.

The theory we present here is slightly more general than the classical one, in the sense that the manifold M is not required to be orientable; hence the connected components of the fixed point sets of (codimension-one) subtori of T are not necessarily orientable. Consequently, there might be two-dimensional submanifolds in the 1-skeleton that are not necessarily spheres which need to be taken into account (see Lemma 3.3 and Example 3.8 below): these new features are reflected in the resulting GKM graph by the presence of some stars, like for instance in Figure 4.

3.1. The abelian 1-skeleton. We set $r := \text{rank} T$ and consider the subspace M_{r-1} of all $p \in M$ whose isotropy group T_p has dimension at least $r-1$, also called the *1-skeleton* of the action. It is the union of the fixed point sets of finitely many codimension-one subtori of T . The following result, which is Theorem 11.6.1 in [12], states that any connected component of the fixed point set of a subtorus of T contains a T -fixed point. We give another proof of this result here which does not use the notions of equivariant Thom and Euler class. Note that the lemma is more general than needed here, i.e., it only requires the action to be equivariantly formal.

Lemma 3.1. *If the action of a torus T on a compact connected manifold M is equivariantly formal and $H \subset T$ is a subtorus, then any connected component of the fixed point set M^H contains a T -fixed point.*

Proof. Suppose that there exists a connected component of M^H , call it N , which does not contain any T -fixed point. We isolate N from the rest of M^H by a T -equivariant open tubular neighborhood, call it U . We set $V := M \setminus N$, which is also a T -invariant open subset of M . By the argument in the proof of [12, Theorem 11.4.1], the support of $H_T^*(U \cap V; \mathbb{C})$ is contained in the union of the complexified Lie algebras of all isotropy groups of points in $U \cap V$. None of these groups contain H , hence (because H is connected) the union above cannot contain $\mathfrak{h} \otimes \mathbb{C}$, where \mathfrak{h} is the Lie algebra of H . In particular, $H_T^*(U \cap V; \mathbb{C})$ is a torsion $H^*(BT; \mathbb{C})$ -module. We now consider the Mayer-Vietoris sequence of the pair U, V in equivariant cohomology. The map $H_T^*(U \cap V; \mathbb{C}) \rightarrow H_T^{*+1}(M; \mathbb{C})$ which arises in this sequence is identically zero, since it is a homomorphism of $H^*(BT; \mathbb{C})$ -modules whose domain is a torsion module and whose range is a free module. Thus the Mayer-Vietoris sequence splits into short exact sequences:

$$(3.1) \quad 0 \longrightarrow H_T^*(M; \mathbb{C}) \longrightarrow H_T^*(U; \mathbb{C}) \oplus H_T^*(V; \mathbb{C}) \longrightarrow H_T^*(U \cap V; \mathbb{C}) \longrightarrow 0.$$

As all the fixed points of the action are contained in V , the natural restriction map $H_T^*(M; \mathbb{C}) \rightarrow H_T^*(M^T; \mathbb{C})$ (which is injective by equivariant formality, see (2.1)) factors through $H_T^*(V; \mathbb{C})$. Thus, $H_T^*(M; \mathbb{C}) \rightarrow H_T^*(V; \mathbb{C})$ is injective, which, via the short exact sequence (3.1), implies that the restriction map $H_T^*(U; \mathbb{C}) \rightarrow H_T^*(U \cap V; \mathbb{C})$ is injective. But then the support of $H_T^*(U \cap V; \mathbb{C})$ contains the support of $H_T^*(U; \mathbb{C})$, which in turn contains $\mathfrak{h} \otimes \mathbb{C}$ (because $H_T^*(U; \mathbb{C}) \simeq H_T^*(N; \mathbb{C}) \simeq H_{T/H}^*(N; \mathbb{C}) \otimes S((\mathfrak{h} \otimes \mathbb{C})^*)$). We obtained a contradiction, which finishes the proof. \square

Remark 3.2. The result stated in the lemma may fail to be true in the case when the group H is not connected. Consider for example the canonical S^1 -action on $\mathbb{R}P^2$ and the subgroup $\mathbb{Z}_2 = \{-1, 1\}$ of S^1 . One of the connected components of $(\mathbb{R}P^2)^{\mathbb{Z}_2}$ is diffeomorphic to a circle and contains no S^1 -fixed points.

From now on we will assume that the T -action on M satisfies the GKM conditions above. Using Lemma 3.1, one shows [12, Theorem 11.8.1] that any positive-dimensional component of a fixed point set of a codimension-one subtorus of T is two-dimensional. We will refer to these components as the *components of the 1-skeleton* M_{r-1} . The following lemma describes these submanifolds.

Lemma 3.3. *Let $T \times N \rightarrow N$ be a nontrivial action of a torus T on a compact, connected, two-dimensional manifold N with nonempty fixed point set. Then:*

- (a) *The kernel K of the action is of codimension one in T .*
- (b) *N is diffeomorphic to S^2 or $\mathbb{R}P^2$ in such a way that the action of the circle T/K corresponds to the standard action of S^1 on S^2 respectively $\mathbb{R}P^2$.*

In particular, the T -action has two or one fixed points, depending on whether N is diffeomorphic to S^2 or $\mathbb{R}P^2$.

Proof. Because the T -action has fixed points, it cannot be transitive. It is also nontrivial, which implies that it is of cohomogeneity-one (group actions of this kind are discussed in Section 4.2 below). The result then follows readily from the classification of cohomogeneity-one actions of compact Lie groups on two dimensional compact manifolds, see, e.g., [21, Section 6]. There are only three such actions that are effective (and have fixed points), namely the actions of S^1 on S^2 , $\mathbb{R}P^2$, respectively the Klein bottle K^2 . The first two actions are standard. To describe the third one, we recall that K^2 can be written as the orbit space $(S^1 \times S^1)/\mathbb{Z}_2$, where the \mathbb{Z}_2 -action is given by $(z_1, z_2) \mapsto (-z_1, z_2^{-1})$. The S^1 -action on K^2 is given by multiplication on the first factor. This action has no fixed points, so we can rule it out. \square

3.2. The GKM graph. By the short exact sequence (2.1), the map $H_T^*(M) \rightarrow H_T^*(M^T)$ is injective and its image is the same as the image of the map $H_T^*(M_{r-1}) \rightarrow H_T^*(M^T)$. To understand this image, we first need to consider the two special situations described in Lemma 3.3 and compute the corresponding equivariant cohomology algebras:

- Lemma 3.4.**
- (a) *Assume that a torus T acts on the sphere S^2 with unique fixed points p and q , such that a codimension-one subgroup $K \subset T$ acts trivially. Then the restriction map $H_T^*(S^2) \rightarrow H_T^*({p}) \oplus H_T^*({q}) = S(\mathfrak{t}^*) \oplus S(\mathfrak{t}^*)$ defines an isomorphism between $H_T^*(M)$ and the space of all pairs $(f, g) \in S(\mathfrak{t}^*) \oplus S(\mathfrak{t}^*)$ such that $f|_{\mathfrak{k}} = g|_{\mathfrak{k}}$. Here \mathfrak{k} is the Lie algebra of K .*
 - (b) *Assume that a torus T acts on the projective space $\mathbb{R}P^2$ with unique fixed point p . Then the restriction map $H_T^*(\mathbb{R}P^2) \rightarrow H_T^*({p}) = S(\mathfrak{t}^*)$ is an isomorphism.*

Proof. Both assertions are consequences of the equivariant Mayer-Vietoris sequence; see also [12, Theorem 11.7.2] for case (a). In case (b), one considers tubular neighborhoods around p and the unique exceptional orbit, and uses that the restriction map from the tubular neighborhood of the exceptional orbit (i.e., whose T/K -isotropy is \mathbb{Z}_2 , see Lemma 3.3) to a regular orbit induces an isomorphism in cohomology. Note that both statements are special cases of [5, Corollary 4.2]. \square

The classical GKM graph encodes the structure of the 1-skeleton of an equivariantly formal torus action under the assumption that it consists of just a union of 2-spheres. In our more general situation, the graph has some new features, as follows. There are two types of vertices, *dots* and *stars*. As in the orientable case, the dots correspond to the T -fixed points. Two dots p and q are joined by an edge if there exists a 2-sphere in the 1-skeleton whose T -fixed points are p and q ; the edge is labeled with K , the kernel of the T -action on the sphere. Each real projective two-plane in the 1-skeleton contains exactly one T -fixed point, say p , and one exceptional T -orbit. We draw a star representing the exceptional orbit, and an edge connecting this star with the dot which corresponds to p .

Remark 3.5. If the dimension of M is $2n$ (the codimension of the fixed point set is necessarily even, and, by assumption, the fixed point set is finite), then from each dot emerge exactly n edges, whereas from a star emerges always exactly one.

The equivariant cohomology can be read off from the GKM graph as follows:

Theorem 3.6. *Via the natural restriction map*

$$H_T^*(M) \rightarrow H_T^*(M^T) = \bigoplus_{p \in M^T} S(\mathfrak{t}^*),$$

the equivariant cohomology $S(\mathfrak{t}^)$ -algebra $H_T^*(M)$ is isomorphic to the set of all tuples $(f_p)_{p \in M^T}$ with the property that if $p, q \in M^T$ are joined by an edge labeled with K , then $f_p|_{\mathfrak{t}} = f_q|_{\mathfrak{t}}$.*

Proof. Using Lemma 3.4, the same proof as in ordinary GKM theory applies, see [8, Theorem 7.2]. \square

Note that the components of the 1-skeleton which are projective planes, that is, the edges connecting a dot with a star in the GKM graph, have no contribution to the equivariant cohomology. Nevertheless, we include them into the GKM graph for two reasons: firstly, we want the GKM graph to encode the entire structure of the 1-skeleton, and secondly, in the natural generalization of the GKM graph to actions of non-abelian groups to be defined below, the stars will in fact contribute nontrivially. Note that one could also label edges connecting dots with stars (and the stars themselves) with the respective isotropy groups, but this is not necessary because this data is of no relevance for the equivariant cohomology.

We end this section with two examples of GKM graphs of torus actions on non-orientable manifolds.

Example 3.7. The simplest example of the type mentioned above is of course $\mathbb{R}P^2$ with the canonical S^1 -action. The GKM graph is presented in Figure 1.



FIGURE 1

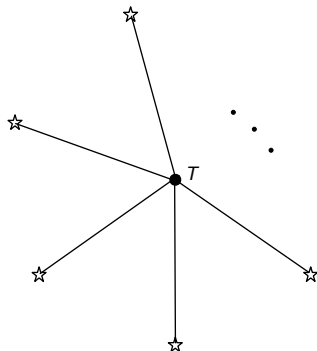
A slightly more general situation arises as follows. We start with the action of $T := (S^1)^n$ on the $2n$ -sphere

$$S^{2n} := \{(z_1, \dots, z_n, x) \in \mathbb{C}^n \times \mathbb{R} \mid |z_1|^2 + \dots + |z_n|^2 + x^2 = 1\},$$

given by

$$(\zeta_1, \dots, \zeta_n) \cdot (z_1, \dots, z_n, x) := (\zeta_1 z_1, \dots, \zeta_n z_n, x),$$

see also [20, Example 3.2]. It descends to an action of T on $\mathbb{R}P^{2n} = S^{2n} / \sim$, where \sim is the obvious equivalence relation on S^{2n} . The GKM conditions (i)-(iii) are clearly satisfied by both T -actions. We are interested here in the second one, which has one fixed point and whose 1-skeleton consists of n copies of $\mathbb{R}P^2$. This is reflected by the GKM graph below, see Figure 2. From here one reads off: $H_T^*(\mathbb{R}P^{2n}) = S(\mathfrak{t}^*)$, i.e., the polynomial ring in n variables.

FIGURE 2. The number of stars in the graph is n .

Example 3.8. We consider the real Grassmannian $G_2(\mathbb{R}^5) = G/K$, where $G := \mathrm{SO}(5)$ and $K := \mathrm{S}(\mathrm{O}(3) \times \mathrm{O}(2))$, with the action of $T := \{1\} \times \mathrm{SO}(2) \times \mathrm{SO}(2)$, which is the standard maximal torus of $\mathrm{SO}(5)$. This Grassmannian is not orientable: in general, $G_k(\mathbb{R}^n)$ is orientable iff n is even. We will show that our action satisfies the GKM assumptions (i)-(iii) above. Then we will construct the GKM graph and describe the equivariant cohomology $S(\mathfrak{t}^*)$ -algebra.

The group K has two components, namely the identity component $K_0 = \mathrm{SO}(3) \times \mathrm{SO}(2)$, along with $g_0 K_0$, where g_0 is the diagonal matrix $\mathrm{Diag}(1, 1, -1, 1, -1)$. We consider the canonical projection map $\pi : G/K_0 \rightarrow G/K$: it is the covering map that arises from the \mathbb{Z}_2 -action on G/K_0 given by multiplication by g_0 from the right.

We start with the GKM description for the T -action on G/K_0 , by following [11] (the theory derived there is applicable, as G is a compact semisimple Lie group and K_0 a connected closed subgroup of maximal rank). The T -fixed point set is $W(G)/W(K_0)$. The Lie algebra of T consists of diagonal blocks of the type $(0, A(\theta_1), A(\theta_2))$, where

$$A(\theta) := \begin{pmatrix} 0 & -\theta \\ \theta & 0 \end{pmatrix}.$$

We regard the roots as elements of the Lie algebra \mathfrak{t} of T , by putting on this a T -invariant metric. Then G and K_0 have a common root, namely $\alpha_1 := (0, A(1), A(0))$. The Weyl group $W(K_0)$ is generated by the reflection through the line perpendicular to α_1 . Thus $W(K_0)$ is the $W(G)$ -stabilizer of the root $\alpha_1 + \alpha_2$ in Figure 3 (the left hand side diagram represents the roots of G , which lie in the Lie algebra of T). That is, the set $(G/K_0)^T$ can be identified with the $W(G)$ -orbit of $\alpha_1 + \alpha_2$. This makes four points altogether, see the right hand side diagram in Figure 3; we can also see there the six T -invariant 2-spheres in G/K_0 , which are represented by thickened lines.

We would like now to describe the fixed points of the T -action on G/K . Since $\pi : G/K_0 \rightarrow G/K$ is a \mathbb{Z}_2 -covering, which is also T -equivariant, we deduce that $(G/K)^T = \pi((G/K_0)^T)$: indeed, if $p = \pi(q) \in G/K$ is T -fixed, where $q \in G/K_0$, then the subspace Tq of G/K_0 is contained in $\pi^{-1}(p)$, thus it must be equal to $\{q\}$.

In order to see concretely what $(G/K)^T$ is, one needs to understand the action of g_0 on $(G/K_0)^T$. Concretely, g_0 induces an element of the Weyl group of G and

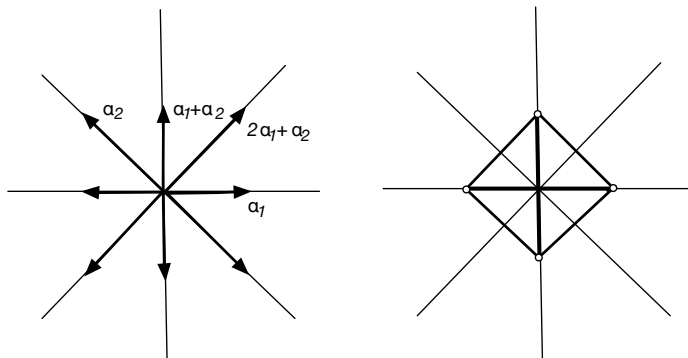


FIGURE 3

we need to understand how does this act on the roots α_1 and $\alpha_1 + \alpha_2$ (see again Figure 3). We have $\alpha_1 = (0, A(1), A(0))$ and $\alpha_1 + \alpha_2 = (0, A(0), A(1))$. Thus

$$(3.2) \quad \text{Ad}_{g_0}(\alpha_1) = -\alpha_1, \quad \text{Ad}_{g_0}(\alpha_1 + \alpha_2) = -(\alpha_1 + \alpha_2).$$

By identifying G/K_0 with the G -adjoint orbit of $\alpha_1 + \alpha_2$, we conclude that $(G/K)^T$ has two fixed points, namely the cosets modulo \mathbb{Z}_2 of α_1 and $\alpha_1 + \alpha_2$.

The weights of the T -isotropy representation at any of these two fixed points are pairwise linearly independent, since the same is true for all isotropy representations of G/K_0 at the fixed points and $\pi : G/K_0 \rightarrow G/K$ is a T -equivariant local diffeomorphism. We deduce that the 1-skeleton of G/K consists of the fixed point sets of the same subtori of T as those which determine the 1-skeleton of G/K_0 . Thus the 1-skeleton of G/K consists of the images under π of the six spheres represented in Figure 3. By equation (3.2), the “horizontal” and the “vertical” spheres are invariant under Ad_{g_0} : indeed, the latter is an element of the Weyl group and any such element permutes the T -invariant spheres. Hence the image of these two spheres are copies of $\mathbb{R}P^2$. For similar reasons, each of the two “oblique” spheres on the left hand side of the vertical axis are transformed under Ad_{g_0} into one of the other two “oblique” spheres; under π , out of the four “oblique” spheres in G/K_0 one obtains only two in G/K . Let T_1 and T_2 be the circles (1-dimensional subtori in T) which leave these two spheres pointwise fixed. Their Lie algebras are the kernels of α_2 , respectively $2\alpha_1 + \alpha_2$. Now, a general element of T is a diagonal block of the form $(1, R(\theta_1), R(\theta_2))$, where

$$R(\theta) := \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}.$$

This gives rise to a natural presentation $T = S^1 \times S^1$ and also to the following expressions:

$$\begin{aligned} T_1 &= \{(z, z) \mid z \in S^1\} \\ T_2 &= \{(z, z^{-1}) \mid z \in S^1\}. \end{aligned}$$

These are the labels carried by the two edges in the GKM graph, see Figure 4.

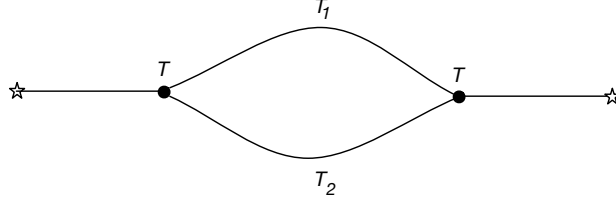


FIGURE 4

In conclusion,

$$\begin{aligned} H_T^*(G_2(\mathbb{R}^5)) &\simeq \{(f, g) \in S(\mathfrak{t}^*) \oplus S(\mathfrak{t}^*) \mid f - g \text{ is divisible by } \alpha_2(2\alpha_1 + \alpha_2)\} \\ &\simeq \{(f, g) \in \mathbb{R}[s_1, s_2]^2 \mid f(s, s) = g(s, s), f(s, -s) = g(s, -s), s \in \mathbb{R}\}. \end{aligned}$$

4. THE NON-ABELIAN 1-SKELETON

Let us consider a group action $G \times M \rightarrow M$, where G is a compact connected Lie group and M a compact connected manifold. Set $r := \text{rank } G$. If the action is equivariantly formal, then, by Theorem 2.2, we can identify $H_G^*(M)$ with an $S(\mathfrak{g}^*)^G$ -subalgebra of $H_G^*(M_{\max})$, namely with the image of the map $H_G^*(M_{r-1}) \rightarrow H_G^*(M_{\max})$. Our next goal is to obtain a description of this image. This will be done under the assumption that the following *non-abelian GKM conditions* are fulfilled:

- (i) The G -action is equivariantly formal.
- (ii) The space M_{\max} consists of finitely many G -orbits.
- (iii) For every $p \in M_{\max}$, the weights of the isotropy representation of G_p on $T_p M$ are pairwise linearly independent.

Condition (i) means, like in the abelian case, that relative to its canonical structure of $H^*(BG)$ -module, $H_G^*(M)$ is free (see, e.g., [9, Section C.4]). Concerning (ii), we recall that the weights of a G_p -representation are, by definition, the weights of the induced representation of a maximal torus in G_p . Conditions (i)-(iii) will be in force throughout the rest of the paper. In this section we obtain some preliminary results, which concern exclusively the structure of M_{r-1} : in the sequel, this space will be referred to as the *non-abelian 1-skeleton* of the G -action.

4.1. Abelian and non-abelian GKM conditions. Let us fix a maximal torus $T \subset G$. Recall from Section 2 the notation $M_i := \{p \in M \mid \text{rank } G_p \geq i\}$, and let us also define $M_{i,T} := \{p \in M \mid \dim T_p \geq i\}$, where T_p is the isotropy group of the T -action at p .

Lemma 4.1. $M_i = GM_{i,T}$. In particular, $M_{\max} = GM^T$.

Proof. If $p \in M_i$, then G_p contains a maximal torus T' of dimension at least i . There exists $g \in G$ such that $gT'g^{-1} \subset T$. Consequently $T_{gp} = G_{gp} \cap T = gG_p g^{-1} \cap T$ contains $gT'g^{-1}$, and thus has dimension at least i . Conversely, if $p \in M_{i,T}$, then G_p contains T_p , hence it has rank at least i . \square

Lemma 4.2. *The G -action satisfies the non-abelian GKM conditions if and only if M is a GKM space relative to the T -action (see Section 3).*

Proof. Equivariant formality of the G - and the T -action are equivalent by [9, Proposition C.26]. By Lemma 4.1, we have $M_{\max} = GM^T$, so if M^T is finite, then M_{\max} consists of finitely many G -orbits. Conversely, if M_{\max} consists of finitely many G -orbits, then each of them is of the form G/H with $\text{rank } G = \text{rank } H$. Since $M^T \subset M_{\max}$ and $(G/H)^T = W(G)/W(H)$ is finite, we deduce that M^T is finite. The conditions on the isotropy representations are obviously equivalent because the weights of the G_p -representation are by definition the weights of the representation of the maximal torus $T \subset G_p$. \square

Example 4.3. Let G be a semisimple compact Lie group and $K, H \subset G$ connected closed subgroups with $\text{rank } K = \text{rank } H = \text{rank } G$ and which share a maximal torus. Then the K -action on G/H by left multiplication satisfies the non-abelian GKM conditions: indeed, by [11, Theorem 1.1], the action of the common maximal torus of K and H satisfies the GKM conditions. The special cases $\text{Sp}(n)/\text{Sp}(1)^n$ and $F_4/\text{Spin}(8)$ with $K = H$ are investigated in [18] and [19] respectively.

Lemma 4.4. *If the G -action satisfies the non-abelian GKM conditions, then the induced action of T on M_{\max} satisfies the usual GKM conditions, see (i)-(iii) in Section 3.*

Proof. Any component of $M_{\max} = GM^T$ is a homogeneous space of the form G/H , where H has maximal rank in G . Since $H^{\text{odd}}(G/H) = 0$, the T -action on this component is equivariantly formal. Conditions (ii) and (iii) in the definition of GKM space, see Section 3, are also clearly satisfied. \square

4.2. A cohomogeneity-one example. Group actions of the type mentioned in the title of this subsection will be frequently used in our paper. Here are a few details concerning this notion (for more, we refer to [16, Section 1.1] or [22, Section 1]). We say that the action of a compact connected Lie group G on a closed manifold M is of cohomogeneity one if the dimension of the orbit space M/G is equal to one (equivalently, there is at least one G -orbit of codimension one in M). Thus, M/G is homeomorphic to the circle S^1 or to a (non-degenerate) closed interval. In this paper we are exclusively interested in the last situation. Let $\pi : M \rightarrow M/G \approx [-1, 1]$ be the orbit projection. Then the orbits $\pi^{-1}(t)$, $t \in (-1, 1)$ are regular, whereas $\pi^{-1}(-1)$ and $\pi^{-1}(1)$ are non-regular. To better understand the geometric structure, we endow M with a G -invariant Riemannian metric. There exists a G -transversal minimal geodesic $c : [-1, 1] \rightarrow M$ such that $c(-1)$ and $c(1)$ are in the non-regular orbits $\pi^{-1}(-1)$ and $\pi^{-1}(1)$, respectively. The isotropy groups $K_{\pm} := G_{c(\pm 1)}$ leave invariant the unit closed disks D_{\pm} in the normal spaces to the orbits $Gc(\pm 1)$, by acting transitively on the unit spheres S_{\pm} . Note that if $H := G_{c(0)}$, then

$$(4.1) \quad K_{\pm}/H \approx S_{\pm}.$$

The manifold M can be written as

$$(4.2) \quad M = G \times_{K_-} D_- \cup_{G/H} G \times_{K_+} D_+,$$

i.e., the result of gluing the tubular neighborhoods $G \times_{K_-} D_-$ and $G \times_{K_+} D_+$ along their common boundary, which is $G \times_{K_{\pm}} K_{\pm}/H = G/H$.

Conversely, assume that

$$(4.3) \quad G \supset K_-, K_+ \supset H$$

are closed subgroups of G such that conditions (4.1) hold for some spheres S_{\pm} . Then the K_{\pm} -actions on S_{\pm} extend linearly to the corresponding disks. The manifold M given by (4.2) has an obvious G -action, which has cohomogeneity one. A collection of the form (4.3) which satisfies the conditions (4.1) is called a *group diagram*. Our discussion shows that any cohomogeneity-one group action with orbit space an interval is determined by a group diagram.

We are now ready to present the example.

Example 4.5. For equivariantly formal torus actions, Lemma 3.1 shows that $M_{\max} = M^T$ is the infinitesimal bottom stratum for the action. For general G -actions this is no longer true, even if they satisfy the non-abelian GKM conditions. We illustrate this claim by an example, which is a cohomogeneity-one group action. Consider the cohomogeneity-one action (of $G = S^3 \times S^3$) with orbit space homeomorphic to a closed interval and whose group diagram is $S^3 \times S^3 \supset S^1 \times S^1, \Delta(S^3) \supset \Delta(S^1)$, where $\Delta(S^3) := \{(g, g) \mid g \in S^3\}$ and $\Delta(S^1)$ consists of pairs (g, g) with $g \in S^1$. One of the two non-regular isotropies is of maximal rank, thus $M_{\max} = S^3 \times S^3 / S^1 \times S^1$. Furthermore, this implies using [5, Corollary 1.3] that the action is equivariantly formal. The second singular orbit is part of the infinitesimal bottom stratum, but not a subset of M_{\max} . Condition (iii) in the definition of a GKM space is also fulfilled: Let $p \in M_{\max}$ with $G_p = S^1 \times S^1$. The three weights of the isotropy representation at p are pairwise linearly independent, as their kernels are given by $\{(x, 0)\} \subset \mathfrak{g}_p, \{(0, x)\} \subset \mathfrak{g}_p$, respectively $\{(x, x)\} \subset \mathfrak{g}_p$.

4.3. Abelian and non-abelian 1-skeletons. Recall from Section 3 that the GKM conditions for the T -action on M imply that $M_{r-1, T}$ can be written as a finite union $\bigcup_{i \in I} S_i \cup \bigcup_{j \in J} P_j$ where S_i are 2-spheres and P_j real projective planes. Each S_i and each P_j is a connected component of the fixed point set of a codimension-one torus $T_i \subset T$, resp. $T_j \subset T$; S_i contains exactly two T -fixed points, say p_i and q_i ; P_j contains one T -fixed point, say r_j . Observe that $M_{r-1} = \bigcup_{i \in I} GS_i \cup \bigcup_{j \in J} GP_j$ by Lemma 4.1. Each space GS_i contains the components Gp_i and Gq_i of M_{\max} ; similarly, GP_j contains Gr_j . The next lemma shows that they contain no other components of M_{\max} ; note, however, that the two components of $GS_i \cap M_{\max}$ might coincide, see Example 4.7 below.

Lemma 4.6. (a) Take $i \in I$ such that $GS_i \neq Gp_i$, i.e., GS_i is not just a G -orbit. Then $S_i \cap M_{\max} = \{p_i, q_i\}$ and consequently $GS_i \cap M_{\max} = Gp_i \cup Gq_i$.
 (b) Take $j \in J$ such that $GP_j \neq Gr_j$, i.e., GP_j is not just a G -orbit. Then $P_j \cap M_{\max} = \{r_j\}$ and consequently $GP_j \cap M_{\max} = Gr_j$.

Proof. (a) Take $p \in S_i \cap M_{\max}$. There exists $g \in G$ such that gTg^{-1} fixes p , hence $gp \in M^T$. The orbit $Ggp = Gp$ has a common point with S_i at p . The T -action on this orbit satisfies the usual GKM conditions, see Lemma 4.4. There are two cases: first, if p is an isolated fixed point of T_i in Gp , then p is fixed by T , hence $p \in \{p_i, q_i\}$; second, if p is not isolated in $(Gp)^{T_i}$, then the positive dimensional connected components of the latter set are 2-spheres or real projective planes, and one of them contains p . On the other hand, S_i is the connected component of M^{T_i} which contains p . Hence $S_i \subset Gp$, which implies that $GS_i = Gp$, contradiction.

(b) We use the same argument as in the proof of (a). \square

Example 4.7. The orbits Gp_i and Gq_i mentioned in point (a) of the lemma are not necessarily different. For example, let us consider the canonical representations

of S^1 on \mathbb{R}^2 , respectively of $\mathrm{SO}(3)$ on \mathbb{R}^3 , along with the induced (direct product) action of $S^1 \times \mathrm{SO}(3)$ on the sphere S^4 . Concretely, if we regard the elements of S^4 as pairs of the form $(x, y) \in \mathbb{R}^2 \times \mathbb{R}^3$, where $|x|^2 + |y|^2 = 1$, then the group action is given by

$$(4.4) \quad (g_1, g_2) \cdot (x, y) := (g_1 x, g_2 y),$$

$(g_1, g_2) \in S^1 \times \mathrm{SO}(3)$. This is in fact a special kind of a cohomogeneity-one action, also called ‘‘sum action’’, see, e.g., [16, Section 1.5.2]. It has orbit space homeomorphic to a closed interval and is determined by the group diagram $S^1 \times \mathrm{SO}(3) \supset S^1 \times S^1, \{1\} \times \mathrm{SO}(3) \supset \{1\} \times S^1$. (The embedding of S^1 into $\mathrm{SO}(3)$ is the one induced by $S^1 \approx \{1\} \times \mathrm{SO}(2)$.) The fixed points of the induced action of $T := S^1 \times S^1$ on S^4 are the pairs $(0, y_0)$ and $(0, -y_0)$, where $y_0 := (1, 0, 0)$. The T -isotropy representation at any of these two points is described by equation (4.4), where x is arbitrary in \mathbb{R}^2 and y lies in $\{0\} \times \mathbb{R}^2 \subset \mathbb{R}^3$: the two weights of this representation are clearly linearly independent. Thus, by Lemma 4.2, the $S^1 \times \mathrm{SO}(3)$ -action on S^4 is of GKM type. We have $(0, y_0) = (g_1, g_2) \cdot (0, -y_0)$, where $g_1 = 1$ and g_2 is the diagonal matrix $\mathrm{Diag}(-1, -1, 1)$. In conclusion, there is just one orbit of the type $Gp, p \in M^T$. See Example 6.5 for a continuation of this example.

The intersection of two submanifolds of the form S_i or P_j consists of at most two points in M^T . We are now interested in the intersection of the G -orbits of two such spaces:

- Lemma 4.8.** (a) *If $i_1, i_2 \in I$ are such that $GS_{i_1} \neq GS_{i_2}$ then the intersection $GS_{i_1} \cap GS_{i_2}$ is empty or consists of one or two G -orbits in M_{\max} .*
 (b) *If $j_1, j_2 \in J$ are such that $GP_{j_1} \neq GP_{j_2}$ then the intersection $GP_{j_1} \cap GP_{j_2}$ is empty or consists of one G -orbit in M_{\max} .*
 (c) *If $i \in I$ and $j \in J$ are such that $GS_i \neq GP_j$ then the intersection $GS_i \cap GP_j$ is empty or consists of one G -orbit in M_{\max} .*

Proof. We only give the details of the proof for (a); the other two points can be proven by similar methods. Because of Lemma 4.6, we only need to show that $GS_{i_1} \cap GS_{i_2}$ is contained in M_{\max} . Indeed, assume that there exists $p \in S_{i_1}$, $p \notin M_{\max}$ such that $gp \in S_{i_2}$ for some $g \in G$. The isotropy groups at p and gp are $T_p = T_{i_1}$ and $T_{gp} = T_{i_2}$, hence $T_{i_2} = T_{gp} = gT_p g^{-1} \cap T = gT_{i_1} g^{-1} \cap T \subset gT_{i_1} g^{-1}$, which actually means $T_{i_2} = gT_{i_1} g^{-1}$. This implies that $M^{T_{i_1}} = g^{-1}M^{T_{i_2}}$. The connected components of p in the previous two spaces are S_{i_1} , respectively $g^{-1}S_{i_2}$, hence $S_{i_1} = g^{-1}S_{i_2}$ and consequently $GS_{i_1} = GS_{i_2}$, contradiction. \square

Example 4.9. It can happen already for torus actions that the intersection $GS_{i_1} \cap GS_{i_2}$ consists of two orbits (i.e., fixed points in this case). A simple example is given by the restriction of the action in Example 4.7 to T .

4.4. G -orbits along T -transversal geodesics. The results of this section are important tools in studying the structure of M_{r-1} . They require the choice of a G -invariant Riemannian metric on M . We fix $i \in I$ and observe that the sphere S_i is T -invariant. We consider an injective geodesic segment $\mu_i : [0, 1] \rightarrow S_i$ which is T -transversal, i.e., perpendicular to each T -orbit it intersects, and such that $\mu_i(0) = p_i$ and $\mu_i(1) = q_i$. Since $S_i = T\mu_i([0, 1])$, we have $GS_i = G\mu_i([0, 1])$. We denote by G_i the isotropy group of μ_i , i.e., the group of elements that fix μ_i pointwise. Clearly, $G_i \subset G_{\mu_i(t)}$ for all t .

We assume that $GS_i \neq Gp_i$. Recall that S_i is a component of the fixed point set of a certain codimension-one subtorus $T_i \subset T$. By Lemma 4.6, the torus T_i is maximal in both G_i and $G_{\mu_i(t)}$ for all $0 < t < 1$; we may therefore form the Weyl groups $W(G_i)$ and $W(G_{\mu_i(t)})$ (for $0 < t < 1$) with respect to T_i . As $W(G_i) \subset W(G_{\mu_i(t)})$ for such t , we can consider

$$H_G^*(G_{\mu_i(t)}) = H^*(BG_{\mu_i(t)}) = H^*(BT_i)^{W(G_{\mu_i(t)})}$$

as a subring of

$$H^*(BG_i) = H^*(BT_i)^{W(G_i)}.$$

Lemma 4.10. *Assume that $GS_i \neq Gp_i$. Then one of the following two cases holds:*

- (a) *For all $0 < t < 1$, we have $G_{\mu_i(t)} \cap S_i = T_{\mu_i(t)}$. In this case, $W(G_{\mu_i(t)}) = W(G_i)$ and $H^*(BG_{\mu_i(t)}) = H^*(BG_i)$.*
- (b) *For all $0 < t < 1$, we have $G_{\mu_i(t)} \cap S_i = T_{\mu_i(t)} \cup T_{\mu_i(1-t)}$. Then there exists $g_i \in W(G_{\mu_i(\frac{1}{2})}) \setminus W(G_i)$ such that $g_i\mu_i(t) = \mu_i(1-t)$. In particular, $Gp_i = Gq_i$. For $0 < t < 1$, $t \neq \frac{1}{2}$ we have $W(G_{\mu_i(t)}) = W(G_i)$ and $H^*(BG_{\mu_i(t)}) = H^*(BG_i)$.*

Proof. Assume that there exists $0 < t_0 < 1$ such that $G_{\mu_i(t_0)} \cap S_i$ is strictly larger than $T_{\mu_i(t_0)}$, i.e., we are not in case (a). Then there exists $h \in G$ such that $h\mu_i(t_0) = \mu_i(t_1)$ for some $t_1 \neq t_0$, $t_1 \in (0, 1)$. We may assume $t_0 < t_1$. Both T_i and $hT_i h^{-1}$ are maximal tori in $G_{\mu_i(t_1)} = hG_{\mu_i(t_0)}h^{-1}$, so there exists $k \in G_{\mu_i(t_1)}$ such that $khT_i(kh)^{-1} = T_i$. Define $g_i := kh$. Because g_i normalizes T_i and sends $\mu_i(t_0)$ to $\mu_i(t_1)$, it maps S_i to itself. Therefore, the curve $g_i\mu_i$ is contained in S_i and is (because g_i acts as an isometry) perpendicular to the T -orbits in S_i . Thus, there are only two possibilities: either $g_i\mu_i(t) = \mu_i(t + t_1 - t_0)$ for all t or $g_i\mu_i(t) = \mu_i(t_0 + t_1 - t)$ for all t . The first case is impossible because $\mu_i(t_1 - t_0)$ is not contained in M_{\max} . Therefore, $g_i\mu_i(t) = \mu_i(t_0 + t_1 - t)$. This implies that $\mu_i(t_0 + t_1) = q_i$, so $t_0 + t_1 = 1$, i.e., $g_i\mu_i(t) = \mu_i(1 - t)$ for all t ; thus, the first statement in case (b) is true.

It remains to show that in case (a), $W(G_i) = W(G_{\mu_i(t)})$ for $0 < t < 1$ and the same equality holds in case (b) for $0 < t < 1$, $t \neq \frac{1}{2}$. But any element in $W(G_{\mu_i(t)})$ that does not fix μ_i pointwise has to reflect the geodesic μ_i in the point $\mu_i(t)$ by the same arguments as above, which is only possible for $t = \frac{1}{2}$ in case (b). \square

Let us now take $j \in J$ and consider the real projective plane P_j , which is a component of M^{T_j} , T_j being a certain codimension-one subtorus of T . We also consider an injective geodesic segment $\eta_j : [0, 1] \rightarrow P_j$ which is T -transversal and such that $\eta_j(0) = r_j$ and $T\eta_j(1)$ is an exceptional T -orbit. Then $\eta_j([0, 1])$ intersects each orbit of the T -action on P_j exactly once. We denote by G_j the subgroup of G which fixes η_j pointwise.

We assume now that $GP_j \neq Gr_j$. By Lemma 4.6, the torus T_j is maximal in both G_j and $G_{\eta_j(t)}$ for $0 < t \leq 1$. For such t , we have $W(G_j) \subset W(G_{\eta_j(t)})$, and therefore

$$H_G^*(G_{\eta_j(t)}) = H^*(BG_{\eta_j(t)}) = H^*(BT_j)^{W(G_{\eta_j(t)})}$$

is a subring of

$$H^*(BG_j) = H^*(BT_j)^{W(G_j)}.$$

With the methods of the previous lemma, we can show as follows:

Lemma 4.11. *Assume that $GP_j \neq Gr_j$. Then for all $0 < t < 1$ we have $G\eta_j(t) \cap P_j = T\eta_j(t)$, $W(G_{\eta_j(t)}) = W(G_j)$, and $H^*(BG_{\eta_j(t)}) = H^*(BG_j)$.*

Remark 4.12. Lemmata 4.10 and 4.11 show that if the spaces GS_i or GP_j are smooth (and not a single G -orbit), then they become cohomogeneity-one manifolds relative to the G -action. There are situations when this is not the case. Let us consider for instance the canonical action of H on K/H , where the latter is an irreducible symmetric space of rank at least two and satisfies $\text{rank } K = \text{rank } H$ (take for instance the Grassmannian $SU(4)/S(U(2) \times U(2))$). This action satisfies the non-abelian GKM hypotheses, by [11] and Lemma 4.2 above. Let $T \subset H$ be a maximal torus. The coset eH is a fixed point of the T -action and we consider a two-dimensional submanifold R in the 1-skeleton of the T -action which contains it. We claim that the space HR is not smooth. Indeed, if it were, then the H -action on it would have cohomogeneity-one with orbit space $[0, 1]$ and eH would be a singular orbit. From the slice theorem, H then acts transitively on the unit sphere in $T_{eH}HR$. Thus $T_{eH}HR$ is an irreducible H -submodule of $T_{eH}K/H$. Since the symmetric space is irreducible, we deduce that $T_{eH}HR = T_{eH}K/H$ and conclude that H acts transitively on the unit sphere in $T_{eH}K/H$: this is in contradiction with the cohomogeneity of the isotropy representation of H on K/H at eH being $\text{rank } K/H$, which was assumed to be at least two.

Another concrete situation when GS_i is not smooth will be presented in Examples 5.1 and 6.7.

5. NON-ABELIAN GKM THEORY

In this section we accomplish the goal mentioned at the beginning of the previous section, that is, we identify the image of the (injective) map $H_G^*(M_{r-1}) \rightarrow H_G^*(M_{\max})$ in the presence of the non-abelian GKM conditions. The main results are Propositions 5.5 and 5.6; for a more systematic presentation, see Section 6, especially Theorem 6.2.

We would like to emphasize that, in contrast with the previous section, all considerations made here will be independent of the choice of a maximal torus T . The first step will be an intrinsic description of the non-abelian 1-skeleton M_{r-1} , which is going to be presented in the following subsection.

5.1. The components of the 1-skeleton. Consider the connected components of $M_{r-1} \setminus M_{\max}$ and their closures: these will be called the *components of the 1-skeleton*. Note that for any maximal torus $T \subset G$, these closures are nothing but the associated spaces GS_i and GP_j from the previous section which are not single G -orbits. We summarize some of the results concerning them which were obtained in the previous section:

- Their union is M_{r-1} .
- Each of them is G -invariant and contains one or two G -orbits in M_{\max} .
- The intersection of any two of them is contained in M_{\max} (possibly empty).
- The G -orbit space of any of them is homeomorphic to a closed interval or a circle.

The last statement follows from Lemmata 4.10 and 4.11. In fact, if N is a component, then it fits into one of the following three cases, see also items (I)-(III) in the introduction:

- (I) $N/G \simeq [0, 1]$ and N contains exactly two orbits in M_{\max} , which correspond to 0 and 1.
- (II) $N/G \simeq [0, 1]$ and N contains exactly one orbit in M_{\max} , which corresponds to 0.
- (III) $N/G \simeq \mathbb{R}/\mathbb{Z} = S^1$ and N contains exactly one orbit in M_{\max} .

By fixing a maximal torus $T \subset G$ like in the previous section, the components are of the form GS_i , $i \in I$ or GP_j , $j \in J$. More precisely, if S_i is like in Lemma 4.10 (a) and $Gp_i \neq Gq_i$, then GS_i is of type (I), whereas if $Gp_i = Gq_i$, then GS_i is of type (III); if S_i is like in Lemma 4.10 (b), then GS_i is of type (II). Also of type (II) is any component of the form GP_j like in Lemma 4.11.

We can illustrate these situations by concrete examples. Of type (I) are: any 2-sphere in the 1-skeleton of a GKM torus action as well as the spaces GS_1 and GS_2 to be mentioned in Example 6.4 below. In Example 3.8, the two $\mathbb{R}P^2$'s in the 1-skeleton are of type (II); of the same type is the component GS in Example 4.7 (and 6.5 below). We will now give an example of a component of type (III). (Note that, by Remark 4.12 and the general theory of cohomogeneity-one actions, a type (III) component is not smooth.)

Example 5.1. Consider the compact, connected, and simply connected Lie group of type G_2 along with M , the adjoint orbit of the point p in Figure 5. That diagram represents the roots of our group, which lie in the Lie algebra \mathfrak{t} of a maximal torus T . Denote by K the connected subgroup which contains T and

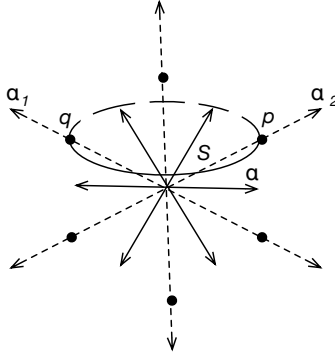


FIGURE 5

whose roots are the vectors represented by dotted lines in the diagram. Note that K is isomorphic to $SU(3)$. The relevant action for us is the one of K on M . It satisfies the GKM conditions, see Example 4.3. The T -fixed points of M are the six dots in the diagram: they turn out to be the orbit of p under the Weyl group $W(K)$. Especially important for us are p and $q := s_\alpha(p)$ (in general, if γ is a root, then s_γ is the reflection in the line γ^\perp). There is a T -invariant 2-sphere in M between p and q , which we denote by S . It is not contained in Kp , because Kp is a K -adjoint orbit and α is not a root of K . But p and q are in the same $W(K)$ -orbit, hence we have $Kp = Kq$. We will now show that S satisfies the conditions in Lemma 4.10

(a): this implies immediately that $KS/K \simeq S^1$. Concretely, if $\mu : [0, 1] \rightarrow M$ is a T -transversal injective geodesic segment in S with $\mu(0) = p$ and $\mu(1) = q$, then $K\mu(t) \cap S = T\mu(t)$, for all $0 < t < 1$. Indeed, let us assume that the contrary is true. Like in the proof of Lemma 4.10, this implies that there exists $k \in K$ such that $k\mu(t) = \mu(1 - t)$ for all $t \in (0, 1)$. In particular, $kp = q$ and $kq = p$. Since $p \in \mathfrak{t}$ is a regular point for the adjoint action of K , the K -stabilizer of p is equal to T . Hence $kp = q$ implies readily that $k \in N_K(T)$. Finally observe, by inspection, that for any of the six elements w of $W(K)$, we have either $wp \neq q$ or $wq \neq p$. See Example 6.7 for a continuation of this example.

The following notion will play an important role:

Definition 5.2. Let N be a component of the non-abelian 1-skeleton. An injective geodesic segment $\gamma : [0, 1] \rightarrow M$ is *adapted* to N if:

- the trace of γ is contained in N
- $\gamma(0) \in M_{\max}$
- γ is G -transversal, i.e., it is perpendicular to any G -orbit it meets at the intersection point
- if N is of type (I) or (II), then the map $[0, 1] \rightarrow N/G$ given by $t \mapsto G\gamma(t)$ is a bijection; if N is of type (III), then the map $[0, 1]/_{0 \sim 1} \simeq S^1 \rightarrow N/G$ given by $t \mapsto G\gamma(t)$ is well defined and bijective.

Assume that $T \subset G$ is fixed and N is of the form GS_i or GP_j like in the previous section. In the situations described by Lemma 4.10 (a), respectively Lemma 4.11, μ_i is adapted to GS_i and η_j is adapted to GP_j ; for S_i like in Lemma 4.10 (b), a geodesic segment adapted to GS_i is $s \mapsto \mu_i(s/2)$, $0 \leq s \leq 1$. Any such curve remains an adapted geodesic segment after translations by elements of G or, if N is of type (I), the reparametrization induced by $s \mapsto 1 - s$, $s \in [0, 1]$. The following converse result holds:

Lemma 5.3. *Let N be a component of the non-abelian 1-skeleton. Then there exists a geodesic segment adapted to N ; it is unique up to G -translations and, if N is of type (I), the reparametrization induced by $s \mapsto 1 - s$, $s \in [0, 1]$.*

Proof. Let $p \in N$ be any point with $\text{rank } G_p = r - 1$. Let $K \subset G_p$ be a maximal torus and consider the component R of M^K containing p . Let also T be a maximal torus in G with $K \subset T$. The point p is not in M^T , hence, by Lemma 4.2, R is a two-dimensional T -invariant submanifold of M . By Lemma 3.1 the T -action on R fits into one of the cases described in Lemma 3.3. Consequently the tangent space $T_p T p$ is contained in $T_p R$, hence $(T_p G p)^\perp \cap T_p R$ has dimension at most 1. The tangent vector at p to any admissible geodesic segment which goes through p is contained, up to a G_p -translation, in this intersection. Indeed, if $v \in T_p M$ is such a vector, then the group $(G_p)_v$ (being equal to the G -stabilizer of points on the geodesic that are sufficiently close to p) has rank $r - 1$ and hence contains a G_p -conjugate of K : thus, the geodesic segment is, up to a G_p -translation, contained in R . The claim follows because adapted geodesic segments are supposed to meet every G -orbit in N . \square

For each component N we choose a geodesic γ_N adapted to it, along with the group G_{γ_N} , the pointwise G -stabilizer of γ_N . Both stabilizers $G_{\gamma_N(0)}$ and $G_{\gamma_N(1)}$ contain G_{γ_N} . Let \mathfrak{g}_{γ_N} , $\mathfrak{g}_{\gamma_N(0)}$ and $\mathfrak{g}_{\gamma_N(1)}$ be the Lie algebras of these groups. We

will make the following identifications:

$$\begin{aligned} H_G^*(G\gamma_N(0)) &= H^*(BG_{\gamma_N(0)}) = S(\mathfrak{g}_{\gamma_N(0)}^*)^{G_{\gamma_N(0)}}, \\ H_G^*(G\gamma_N(1)) &= H^*(BG_{\gamma_N(1)}) = S(\mathfrak{g}_{\gamma_N(1)}^*)^{G_{\gamma_N(1)}}, \\ H_G^*(G\gamma_N(t)) &= H^*(BG_{\gamma_N}) = S(\mathfrak{g}_{\gamma_N}^*)^{G_{\gamma_N}}, \end{aligned}$$

where $0 < t < 1$. In this way, the canonical maps $H^*(BG_{\gamma_N(0)}) \rightarrow H^*(BG_{\gamma_N})$ and $H^*(BG_{\gamma_N(1)}) \rightarrow H^*(BG_{\gamma_N})$ are those induced by restriction from $\mathfrak{g}_{\gamma_N(0)}$ respectively $\mathfrak{g}_{\gamma_N(1)}$ to \mathfrak{g}_{γ_N} .

5.2. The 1-skeleton and equivariant cohomology. In Propositions 5.5 and 5.6 we will give a complete description of the $S(\mathfrak{g}^*)^G$ -algebra $H_G^*(M)$. First we need a lemma, whose proof is essentially the same as that of [7, Proposition 4.2]¹.

Lemma 5.4. *Let N be a component of the nonabelian 1-skeleton, and $(a, b) \subset [0, 1]$ be any interval. Let T_N be a maximal torus in G_{γ_N} (of rank $r - 1$). Then*

$$H_G^*(G\gamma_N((a, b))) \simeq S(\mathfrak{t}_N^*)^{W(G_{\gamma_N})}$$

as graded rings.

Proof. Let T be a maximal torus in G such that $T_N \subset T$. We first calculate the fibers of the natural map

$$(5.1) \quad G \times_K (T\gamma_N((a, b))) \rightarrow G\gamma_N((a, b)),$$

where $K = \{g \in N_G(T_N) \mid g(T\gamma_N((a, b))) \subset T\gamma_N((a, b))\}$. Assume that $[g, t\gamma_N(s_0)]$ is mapped to $\gamma_N(s_1) \in \gamma_N((a, b))$, i.e., $gt\gamma_N(s_0) = \gamma_N(s_1)$. By the last condition in the definition of an adapted geodesic segment, this implies that $s_0 = s_1$. Thus, the fiber of $\gamma_N(s_1)$ is the set of elements of the form $[h, \gamma_N(s_1)]$ with $h \in G_{\gamma_N(s_1)}$, which is isomorphic to $G_{\gamma_N(s_1)}/G_{\gamma_N(s_1)} \cap K$. We clearly have $G_{\gamma_N(s_1)} \cap K \subset N_{G_{\gamma_N(s_1)}}(T_N)$. The converse inclusion holds, because the connected component of $\gamma_N(s_1)$ in M^{T_N} is in fact a two-dimensional component of the abelian 1-skeleton of the T -action on M and is left invariant by any $g \in G$ which fixes $\gamma_N(s_1)$ and normalizes T_N ; the action of T on this component is of one of the two types described in Lemma 3.3 and both γ_N and $g\gamma_N$ are contained in the component and T -transversal relative to this action, which implies readily that $g \in G_{\gamma_N}$ and $g(T\gamma_N(s)) \subset T\gamma_N(s)$, for all $s \in (a, b)$. Thus, the fiber of $\gamma_N(s_1)$ is the quotient $G_{\gamma_N(s_1)}/N_{G_{\gamma_N(s_1)}}(T_N)$, which is acyclic by [7, Lemma 3.2] (if $G_{\gamma_N(s_1)}$ is connected, see [17, Sect. III.1, Lemma (1.1)]). The Leray spectral sequence of the Borel construction associated to (5.1) thus collapses at the E_2 -term, which implies that

$$H_G^*(G\gamma_N((a, b))) = H_K^*(T\gamma_N((a, b))).$$

Next we note that K is the subgroup of G generated by T and $N_{G_{\gamma_N}}(T_N)$ (given $k \in K$, we have $k\gamma_N(s_1) \in T\gamma_N(s_1)$, hence there exists $t \in T$ such that tk is in $G_{\gamma_N(s_1)}$ and normalizes T_N , which implies that $tk \in G_{\gamma_N}$, by an argument already used before). Since the identity component of $N_{G_{\gamma_N}}(T_N)$ is T_N , the identity component of K is T . It follows that

$$H_K^*(T\gamma_N((a, b))) = (S(\mathfrak{t}_N^*) \otimes H_{T/T_N}^*(T\gamma_N((a, b))))^{K/T} = S(\mathfrak{t}_N^*)^{W(G_{\gamma_N})}$$

¹Note that there is a slight error in [7, Proposition 4.2]: on the right hand side of the equation one has to consider the subgroup of $N_G(\mathfrak{t}_p)$ consisting of those elements which leave invariant $M^{L_p \cdot p}$ instead of the whole normalizer $N_G(\mathfrak{t}_p)$.

because the orbit space of the T/T_N -action on $T\gamma_N((a, b))$ is just (a, b) , which is contractible. \square

Proposition 5.5. *Let N be a component of the non-abelian 1-skeleton.*

(a) *If N is of type (I), then the image of the restriction map*

$$H_G^*(N) \rightarrow H_G^*(N \cap M_{\max}) = S(\mathfrak{g}_{\gamma_N(0)}^*)^{G_{\gamma_N(0)}} \oplus S(\mathfrak{g}_{\gamma_N(1)}^*)^{G_{\gamma_N(1)}}$$

consists of all pairs (f, g) such that $f|_{\mathfrak{g}_{\gamma_N}} = g|_{\mathfrak{g}_{\gamma_N}}$.

(b) *If N is of type (II), then the image of the restriction map*

$$H_G^*(N) \rightarrow H_G^*(N \cap M_{\max}) = S(\mathfrak{g}_{\gamma_N(0)}^*)^{G_{\gamma_N(0)}}$$

consists of all f for which there exists $g \in S(\mathfrak{g}_{\gamma_N(1)}^)^{G_{\gamma_N(1)}}$ such that $f|_{\mathfrak{g}_{\gamma_N}} = g|_{\mathfrak{g}_{\gamma_N}}$.*

(c) *If N is of type (III), then the image of the restriction map*

$$H_G^*(N) \rightarrow H_G^*(N \cap M_{\max}) = S(\mathfrak{g}_{\gamma_N(0)}^*)^{G_{\gamma_N(0)}}$$

consists of all f such that $f|_{\mathfrak{g}_{\gamma_N}} = (\text{Ad}_h^ f)|_{\mathfrak{g}_{\gamma_N}}$, where $h \in G$ is such that $h\gamma_N(0) = \gamma_N(1)$.*

Proof. We prove (a) and (b) at the same time. Consider the covering

$$(5.2) \quad N = (N \setminus G\gamma_N(1)) \cup (N \setminus G\gamma_N(0)).$$

We claim first that $H_G^*(N \setminus G\gamma_N(1)) \simeq H_G^*(G\gamma_N(0)) = S(\mathfrak{g}_{\gamma_N(0)}^*)^{G_{\gamma_N(0)}}$. For that, let $\varepsilon > 0$ be so small that for all $0 < t < \varepsilon$, the isotropy group $G_{\gamma_N(t)}$ is equal to G_{γ_N} . We consider the covering of $N \setminus G\gamma_N(1)$ given by

$$(5.3) \quad N \setminus G\gamma_N(1) = U \cup V,$$

where $U = G\gamma_N([0, \varepsilon))$ and $V = G\gamma_N((\varepsilon/2, 1))$. Then, $G\gamma_N(0)$ is a deformation retract of U via the G -equivariant retraction $r_s(g\gamma_N(t)) = g\gamma_N(st)$, $s \in [0, 1]$. By Lemma 5.4, both $H_G^*(V)$ and $H_G^*(U \cap V)$ are isomorphic to $S(\mathfrak{t}_N^*)^{W(G_{\gamma_N})}$, and hence isomorphic to each other. Thus, the equivariant Mayer-Vietoris sequence of the covering (5.3) becomes short exact

$$0 \longrightarrow H_G^*(N \setminus G\gamma_N(1)) \longrightarrow H_G^*(U) \oplus H_G^*(V) \longrightarrow H_G^*(U \cap V) \longrightarrow 0$$

and shows that the restriction map $H_G^*(N \setminus G\gamma_N(1)) \rightarrow H_G^*(U) \simeq S(\mathfrak{g}_{\gamma_N(0)}^*)^{G_{\gamma_N(0)}}$ is an isomorphism. Using this, together with the analogous statement for $N \setminus G\gamma_N(0)$ and another application of Lemma 5.4, this time to $G\gamma_N((0, 1))$, we see that the equivariant Mayer-Vietoris sequence of the covering (5.2) takes the form

$$\dots \longrightarrow H_G^*(N) \longrightarrow S(\mathfrak{g}_{\gamma_N(0)}^*)^{G_{\gamma_N(0)}} \oplus S(\mathfrak{g}_{\gamma_N(1)}^*)^{G_{\gamma_N(1)}} \longrightarrow S(\mathfrak{t}_N^*)^{W(G_{\gamma_N})} \longrightarrow \dots$$

This implies (a) and (b).

For (c), we use the same method of proof, but this time with the covering

$$(5.4) \quad N = G\gamma_N([0, \varepsilon) \cup (1 - \varepsilon, 1]) \cup G\gamma_N((\varepsilon/2, 1 - \varepsilon/2)).$$

By Lemma 5.4 and since $G\gamma_N(0) = G\gamma_N(1)$, the corresponding equivariant Mayer-Vietoris sequence takes the form

$$(5.5) \quad \dots \longrightarrow H_G^*(N) \longrightarrow S(\mathfrak{g}_{\gamma_N(0)}^*)^{G_{\gamma_N(0)}} \oplus S(\mathfrak{t}_N^*)^{W(G_{\gamma_N})} \longrightarrow S(\mathfrak{t}_N^*)^{W(G_{\gamma_N})} \oplus S(\mathfrak{t}_N^*)^{W(G_{\gamma_N})} \longrightarrow \dots$$

Let $h \in G$ be such that $h\gamma_N(0) = \gamma_N(1)$. Then, $G_{\gamma_N(1)} = hG_{\gamma_N(0)}h^{-1}$ and hence, $\text{Ad}_h^* : S(\mathfrak{g}_{\gamma_N(0)})^{G_{\gamma_N(0)}} \rightarrow S(\mathfrak{g}_{\gamma_N(1)})^{G_{\gamma_N(1)}}$ is an isomorphism. The claim follows because an element (f, g) contained in the middle term of (5.5) is mapped to $(f|_{t_N} - g, (\text{Ad}_h^* f)|_{t_N} - g)$. \square

Let us denote the components of M_{\max} by O_A , where A is in a certain (finite) set \mathcal{A} .

Proposition 5.6. *The image of the canonical map $H_G^*(M_{r-1}) \rightarrow H_G^*(M_{\max}) = \bigoplus_{A \in \mathcal{A}} H_G^*(O_A)$ consists of all $(f_A)_{A \in \mathcal{A}}$ with the property that:*

- if there exist a component N such that $N \cap M_{\max} = O_A \cup O_B$, for $A, B \in \mathcal{A}$, $A \neq B$, then (f_A, f_B) is in the image of the restriction map $H_G^*(N) \rightarrow H_G^*(O_A) \oplus H_G^*(O_B)$.
- if there exist a component N such that $N \cap M_{\max} = O_A$, for $A \in \mathcal{A}$, then f_A is in the image of the restriction map $H_G^*(N) \rightarrow H_G^*(O_A)$.

Proof. From the long exact sequence of the pair (M_{r-1}, M_{\max}) we deduce that the image of the map $H_G^*(M_{r-1}) \rightarrow H_G^*(M_{\max})$ is equal to the kernel of $H_G^*(M_{\max}) \rightarrow H_G^{*+1}(M_{r-1}, M_{\max})$. We set

$$U := \bigcup_N G\gamma_N([0, \varepsilon]) \cup G\gamma_N((1 - \varepsilon, 1])$$

where N runs through the components of the non-abelian 1-skeleton of the G -action, and $\varepsilon > 0$ is chosen so small that M_{\max} is a deformation retract of U (see the proof of Lemma 5.5). Moreover, U is open in M_{r-1} : the complement of U in $M_{r-1} = \bigcup_N G\gamma_N([0, 1])$ is equal to $\bigcup_N G\gamma_N([\varepsilon, 1 - \varepsilon])$, which is a compact space.

By using the homotopy, respectively the excision property we obtain:

$$H_G^{*+1}(M_{r-1}, M_{\max}) = H_G^{*+1}(M_{r-1}, U) = H_G^{*+1}(M_{r-1} \setminus M_{\max}, U \setminus M_{\max}).$$

Since $M_{r-1} \setminus M_{\max}$ is equal to the disjoint union of all $N \setminus M_{\max}$, the last term in the previous equation is equal to $\bigoplus_N H_G^{*+1}(N \setminus M_{\max}, G\gamma_N([0, \varepsilon]) \cup G\gamma_N((1 - \varepsilon, 1)))$. By using again the excision, respectively homotopy property, the last sum is equal to

$$\bigoplus_N H_G^{*+1}(N, N \cap M_{\max}).$$

The result stated by the proposition follows now from the long exact cohomology sequence of the pairs $(N, N \cap M_{\max})$. \square

6. NON-ABELIAN GKM GRAPHS

6.1. Abstract non-abelian GKM graphs. The following abstract notion is relevant for us:

Definition 6.1. A *non-abelian GKM graph* (associated to G) is a diagram which consists of:

- finitely many circles (sometimes just simple closed contours), indexed by $A \in \mathcal{A}$, whose interiors are pairwise disjoint
- finitely many dots, each dot a lying inside a circle and labeled with a closed subgroup $G_a \subset G$
- finitely many stars, lying outside the circles and which are labeled with closed subgroups of G as well

- edges that join pairs of dots or a dot with a star; each edge is labeled with a group which is contained in the groups corresponding to the two end-points
- arrows that join two dots; an arrow going from a to b , where a and b are dots, is labeled with an isomorphism $\mathfrak{g}_a \rightarrow \mathfrak{g}_b$.

Each circle contains a distinguished dot, the *representative*, which is joined with any other dot inside that circle by exactly one arrow emerging at the representative. There are no other arrows.

Recall that Propositions 5.5 and 5.6 give a complete description of the G -equivariant cohomology of M associated to the group action $G \times M \rightarrow M$ which satisfies the non-abelian GKM conditions. The main observation of this section is that the output, i.e., $H_G^*(M)$ as an $S(\mathfrak{g}^*)^G$ -algebra, can be encoded into a non-abelian GKM graph. Here comes the precise construction.

6.2. Construction of the non-abelian GKM graph. The input consists of the components N of the non-abelian 1-skeleton M_{r-1} ; each of them comes equipped with an adapted geodesic segment γ_N .

Step 1. We draw one big circle for each connected component O_A of M_{\max} . Inside the circle we draw one dot, which corresponds to an endpoint contained in O_A of one of the chosen adapted geodesic segments. We call this point a *representative* of O_A . We attach to it the label consisting of the G -isotropy group of that point. In the following steps there will potentially be drawn more dots inside the circle, corresponding to other endpoints contained in O_A of adapted geodesic segments.

Step 2. Assume that N is a component of type (I) (see Section 5.1 for the description of the types (I)-(III)). We proceed in the following way: Say that the two orbits in M_{\max} contained in N are O_A and O_B . At this step, we attach to N a line segment and possibly one or two arrows as follows. If one of the two endpoints of γ_N does not correspond yet to a dot inside the corresponding circle, then we add an additional dot inside this circle, label it with the isotropy group of that endpoint, and draw an arrow with tail at the representative of O_A and tip at the new dot. Say that the representative and the new dot correspond to the points p_a and hp_a , where $h \in G$, and thus carry the labels G_{p_a} , respectively $G_{hp_a} = hG_{p_a}h^{-1}$: then the arrow will carry the label Ad_h . (More precisely, the conjugation map $c_h : g \mapsto hgh^{-1}$ is an isomorphism between G_{p_a} and G_{hp_a} , hence Ad_h , i.e., its differential at e , is a linear isomorphism between the Lie algebras \mathfrak{g}_{p_a} and \mathfrak{g}_{hp_a} .) The same procedure is used for O_B , i.e., we may need to draw an arrow inside the corresponding circle as well. We next join the two endpoints of γ_N by a line segment and label it with G_{γ_N} . Note that, by Lemma 5.3, we may have translated γ_N by an element of G in such a way that at least one of its two endpoints is the representative of its orbit in M_{\max} and avoid in this way drawing the corresponding arrow.

Step 3. Here, we consider the components of type (III). For any such component, the same procedure as in Step 2 applies, the only difference being that the two dots that are connected by the line segment are contained in the same circle, because the endpoints of the corresponding adapted geodesic segment are contained in the same orbit in M_{\max} . As before, we may have translated this geodesic segment to make one of its endpoints coincide with the representative of its orbit.

Step 4. Finally, we treat the components of type (II). For any such component N , we first draw a star outside all circles and label it with the isotropy group $G_{\gamma_N(1)}$. We then connect it by a line segment with the dot corresponding to $\gamma_N(0)$ (if this

dot does not exist yet, then introduce it, together with an arrow, as in Step 2), and label that line segment with G_{γ_N} . Again, we may first replace γ_N by a G -translate of it in such a way that $\gamma_N(0)$ becomes the representative of its orbit.

6.3. The non-abelian GKM theorem. To any non-abelian GKM graph we associate an $S(\mathfrak{g}^*)^G$ -algebra as follows: First for each circle A we set $G_A := G_a$, where a is the representative of A . Then we consider the subalgebra of $\bigoplus_A S(\mathfrak{g}_A^*)^{G_A}$ (where the $S(\mathfrak{g}^*)^G$ -algebra structure is given by the natural restriction maps) consisting of all tuples (f_A) which satisfy the following two types of relations:

- Consider an edge whose endpoints are the dots a and b . We take the (possibly equal) circles A and B in which a , respectively b are contained, as well as the arrows from their representatives to a , respectively b , labeled by the isomorphisms $h_{Aa} : \mathfrak{g}_A \rightarrow \mathfrak{g}_a$, respectively $h_{Bb} : \mathfrak{g}_B \rightarrow \mathfrak{g}_b$. (If a or b is the representative of its circle, then we set h_{Aa} or h_{Bb} to be the identity.) If H is the label of the edge and \mathfrak{h} its Lie algebra, then we must have

$$f_A \circ h_{Aa}^{-1}|_{\mathfrak{h}} = f_B \circ h_{Bb}^{-1}|_{\mathfrak{h}}.$$

- Consider an edge, whose endpoints are the dot a and a certain star. We take the circle A in which a is contained and also the arrow from its representative to a , labeled by the isomorphism $h_{Aa} : \mathfrak{g}_A \rightarrow \mathfrak{g}_a$. (If a is the representative, then we set h_{Aa} to be the identity.) If K is the label of the star, H the label of the edge, and \mathfrak{k} , respectively \mathfrak{h} their Lie algebras, then we must have

$$f_A \circ h_{Aa}^{-1}|_{\mathfrak{h}} = g|_{\mathfrak{h}},$$

for some $g \in S(\mathfrak{k}^*)^K$.

In the previous subsection we have attached a non-abelian GKM graph to a group action of G on M which satisfies the GKM conditions. From Propositions 5.5 and 5.6 we immediately deduce:

Theorem 6.2. *If the group action of G on M satisfies the GKM conditions, then $H_G^*(M)$ is isomorphic to the $S(\mathfrak{g}^*)^G$ -algebra induced by the non-abelian GKM graph attached to the action.*

In what follows we will give some applications of this theorem to certain concrete situations.

Example 6.3. Consider the (transitive) action of G on G/H , where G is a compact connected Lie group and H a closed subgroup with $\text{rank } H = \text{rank } G$. This action satisfies the non-abelian GKM conditions, see Example 4.3. The GKM graph is presented in Figure 6: this is just another way of expressing the well-known fact that $H_G^*(G/H) = S(\mathfrak{h}^*)^H$.

Example 6.4. Let $G := \text{Sp}(1) \times \text{Sp}(1)$ be the subgroup of all diagonal matrices in $\text{Sp}(2)$, and $T := S^1 \times S^1$, where $S^1 = \{a + bi \in \mathbb{H} \mid a, b \in \mathbb{R}, a^2 + b^2 = 1\}$. The torus T is clearly maximal in both $\text{Sp}(2)$ and $\text{Sp}(1) \times \text{Sp}(1)$. We are interested in the action of $G := \text{Sp}(1) \times \text{Sp}(1)$ on the quotient $M := \text{Sp}(2)/T$. The latter space is a principal orbit of the adjoint representation of $\text{Sp}(2)$. It is the orbit of a regular element in \mathfrak{t} , the Lie algebra of T . The fixed point set M^T is the same as the intersection of the orbit with \mathfrak{t} , and actually equal to a Weyl group orbit. The situation is described in Figure 7: the Weyl group $W(\text{Sp}(2))$ is generated by the

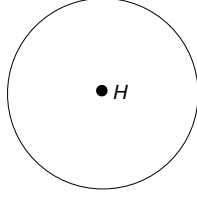


FIGURE 6

reflections through the lines $l_1, l_2, l_3,$ and l_4 , whereas $W(\mathrm{Sp}(1) \times \mathrm{Sp}(1)) \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$ is its subgroup generated by the reflections through l_1 and l_3 .

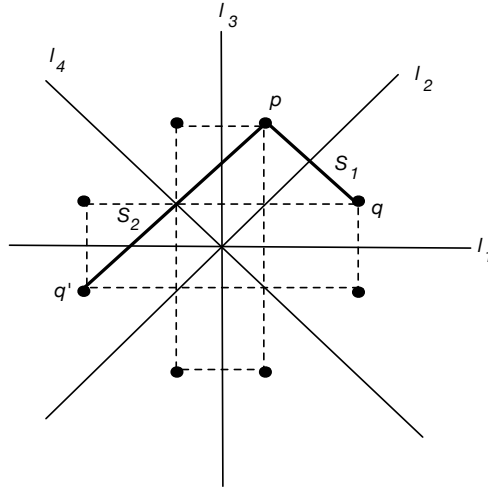


FIGURE 7

The corners of the two rectangles in the diagram lie respectively in the same $\mathrm{Sp}(1) \times \mathrm{Sp}(1)$ -orbit. That is, the set $\{Gx \mid x \in M^T\}$ has two elements, which are Gp and Gq . There are sixteen T -invariant spheres S_i in $\mathrm{Sp}(2)/T$, which join pairs of the type x, sx , where x is one of the eight T -fixed points and s one of the reflections in $W(\mathrm{Sp}(2))$. However, the set $\{GS_i \mid 1 \leq i \leq 16\}$ has only two elements, which are induced by the spheres S_1 and S_2 represented in the diagram. The intersection of GS_1 with GS_2 is the union $Gp \cup Gq$.

The GKM graph therefore has two line segments, induced by GS_1 and GS_2 . We have $S_1 \cap M_{\max} = \{p, q\}$ and $S_2 \cap M_{\max} = \{p, q'\}$. As representatives of the two orbits Gp and Gq we choose p and q . The G -isotropy groups of $p, q,$ and q' are T . Recall that T is by definition a direct product $S^1 \times S^1$. One can see that the principal T -isotropy groups along the spheres S_1 and S_2 are $\Delta := \{(z, z) \mid z \in S^1\}$, respectively $\Delta' := \{(z, z^{-1}) \mid z \in S^1\}$. The Lie algebra \mathfrak{t} is the direct sum of two copies of \mathbb{R} , which is the Lie algebra of S^1 . In this way we obtain a system of coordinates (t_1, t_2) on \mathfrak{t} , such that the lines $l_1, l_2, l_3,$ and l_4 are described by the equations $t_2 = 0, t_1 - t_2 = 0, t_1 = 0,$ respectively $t_1 + t_2 = 0$. We have $Gq' = Gq$,

so the only thing we still need in order to complete the graph is to find $g_0 \in G$ such that $q' = g_0 q$ and then determine the differential of the map $c_{g_0} : T \rightarrow T$. In fact, g_0 is in the G -normalizer of T and the differential of c_{g_0} at e is just the composition $s_1 s_3$ (where s_i is the reflection through l_i), that is, the map $(t_1, t_2) \mapsto (-t_1, -t_2)$. The GKM graph is given in Figure 8.

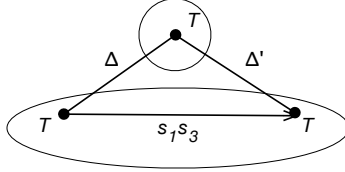


FIGURE 8

Thus $H_{\mathrm{Sp}(1) \times \mathrm{Sp}(1)}^*(\mathrm{Sp}(2)/T)$ is the subalgebra of $\mathbb{R}[t_1, t_2] \oplus \mathbb{R}[t_1, t_2]$ consisting of all pairs (f_1, f_2) with the following two properties:

- $f_1(t_1, t_2) - f_2(t_1, t_2)$ is divisible by $t_1 - t_2$
- $f_1(t_1, t_2) - f_2(-t_1, -t_2)$ is divisible by $t_1 + t_2$

Example 6.5. Let us consider again the action of $G = S^1 \times \mathrm{SO}(3)$ on S^4 which has been defined in Example 4.7. One can distinguish two T -invariant spheres in S^4 between the two T -fixed points $(0, y_0)$ and $(0, -y_0)$. They are

$$S_1 := \{(0, y) \in S^4\} \quad \text{and} \quad S_2 := \{(x, (a, 0, 0)) \in S^4 \mid a \in \mathbb{R}\}.$$

Note that S_1 is in fact the $S^1 \times \mathrm{SO}(3)$ -orbit of $(0, y_0)$, thus GS_1 has no contribution to the GKM graph. Consider the parametrized meridian in S_2 given by

$$\mu(t) := (\sin t(1, 0), \cos t(1, 0, 0)), \quad 0 \leq t \leq \pi,$$

which joins the two T -fixed points. For $t \notin \{0, \pi/2, \pi\}$, the G -isotropy is $\{1\} \times S^1$, whereas at the midpoint $\mu(\pi/2)$ it is equal to $\{1\} \times \mathrm{SO}(3)$. Hence S_2 fits into the situation described by Lemma 4.10 (b), which means that GS_2 is a component of type (II) of the non-abelian 1-skeleton. (In fact, we have $GS_2 = S^4$, so this is the only component of the non-abelian 1-skeleton.) The non-abelian GKM graph is presented in Figure 9.

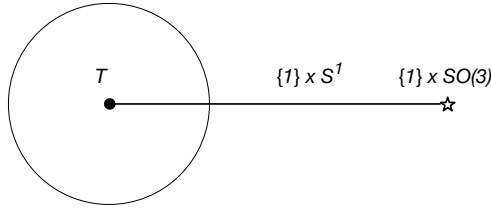


FIGURE 9

A simpler way to deduce this is by directly applying Remark 6.6 below, especially equation (6.2).

Let us now calculate explicitly the corresponding G -equivariant cohomology algebra. To this end, we choose the coordinates (t_1, t_2) on the Lie algebra \mathfrak{t} of T which arise naturally from the splitting $T = S^1 \times S^1$. We have as follows:

$$H^*(BT) = \mathbb{R}[t_1, t_2], \quad H^*(B(\{1\} \times S^1)) = \mathbb{R}[t_2], \quad H^*(B(\{1\} \times \mathrm{SO}(3))) = \mathbb{R}[t_2^2].$$

We obtain

$$H_{S^1 \times \mathrm{SO}(3)}^*(S^4) \simeq \{f \in \mathbb{R}[t_1, t_2] \mid f(0, t_2) \in \mathbb{R}[t_2^2]\}.$$

Remark 6.6. The group action in the previous example is of cohomogeneity one. Consider the general situation when a compact connected Lie group G acts on a compact connected manifold M with cohomogeneity equal to one, such that M/G is homeomorphic to a closed interval (see Section 4.2). Let $G \supset K_+, K_- \supset H$ be the associated group diagram. Assume that the group action satisfies the non-abelian GKM conditions. The condition of being equivariantly formal is equivalent to at least one of the groups K_+, K_- having rank equal to the rank of G , see [5, Corollary 1.3]. As the 1-skeleton coincides with the whole M , the GKM graph consists of only one line segment, which either connects two dots (if both K_+ and K_- are of maximal rank) or a dot and a star (if only one of K_+ and K_- is of maximal rank). In the first case, we have

$$(6.1) \quad H_G^*(M) = \{(f, g) \in S(\mathfrak{k}_+^*)^{K_+} \oplus S(\mathfrak{k}_-^*)^{K_-} \mid f|_{\mathfrak{h}} = g|_{\mathfrak{h}}\},$$

and in the second (assume that K_+ is of maximal rank) we have

$$(6.2) \quad H_G^*(M) = \{f \in S(\mathfrak{k}_+^*)^{K_+} \mid f|_{\mathfrak{h}} = g|_{\mathfrak{h}} \text{ for some } g \in S(\mathfrak{k}_-^*)^{K_-}\}.$$

Note that equation (6.1) holds true for any cohomogeneity-one action which satisfies $\mathrm{rank} G = \mathrm{rank} K_+ = \mathrm{rank} K_- = \mathrm{rank} H + 1$, even if the GKM conditions are not satisfied, see [5, Corollary 4.2]. For example, consider the $\mathrm{SO}(3)$ -action on the sphere S^4 which is induced by the group diagram $\mathrm{SO}(3) \supset \mathrm{S}(\mathrm{O}(2) \times \mathrm{O}(1)), \mathrm{S}(\mathrm{O}(1) \times \mathrm{O}(2)) \supset \mathbb{Z}_2 \times \mathbb{Z}_2$, see, e.g., [22, Section 2]; the third GKM condition is obviously not satisfied by this action. Similarly, (6.2) holds true whenever $\mathrm{rank} G = \mathrm{rank} K_+ = \mathrm{rank} K_- + 1 = \mathrm{rank} H + 1$, (to perceive this, combine [5, Corollary 4.2] with the injectivity of $H_G^*(M) \rightarrow H_G^*(G/K^+)$).

Example 6.7. We now consider again the action of K on M which is described in Example 5.1. First, all T -fixed points are on the same K -orbit, which means that M_{\max} consists of exactly one component. The locus M_1 of all points in M of isotropy corank at least 1 consists only of KS . Indeed, the 1-skeleton of the T -action on M is the union of fifteen 2-spheres among which only six have the K -orbit different from the K -orbit of a point. Moreover, the K -orbit of any of those six spheres is equal to KS : the reason is that the two T -fixed points of such a sphere can be mapped to p , respectively q by an element of $W(K)$; we then use [11], namely the last paragraph in Section 2.2.5 and Theorem 2.5. To construct the graph, we only need to perform Step 3 once. The K -stabilizers of p and q are both equal to T . Also note that S is a component of the fixed point set of the subtorus T_α of T whose Lie algebra is α^\perp . Finally, observe that $q = s_{\alpha_2} \circ s_{\alpha_1} p$. The GKM graph is presented in the figure below.

We obtain the following presentation:

$$H_K^*(M) \simeq \{f \in S(\mathfrak{t}^*) \mid f - f \circ s_{\alpha_1} \circ s_{\alpha_2} \text{ is divisible by } \alpha\}.$$

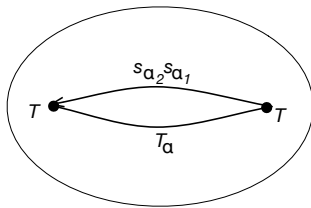


FIGURE 10

Remark 6.8. In the general context of a non-abelian GKM action of G on M , one can also compute the corresponding G -equivariant cohomology algebra by taking a maximal torus $T \subset G$ along with the corresponding Weyl group $W(G)$ and using the formula $H_G^*(M) = H_T^*(M)^{W(G)}$. The result stated in Lemma 4.2 is crucial here: by the classical GKM theorem, see, e.g., Theorem 3.6, the algebra $H_T^*(M)$ can be expressed as a subalgebra of $\bigoplus_{p \in M^T} S(\mathfrak{t}^*)$, which we refer to as the GKM algebra. The Weyl group $W(G)$ acts canonically on M^T and the action of $W(G)$ on the GKM algebra is given by

$$w \cdot (f_p)_{p \in M^T} = (g_p)_{p \in M^T}, \text{ where } g_p = f_{w^{-1}p} \circ w^{-1} \text{ for all } p \in M^T.$$

We conclude that $H_G^*(M)$ consists of all $(f_p)_{p \in M^T}$ in the GKM algebra with the property that

$$(6.3) \quad f_{wp} = f_p \circ w^{-1},$$

for all $p \in M^T$ and all $w \in W(G)$. It is interesting to compare this presentation with the one obtained by using our method. For instance, the equivalence is easy to establish in Examples 6.4 and 6.5 and we leave it as an exercise for the reader. Example 6.7 illustrates the power of our method: here, instead of fifteen divisibility relations involving six polynomials f_p , combined with the $W(K)$ -invariance conditions (6.3), non-abelian GKM theory leads to a presentation with just one divisibility relation.

REFERENCES

- [1] C. Allday and V. Puppe, *Cohomological methods in transformation groups*, Cambridge Studies in Advanced Mathematics, 32. Cambridge University Press, Cambridge, 1993.
- [2] G. E. Bredon, *The free part of a torus action and related numerical equalities*, Duke Math. J. **41** (1974), 843-854.
- [3] T. Chang and T. Skjelbred, *The topological Schur lemma and related results*, Ann. Math. **100** (1974), 307-321.
- [4] M. Franz and V. Puppe, *Exact sequences for equivariantly formal spaces*, C. R. Math. Acad. Sci. Soc. R. Can. **33** (2011), no. 1, 1-10.
- [5] O. Goertsches and A.-L. Mare, *Equivariant cohomology of cohomogeneity-one actions*, preprint, [arXiv:1110.6310](https://arxiv.org/abs/1110.6310).
- [6] O. Goertsches, H. Nozawa, and D. Töben, *Equivariant cohomology of K -contact manifolds*, Math. Ann. **354** (2012), no. 4, 1555-1582.
- [7] O. Goertsches and S. Rollenske, *Torsion in equivariant cohomology and Cohen-Macaulay actions*, Transform. Groups. **16** (2011), no. 4, 1063-1080.
- [8] M. Goresky, R. Kottwitz, and R. MacPherson, *Equivariant cohomology, Koszul duality, and the localization theorem*, Invent. Math. **131** (1998), no. 1, 25-83.

- [9] V. W. Guillemin, V. L. Ginzburg, and Y. Karshon, *Moment maps, cobordisms, and Hamiltonian group actions*, Mathematical Surveys and Monographs, 96. AMS, Providence, RI, 2002.
- [10] V. W. Guillemin and T. Holm, *GKM theory for torus actions with nonisolated fixed points*, Int. Math. Res. Not. 2004, no. 40, 2105-2124.
- [11] V. W. Guillemin, T. Holm, and C. Zara, *A GKM description of the equivariant cohomology ring of a homogeneous space*, J. Algebraic Combin. **23** (2006), no. 1, 21-41.
- [12] V. W. Guillemin and S. Sternberg, *Supersymmetry and equivariant de Rham theory*, Springer-Verlag, Berlin 1999.
- [13] V. W. Guillemin and C. Zara, *Equivariant de Rham theory and graphs*, Asian J. Math., **3** (1999), 49-76.
- [14] V. W. Guillemin and C. Zara, *One-skeleta, Betti numbers, and equivariant cohomology*, Duke Math. J., **107** (2001), 283-349.
- [15] M. Harada, A. Henriques, and T. Holm, *Computation of generalized equivariant cohomologies of Kac-Moody flag varieties*, Adv. Math. **197** (2005), 198-221.
- [16] C. A. Hoelscher, *Classification of cohomogeneity one manifolds in low dimensions*, Pacific J. Math. **246** (2010), no. 1, 129-185.
- [17] W.-Y. Hsiang, *Cohomology Theory of Topological Transformation Groups*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Vol. 85, Springer-Verlag, New-York 1975.
- [18] A.-L. Mare, *Equivariant cohomology of quaternionic flag manifolds*, J. Algebra **319** (2008), no. 7, 2830-2844.
- [19] A.-L. Mare and M. Willems, *Topology of the octonionic flag manifold*, preprint, [arXiv:0809.4318](https://arxiv.org/abs/0809.4318).
- [20] M. Masuda and T. Panov, *On the cohomology of torus manifolds*, Osaka J. Math., **43** (2006), 711-746.
- [21] P. S. Mostert, *On a compact Lie group acting on a manifold*, Ann. Math. **65** (1957), no. 3, 447-455.
- [22] W. Ziller, *On the geometry of cohomogeneity one manifolds with positive curvature*, Riemannian topology and geometric structures on manifolds, 233-262, Progr. Math., Vol. 271, Birkhäuser Boston, Boston, MA, 2009.

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