KOSTANT CONVEXITY FOR SYMMETRIC R-SPACES REVISITED

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An embedding of a connected and compact manifold M into Euclidean space $(V, \langle\ ,\ \rangle)$ is called *extrinsically symmetric* if for any $x \in M$, the reflection r_x of V about the affine normal space at x leaves M invariant. If we equip M with the submanifold metric, then the restriction of r_x to M is an involutive isometry. Relative to these isometries, M is an (intrinsically) symmetric space. One knows precisely what these embeddings are: by a theorem of Ferus [3] (see also [4]), they are the so-called symmetric R-spaces, which we present in what follows. We start with a simply connected symmetric space of non-compact type G/K and consider the Lie algebras of G and K, which are $\mathfrak{g}$ and $\mathfrak{k}$, respectively. Let also $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the corresponding Cartan decomposition and $\mathfrak{a} \subset \mathfrak{p}$ a maximal abelian subspace. The roots are elements of $\mathfrak{a}^*$; we denote by W be the corresponding Weyl group, which acts linearly on $\mathfrak{a}$. Pick $\xi \in \mathfrak{a}$ such that $\alpha(\xi) \in \{-1, 0, 1\}$, for any root α. The adjoint orbit $K\xi := \mathrm{Ad}_G(K)\xi$ turns out to be an extrinsic symmetric submanifold of $\mathfrak{p}$, the latter being equipped with the inner product given by the negative of the Killing form. Such orbits are called *symmetric R-spaces*. The aforementioned theorem of Ferus says that all extrinsic symmetric submanifolds in Euclidean space are of this type.

The following result is a special case of a theorem of Kostant [6] (see also [2] for a symplectic version and [9] for a generalization to isoparametric submanifolds). It describes the image of $M := K\xi$ under the orthogonal projection map $\pi : \mathfrak{p} \to \mathfrak{a}$. To make the statement more clear, we first recall that $M \cap \mathfrak{a} = W\xi$.

Theorem 1. (Kostant) *If $M = K\xi$ is a symmetric R-space, then $\pi(M) =$ convex hull of $W\xi$.*

Our goal here is to give an alternative proof of this theorem.

Claim 1. For any $w \in W$ and $x \in M$, the line segment between $w\xi$ and $\pi(x)$ is contained in $\pi(M)$.

There exists a flat F in the symmetric space M such that both $w\xi$ and x are in F. By a theorem of Eschenburg, Quast, and Tanaka [5] (see also [8, Theorem 7], cf. also [7]), F equipped with the submanifold metric is a direct product of round circles, each of them contained in an affine 2-plane, these planes being orthogonal to each other. Moreover, $\mathfrak{a}$ is contained in the normal space $\nu_{w\xi}M$, hence also in $\nu_{w\xi}F$. The projection of F onto the latter space is convex (concretely, a hypercuboid), hence its projection onto $\mathfrak{a}$ is convex as well. This implies Claim 1.

Claim 2. The convex hull of $W\xi$ is contained in $\pi(M)$.

This follows readily from Claim 1: first, the convex polytope in the claim has its edges contained in $\pi(M)$; then we prove inductively that the 2-faces, 3-faces etc. are contained in $\pi(M)$.

Claim 3. $\pi(M)$ is contained in the convex hull of $W\xi$.

If $a \in \mathfrak{a}$ is a regular vector (i.e., not canceled by any root), then the critical set of the height function $h_a : M \to \mathbb{R}$, $h_a(x) := \langle a, x\rangle$ is $W\xi$. Thus the largest height relative to a is reached somewhere in $W\xi$, possibly at more than one point in there. But then, for any $y \in \pi(M)$, one has

$$\langle a, y\rangle \le \max\{\langle a, w\xi\rangle \mid w \in W\}. \tag{1}$$

The latter inequality holds true even when a is not regular: otherwise, there exists $y \in \pi(M)$ such that $\langle a, y\rangle > \langle a, w\xi\rangle$ for all $w \in W$; by continuity, this remains true if we replace a by a regular vector sufficiently close to it, which is a contradiction. Now, that we know that (1) holds for any $a \in \mathfrak{a}$, one deduces that y is the convex hull of $W\xi$ (recall that any polytope in Euclidean space is completely determined by its “valid” inequalities, see e.g. [1, Theorem 6.11]).

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