Equivariant cohomology of real flag manifolds

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Abstract. Let $P = G/K$ be a semisimple non-compact Riemannian symmetric space, where $G = I_0(P)$ and $K = G_p$ is the stabilizer of $p \in P$. Let X be an orbit of the (isotropy) representation of K on $T_p(P)$ (X is called a real flag manifold). Let $K_0 \subset K$ be the stabilizer of a maximal flat, totally geodesic submanifold of P which contains p. We show that if all the simple root multiplicities of G/K are at least 2 then K_0 is connected and the action of K_0 on X is equivariantly formal. In the case when the multiplicities are equal and at least 2, we will give a purely geometric proof of a formula of Hsiang, Palais and Terng concerning $H^*(X)$. In particular, this gives a conceptually new proof of Borel's formula for the cohomology ring of an adjoint orbit of a compact Lie group.

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1. Introduction

Let G/K be a non-compact symmetric space, where G is a non-compact connected semisimple Lie group and $K \subset G$ a maximal compact subgroup. Then K is connected [He, Thm. 1.1, Ch. VI] and there exists a Lie group automorphism τ of G which is involutive and whose fixed point set is $G^\tau = K$. The involutive automorphism $d(\tau)_e$ of $\mathfrak{g} = \mathrm{Lie}(G)$ induces the Cartan decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p},$$

where $\mathfrak{k}$ (the same as $\mathrm{Lie}(K)$) and $\mathfrak{p}$ are the $(+1)$-, respectively (-1)-eigenspaces of $(d\tau)_e$. Since $[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}$, the space $\mathfrak{p}$ is $Ad_G(K) := Ad(K)$-invariant. The orbits of the action of $Ad(K)$ on $\mathfrak{p}$ are called *real flag manifolds*, or *s-orbits*. The restriction of the Killing form of $\mathfrak{g}$ to $\mathfrak{p}$ is an $Ad(K)$-invariant inner product on $\mathfrak{p}$, which we denote by $\langle \ , \ \rangle$.

Fix $\mathfrak{a} \subset \mathfrak{p}$ a maximal abelian subspace. Recall that the roots of the symmetric space G/K are linear functions $\alpha : \mathfrak{a} \to \mathbb{R}$ with the property that the space

$$\mathfrak{g}_\alpha := \{z \in \mathfrak{g} : [x, z] = \alpha(x)z \text{ for all } x \in \mathfrak{a}\}$$

is non-zero. The set Π of all roots is a root system in $(\mathfrak{a}^*, \langle \ , \ \rangle)$. Pick $\Delta \subset \Pi$ a simple root system and let $\Pi^+ \subset \Pi$ be the corresponding set of positive roots. For any $\alpha \in \Pi^+$ we have

$$\mathfrak{g}_\alpha + \mathfrak{g}_{-\alpha} = \mathfrak{k}_\alpha + \mathfrak{p}_\alpha,$$

where $\mathfrak{k}_\alpha = (\mathfrak{g}_\alpha + \mathfrak{g}_{-\alpha}) \cap \mathfrak{k}$ and $\mathfrak{p}_\alpha = (\mathfrak{g}_\alpha + \mathfrak{g}_{-\alpha}) \cap \mathfrak{p}$. We have the direct decompositions

$$\mathfrak{p} = \mathfrak{a} + \sum_{\alpha \in \Pi^+} \mathfrak{p}_\alpha, \quad \mathfrak{k} = \mathfrak{k}_0 + \sum_{\alpha \in \Pi^+} \mathfrak{k}_\alpha,$$

where $\mathfrak{k}_0$ denotes the centralizer of $\mathfrak{a}$ in $\mathfrak{k}$. The *multiplicity* of a root $\alpha \in \Pi^+$ is

$$m_\alpha = \dim \mathfrak{k}_\alpha + \dim \mathfrak{k}_{2\alpha}.$$

We note that this definition is slightly different from the standard one (see e.g. [Lo, Ch. VI, section 4]) which says that the multiplicity of α is just $\dim \mathfrak{k}_\alpha$.

Now $\mathfrak{k}_0$ is the Lie algebra of the Lie group $K_0 := C_K(\mathfrak{a})$ as well as of $K_0' := N_K(\mathfrak{a})$. One can see that K_0 is a normal subgroup of K_0'; the *Weyl group* of the symmetric space G/K is

$$W = K_0'/K_0.$$

It can be realized geometrically as the (finite) subgroup of $O(\mathfrak{a}, \langle\ ,\ \rangle)$ generated by the reflections about the hyperplanes $\ker \alpha$, $\alpha \in \Pi^+$.

Take $x_0 \in \mathfrak{a}$ and let $X = Ad(K)x_0$ be the corresponding flag manifold. The goal of our paper is to describe the cohomology, always with coefficients in $\mathbb{R}$, of X. The first main result concerns the action of K_0 on X.

Theorem 1.1. *If the symmetric space G/K has all root multiplicities m_α, $\alpha \in \Delta$, strictly greater than 1 then:*

(a) K_0 is connected;

(b) the action of K_0 on $X = Ad(K)x_0$ is equivariantly formal, in the sense that

$$H^*_{K_0}(X) \simeq H^*(X) \otimes H^*_{K_0}(\mathrm{pt})$$

by an isomorphism of $H^*_{K_0}(\mathrm{pt})$-modules;

(c) we have the isomorphisms of $\mathbb{R}$-vector spaces

$$H^*(X) \simeq \sum_{w \in W} H^{*-d_w}(w.x_0), \quad H^*_{K_0}(X) \simeq \sum_{w \in W} H^{*-d_w}_{K_0}(w.x_0).$$

Here

$$d_w = \sum m_\alpha$$

where the sum runs after all $\alpha \in \Pi^+$ such that $\alpha/2 \notin \Pi^+$ and the line segment $[x_0, w.x_0)$ crosses the hyperplane $\ker \alpha$.

Remark. Let U be the (compact) Lie subgroup of $G^{\mathbb{C}}$ whose Lie algebra is $\mathfrak{k} \oplus i\mathfrak{p}$. Then the manifold $X = Ad(K)x_0$ is the "real locus" [Go-Ho], [Bi-Gu-Ho] of an anti-symplectic involution on the adjoint orbit $Ad(U)ix_0$ (see e.g. [Du, section 5]). The natural action of the torus $T := \exp(i\mathfrak{a})$ on this orbit is Hamiltonian. In this way, X fits into the more general framework of [Go-Ho] and [Bi-Gu-Ho]. But these papers investigate X from the perspective of the action of $T_{\mathbb{R}} = T \cap K = T \cap K_0$, whereas we are interested here in the action on X of a group which may be larger than $T_{\mathbb{R}}$, namely K_0.

In the second part of our paper we will deal with the ring structure of the usual cohomology of X, under the supplementary assumption that the symmetric space has all root multiplicities equal. By [He, Ch. X, Table VI], their common value can be only 2, 4 or 8. An important ingredient is the action of $W = K_0'/K_0$ on X given by

$$hK_0.Ad(k)x_0 = Ad(k)Ad(h^{-1})x_0, \tag{1}$$

for any $h \in K_0'$ and $k \in K$. By functoriality, this induces an action of W on $H^*(X)$. We also note that W acts in a natural way on $\mathfrak{a}^*$.

Theorem 1.2. *Assume that G/K is an irreducible non-compact symmetric space whose simple root multiplicities are equal to the same number, call it m, which is at least 2. Take $X = Ad(K)x_0$.*

(i) If x_0 is a regular point of $\mathfrak{a}$, then there exists a canonical linear W-equivariant isomorphism $\Phi : \mathfrak{a}^* \to H^m(X)$. Its natural extension $\Phi : S(\mathfrak{a}^*) \to H^*(X)$ is a surjective ring homomorphism whose kernel is the ideal $\langle S(\mathfrak{a}^*)_+^W \rangle$ generated by all nonconstant W-invariant elements of $S(\mathfrak{a}^*)$. Consequently we have the $\mathbb{R}$-algebra isomorphism

$$H^*(X) \simeq S(\mathfrak{a}^*)/\langle S(\mathfrak{a}^*)_+^W \rangle.$$

(ii) If x_0 is an arbitrary point in $\mathfrak{a}$, then we have the $\mathbb{R}$-algebra isomorphism

$$H^*(X) \simeq S(\mathfrak{a}^*)^{W_{x_0}}/\langle S(\mathfrak{a}^*)_+^W \rangle,$$

where W_{x_0} is the W-stabilizer of x_0.

Remark. Any real flag manifold $X = Ad(K)x$ with the canonical embedding in $(\mathfrak{p}, \langle\ ,\ \rangle)$ is an element of an *isoparametric foliation* [Pa-Te]. The topology of such manifolds, including their cohomology rings, has been investigated by Hsiang, Palais and Terng in [Hs-Pa-Te] (see also [Ma]). The formulas for $H^*(X)$ given by Theorem 1.2 have been proved by them in that paper. Even though we do use some of their ideas (originating in [Bo-Sa]), our proof is different: they rely on Borel's formula [Bo] for the cohomology of a generic adjoint orbit of a compact Lie group, whereas we actually prove it.

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2. Symmetric spaces with multiplicities at least 2 and their s-orbits

Let G/K be an arbitrary non-compact symmetric space, $x_0 \in \mathfrak{a}$ and $X = Ad(K)x_0$ the corresponding s-orbit. The latter is a submanifold of the Euclidean space $(\mathfrak{p}, \langle\ ,\ \rangle)$. The Morse theory of height functions on X will be an essential instrument. The following proposition summarizes results from [Bo-Sa] or [Hs-Pa-Te] (see also [Ma]).

Proposition 2.1. *(i) If $a \in \mathfrak{a}$ is a general vector (i.e. not contained in any of the hyperplanes* $\ker \alpha$*, $\alpha \in \Pi^+$), then the height function $h_a(x) = \langle a, x\rangle$, $x \in X$ is a Morse function. Its critical set is the orbit $W.x_0$.*

(ii) Assume that a and x_0 are contained in the same Weyl chamber in $\mathfrak{a}$. Then the index of h_a at the critical point $w.x_0$ is

$$d_w = \sum m_\alpha \tag{2}$$

where the sum runs after all $\alpha \in \Pi^+$ such that $\alpha/2 \notin \Pi^+$ and the line segment $[a, wx_0)$ crosses the hyperplane $\ker \alpha$.

In the next lemma we consider the situation when all root multiplicities are at least 2.

Lemma 2.2. *Assume that the root multiplicities m_α, $\alpha \in \Delta$, of the symmetric space G/K are all strictly greater than 1. Then:*

(i) for any general vector $a \in \mathfrak{a}$, the height function $h_a : X \to \mathbb{R}$ is $\mathbb{Z}$-perfect,

(ii) the space K_0 is connected,

(iii) if $X = Ad(K)x_0$, then the orbit $W.x_0$ is contained in the fixed point set X^{K_0}.

Proof. (i) According to [Ko, Theorem 1.1.4], there exists a metric on X such that if two critical points x and y can be joined by a gradient line, then $x = s_\gamma y$, where $\gamma \in \Pi^+$. By (2), the difference of the indices of x and y is different from ± 1. Because the stable and unstable manifolds intersect transversally [Ko, Corollary 2.2.7], the Morse complex of h_a has all boundary operators identically zero, hence h_a is $\mathbb{Z}$-perfect.

(ii) Take $a \in \mathfrak{a}$ a general vector. The height function h_a on $Ad(K)a$ is $\mathbb{Z}$-perfect. From (2) we deduce that $H_1(Ad(K)a, \mathbb{Z}) = 0$, thus $Ad(K)a$ is simply connected. On the other hand, the stabilizer $C_K(a)$ is just K_0 (see e.g. [Bo-Sa]). Because K/K_0 is simply connected and K is connected, we deduce that K_0 is connected.

(iii) The height function h_a is $Ad(K_0)$-invariant, thus $Crit(h_a) = W.x_0$ is also $Ad(K_0)$-invariant. The result follows from the fact that K_0 is connected. □

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. Point (a) was proved in Lemma 2.2 (ii).

(b) According to [Gu-Gi-Ka, Proposition C.25] it is sufficient to show that $H^*_{K_0}(X)$ is free as a $H^*_{K_0}(\text{pt})$-module. In order to do that we consider the height function $h_a : X \to \mathbb{R}$ corresponding to a general $a \in \mathfrak{a}$. We use the same arguments as in the proof of Lemma 2.2, (i). The function h_a is a K_0-invariant. By the same reasons as above, the K_0-equivariant Morse complex [Au-Br, Sections 5 and 6] has all boundary operators identically zero. Thus $H^*_{K_0}(X)$ is a free $H^*_{K_0}(\text{pt})$-module (with a basis indexed by $\text{Crit}(h_a) = W.x_0$).

(c) The space $H_*(X)$ has a basis $\{[X_{w.x_0}] : w \in W\}$, where $X_{w.x_0}$ is some d_w-dimensional cycle in X, $w \in W$. The evaluation pairing $H^*(X) \times H_*(X) \to \mathbb{R}$ is non-degenerate; consider the basis of $H^*(X)$ dual to $\{[X_{w.x_0}] : w \in W\}$, which gives one element of degree d_w for each $w.x_0$. The result follows. □

3. Cohomology of s-orbits of symmetric spaces with uniform multiplicities at least 2

Throughout this section G/K is a non-compact irreducible symmetric space whose simple root multiplicities are all equal to m, where $m \geq 2$; $x_0 \in \mathfrak{a}$ is a regular element and

$$X = Ad(K)x_0 \simeq K/K_0$$

is the corresponding real flag manifold. There are three such symmetric spaces; their compact duals are (see e.g. [Hs-Pa-Te, Section 3]):

1. any connected simple compact Lie group K; we have $m = 2$; the flag manifold is $X = K/T$, where T is a maximal torus in K;
2. $SU(2n)/Sp(n)$ where $m = 4$; the flag manifold is $X = Sp(n)/Sp(1)^{\times n}$;
3. E_6/F_4 where $m = 8$; the flag manifold is $X = F_4/Spin(8)$.

Let $\Delta = \{\gamma_1, \dots, \gamma_l\}$ be a simple root system of Π. To each γ_j corresponds the distribution E_j on X, defined as follows: its value at x_0 is

$$E_j(x_0) = [\mathfrak{k}_{\gamma_j}, x_0]$$

and E_j is K-invariant, i.e.

$$E_j(Ad(k)x_0) = Ad(k)E_j(x_0),$$

for all $k \in K$.

A basis of $H_m(X)$ can be obtained as follows: Assume that x_0 is in the (interior of the) Weyl chamber $C \subset \mathfrak{a}$ which is bounded by the hyperplanes $\ker \gamma_j$, $1 \le j \le l$. The Weyl group W is generated by s_j, which is the reflection of $\mathfrak{a}$ about the wall $\ker \gamma_j$, $1 \le j \le l$. For each $1 \le j \le l$ we consider the Lie subalgebra $\mathfrak{k}_0 + \mathfrak{k}_{\gamma_j}$ of $\mathfrak{k}$; denote by K_j the corresponding connected subgroup of K. It turns out that the orbit $Ad(K_j)x_0$ is a round m-dimensional metric sphere in $(\mathfrak{p}, \langle\ ,\ \rangle)$. To any $x = Ad(k)x_0 \in X$ we attach the round sphere

$$S_j(x) = Ad(k)Ad(K_j)x_0.$$

The spheres S_j are integral manifolds of the distribution E_j. We denote by $[S_j]$ the homology class carried by any of the spheres $S_j(x)$, $x \in X$. It turns out that $S_1(x_0), \dots, S_l(x_0)$ are cycles of Bott-Samelson type (see [Bo-Sa], [Hs-Pa-Te]) for the index m critical points of the height function h_a, thus $[S_1], \dots, [S_l]$ is a basis of $H_m(X)$.

The following result concerning the action of W on $H_m(X)$ was proved in [Hs-Pa-Te, Corollary 6.10] (see also [Ma, Theorem 2.1.1]):

Proposition 3.1. *We can choose an orientation of the spheres S_j, $1 \le j \le l$, such that the linear isomorphism $\mathfrak{a} \to H_m(X)$ determined by*

$$\gamma_j^\vee := \frac{2\gamma_j}{\langle \gamma_j, \gamma_j \rangle} \mapsto [S_j],$$

$1 \le j \le l$, is W-equivariant.

We need one more result concerning the action of W on $H^*(X)$:

Lemma 3.2. *Let $x \in \mathfrak{a}$ be an arbitrary element, $C = C_K(x)$ its centralizer in K, and let*

$$p : X = K/K_0 \to Ad(K)x = K/C$$

be the natural map induced by the inclusion $K_0 \subset C$. Then the map $p^* : H^*(Ad(K)x) \to H^*(X)$ is injective. Its image is

$$p^* H^*(Ad(K)x) = H^*(X)^{W_x}$$

where the right hand side denotes the set of all W_x-invariant elements of $H^*(X)$. Here W_x denotes the W-stabilizer of x. In particular, the only elements in $H^*(X)$ which are W-invariant are those of degree 0, i.e.

$$H^*(X)^W = H^0(X).$$

Proof. The map $p : K/K_0 \to K/C$ is a fibre bundle. The fiber C/K_0 is an s-orbit of the symmetric space $C_G(x)/C_K(x)$. The latter has all root multiplicities equal to m, as they are all root multiplicities of some roots of G/K. By Theorem 1.1 (ii), C/K_0 can have non-vanishing cohomology groups only in dimensions which are multiples of m. The same can be said about the cohomology of the space K/C. Because $m \in \{2, 4, 8\}$, the spectral sequence of the bundle $p : K/K_0 \to K/C$ collapses, which implies that p^* is injective.

The map p is W-equivariant with respect to the actions of W on $Ad(K)x_0$, respectively $Ad(K)x$ defined by (1). Thus if $w \in W_x$, then $w|_{Ad(K)x}$ is the identity map, hence we have $p \circ w = p$. This implies the inclusion

$$p^* H^*(Ad(K)x) \subset H^*(X)^{W_x}.$$

On the other hand, the action of W on X defined by (1) is free, as the $Ad(K)$ stabilizer of the general point x_0 reduces to K_0. Consequently we have

$$H^*(X)^{W_x} = H^*(X/W_x)$$

and

$$\chi(X/W_x) = \frac{\chi(X)}{|W_x|} = \frac{|W|}{|W_x|},$$

where χ denotes the Euler-Poincaré characteristic. It follows from Theorem 1.1 (c) that

$$\dim H^*(X)^{W_x} = \frac{|W|}{|W_x|} = \dim H^*(Ad(K)x).$$

Now we use that p^* is injective.

In order to prove the last statement of the lemma, we take $x = 0 \in \mathfrak{a}$. □

Let us consider the Euler class $\tau_i = e(E_i) \in H^m(X)$, $1 \le i \le l$. We will prove that:

Lemma 3.3. *(i) The cohomology classes τ_i, $1 \le i \le l$ are a basis of $H^m(X)$.*

(ii) The linear isomorphism $\Phi : \mathfrak{a}^* \to H^m(X)$ determined by

$$\gamma_i \mapsto e(E_i),$$

$1 \le i \le l$, is W-equivariant.

Proof. By Proposition 3.1 we know that

$$s_{i*}[S_j] = [S_j] - d_{ji}[S_i],$$

where

$$d_{ji} = 2\frac{\langle \gamma_j^\vee, \gamma_i^\vee \rangle}{\langle \gamma_i^\vee, \gamma_i^\vee \rangle}.$$

Denote by $\langle\ ,\ \rangle$ the evaluation pairing $H^m(M) \times H_m(M) \to \mathbb{R}$. Consider $\alpha_j \in H^m(M)$ such that $\langle \alpha_j, [S_i] \rangle = \delta_{ij}$, $1 \le i, j \le l$. Take the expansion

$$\tau_i = \sum_{j=1}^{l} t_{ij} \alpha_j.$$

The automorphism s_i of X maps the distribution E_i onto itself and changes its orientation (since so does the antipodal map on an m-dimensional sphere). Thus

$$s_i^*(\tau_i) = -\tau_i.$$

Consequently we have

$$t_{ij} = \langle \tau_i, [S_j] \rangle = \langle -s_i^*(\tau_i), [S_j] \rangle = -\langle \tau_i, s_{i*}[S_j] \rangle = -\langle \tau_i, [S_j] - d_{ji}[S_i] \rangle = -t_{ij} + 2d_{ji}$$

which implies $t_{ij} = d_{ji}$. By Proposition 3.1, the matrix (d_{ij}) is the Cartan matrix of the root system dual to Π, hence it is non-singular. Consequently τ_i, $1 \leq i \leq l$ is a basis of $H^m(X)$. Again by Proposition 3.1 we have

$$\langle s_j^*(\tau_i), [S_k] \rangle = \langle \tau_i, [S_k] - d_{kj}[S_j] \rangle = t_{ik} - d_{kj}t_{ij} = t_{ik} - t_{jk}d_{ji},$$

thus

$$s_j^*(\tau_i) = \tau_i - d_{ji}\tau_j.$$

It remains to notice that d_{ji} can also be expressed as

$$d_{ji} = 2\frac{\langle \gamma_i, \gamma_j \rangle}{\langle \gamma_j, \gamma_j \rangle}.$$

□

We are now ready to prove Theorem 1.2:

Proof of Theorem 1.2 (i) Consider the ring homomorphism $\Phi : S(\mathfrak{a}^*) \to H^*(X)$ induced by $\gamma_i \mapsto e(E_i)$, $1 \leq i \leq l$. By Lemma 3.3, Φ is W-equivariant and from Lemma 3.2 we deduce that $\langle S(\mathfrak{a}^*)^W_+ \rangle \subset \ker \Phi$. By Lemma 3.4 (see below), it is sufficient to prove that

$$\Phi(\prod_{\alpha \in \Pi^+} \alpha) \neq 0.$$

To this end, we will describe explicitly $\Phi(\alpha)$, for $\alpha \in \Pi^+$. Write $\alpha = w.\gamma_j$, where $w \in W$. The latter is of the form $w = hK_0$, with $h \in K_0'$. The image of $S_j(x_0)$ by the automorphism w of X is

$$\begin{aligned} w(S_j(x_0)) =& Ad(K_j)Ad(h^{-1})x_0 = Ad(h^{-1})Ad(hK_jh^{-1})x_0 \\ =& Ad(h^{-1})Ad(K_\alpha)x_0 = Ad(h^{-1})S_\alpha(x_0) = S_\alpha(Ad(h^{-1})x_0) \\ =& S_\alpha(w.x_0). \end{aligned}$$

Here K_α is the connected subgroup of K of Lie algebra $\mathfrak{k}_0 + \mathfrak{k}_\alpha$ and $S_\alpha(x_0) := Ad(K_\alpha)x_0$ is a round metric sphere through x_0; for any $x = Ad(k)x_0 \in X$ we have $S_\alpha(x) := Ad(k)S_\alpha(x_0)$, which is an integral manifold of

$$E_\alpha(x) = Ad(k)[\mathfrak{k}_\alpha, x_0].$$

It is worth mentioning in passing that the spheres S_α and the distributions E_α are the curvature spheres, respectively curvature distributions of the isoparametric submanifold $X \subset \mathfrak{p}$ (see the remark following Theorem 1.2 in the introduction). Thus the differential of w satisfies $(dw)(E_j) = E_\alpha$, which implies

$$e(E_j) = w^*e(E_\alpha).$$

Consequently

$$\Phi(\alpha) = \Phi(w.\gamma_j) = w^{-1}.\Phi(\gamma_j) = (w^{-1})^*(e(E_j)) = e(E_\alpha).$$

We deduce that

$$\Phi(\prod_{\alpha\in\Pi^+} \alpha) = \prod_{\alpha\in\Pi^+} e(E_\alpha) = e(\sum_{\alpha\in\Pi^+} E_\alpha).$$

On the other hand,

$$\sum_{\alpha\in\Pi^+} E_\alpha(x_0) = \sum_{\alpha\in\Pi^+} [\mathfrak{k}_\alpha, x_0] = [\mathfrak{k}, x_0] = T_{x_0}X$$

thus

$$\sum_{\alpha\in\Pi^+} E_\alpha = TX.$$

It follows that

$$\Phi(\prod_{\alpha\in\Pi^+} \alpha) = e(TX),$$

which is different from zero, as

$$e(TX)([X]) = \chi(X) = |W|,$$

where $\chi(X)$ is the Euler-Poincaré characteristic of X.

(ii) We apply Lemma 3.2. □

The following lemma has been used in the proof:

Lemma 3.4. ([Hi, Lemma 2.8]) *Let I be a graded ideal of $S(\mathfrak{a}^*)$ which is also a vector subspace and such that $\langle S(\mathfrak{a}^*)^W_+\rangle \subset I$. We have $I = \langle S(\mathfrak{a}^*)^W_+\rangle$ if and only if*

$$\prod_{\alpha\in\Pi^+} \alpha \notin I.$$

A proof of this lemma can also be found in the appendix.

4. Appendix: Proof of Lemma 3.4

The goal of this appendix is to provide a proof of Lemma 3.4, which is stated without a proof in [Hi]. As mentioned in the introduction, the Weyl group W can be realized as the group of orthogonal transformations of $\mathfrak{a}$ generated by the reflections s_α, $\alpha \in \Pi^+$. In fact, if $\{\gamma_1, \dots, \gamma_l\}$ is a simple root system, then W is generated by $s_i := s_{\gamma_i}$, $1 \le i \le l$. Denote by w_0 the longest element of W, where the length is measured with respect to the generating set $\{s_1, \dots, s_l\}$. We will use the notations

$$\mathcal{S} := S(\mathfrak{a}^*), \quad I_W := \langle S(\mathfrak{a}^*)^W_+\rangle.$$

First of all we note that the action of W on the polynomial ring $\mathcal{S}$ is given by

$$(w.f)(x) = f(w^{-1}.x),$$

where $w \in W$, $f \in \mathcal{S}$, $x \in \mathfrak{a}$. This action preserves the grading of $\mathcal{S}$, hence the ideal I_W generated by the nonconstant W-invariant polynomials is also graded. The most prominent example of a polynomial which is not W-invariant is

$$d = \prod_{\alpha\in\Pi^+} \alpha.$$

In fact d is skew-invariant, in the sense that $w.d = (-1)^{l(w)}d$, for any $w \in W$.

If $\alpha \in \Pi^+$, we consider the operator $\Delta_\alpha : \mathcal{S} \to \mathcal{S}$ defined as follows:

$$\Delta_\alpha(f) = \frac{f - s_\alpha.f}{\alpha},$$

$f \in \mathcal{S}$. Note that $f - s_\alpha.f$ vanishes on the space $\ker \alpha$, hence $\Delta_\alpha(f)$ is really a polynomial. The following result is straightforward:

Lemma 4.1. *If $w \in W$, $\alpha \in \Pi^+$, $f, g \in \mathcal{S}$, then we have:*
(a) $\Delta_\alpha(fg) = \Delta_\alpha(f)g + s_\alpha(f)\Delta_\alpha(g)$;
(b) $\Delta_\alpha(I_W) \subset I_W$.

To any $w \in W$ we can associate the operator $\Delta_w : \mathcal{S} \to \mathcal{S}$, which has degree $-l(w)$, and is defined as follows: take $w = s_{i_1} \dots s_{i_k}$ a reduced expression and put $\Delta_w = \Delta_{\gamma_{i_1}} \cdots \Delta_{\gamma_{i_k}}$. We note that Δ_w does not depend on the choice of the reduced expression (see e.g. [Hi, Proposition 2.6]). The operators obtained in this way have the following property (see [Hi, Lemma 3.1]):

$$\Delta_w \circ \Delta_{w'} = \begin{cases} \Delta_{ww'}, \text{ if } l(ww') = l(w) + l(w') \\ 0, \text{ otherwise} \end{cases} \tag{3}$$

A classical result which goes back to Chevalley, says that the ideal I_W is generated by l homogeneous polynomials, which are algebraically independent. Let $d_1, \cdots, d_l$ denote their degrees. It follows that the Poincaré polynomial of $\mathcal{S}/I_W$ is:

$$P(\mathcal{S}/I_W) = \sum_{k=0}^{\infty} (\dim \mathcal{S}^k - \dim I_W^k) t^k = \prod_{j=1}^{l} (1 + t + \cdots + t^{d_j - 1}).$$

Combined with the fact that $d_1 + \cdots d_l = N + l$ (see for instance [Hu, Theorem 3.9]), this tells us that $I^k = \mathcal{S}^k$, for $k \geq N + 1$. The same polynomial can be expressed as (see [Hu, Theorem 3.15]):

$$P(S/I_W) = \sum_{w \in W} t^{l(w)}.$$

We deduce that $\dim \mathcal{S}^k - \dim I_W^k$ equals the number of $w \in W$ with $l(w) = k$, $0 \leq k \leq N$. The following result describes a direct complement of I_W^k in $\mathcal{S}^k$:

Proposition 4.2. *For any $0 \leq k \leq N$, the elements $\Delta_w(d)$, $w \in W$, $l(w) = N - k$ are linearly independent and span a direct complement of I_W^k in $\mathcal{S}^k$.*

Proof. The number of elements of W of length k equals the number of elements of length $N - k$, hence we only have to prove that the polynomials $\Delta_w(d)$, where $l(w) = N - k$ are linearly independent and their span intersected with I_W is $\{0\}$. To this end, it is sufficient to show that if

$$\sum_{l(w)=N-k} \lambda_w \Delta_w(d) \in I_W^k$$

then all λ_w must vanish. Indeed, if we fix $v \in W$ with $l(v) = N - k$, then by (3), we have

$$\Delta_{w_0 v^{-1}}\Big(\sum_{l(w)=N-k} \lambda_w \Delta_w(d)\Big) = \lambda_v.$$

The left hand side of this equation is in I_W^0, hence it must be 0.

We are ready to prove Lemma 3.4:

Proof of Lemma 3.4 We prove by induction on k that $I_W^k = I^k$, $0 \leq k \leq N$. Things are clear for $k = N$: I_W^N equals I^N because $I_W^N \subset I^N \neq \mathcal{S}^N$ and the codimension of I_W^N in $\mathcal{S}^N$ is 1 (see Proposition 4.2). Now, from $I^{k+1} = I_W^{k+1}$ we deduce that $I^k = I_W^k$. Suppose that we have

$$f := \sum_{l(w)=N-k} \lambda_w \Delta_w(d) \in I^k,$$

where $\lambda_w \in \mathbb{R}$, not all of them equal to 0. We will prove by induction on $m \in \{0, \ldots, k\}$ the following claim

Claim. For any $h_m \in \mathcal{S}^m$ and any $\alpha_1, \ldots, \alpha_m \in \Pi^+$, we have

$$h_m \Delta_{\alpha_1} \circ \ldots \circ \Delta_{\alpha_m}(f) \in I^k.$$

For $m = 0$, this is trivial. Suppose it is true for a certain m and prove it for $m+1$. If $h_m \in \mathcal{S}_m$, $\alpha_1, \ldots, \alpha_m \in \Pi^+$, h an arbitrary homogeneous polynomial of degree 1, and α a positive root, then we have

$$hh_m \Delta_{\alpha_1} \circ \ldots \circ \Delta_{\alpha_m}(f) \in I^{k+1} = I_W^{k+1},$$

hence its image by Δ_α is in $I_W^k \subseteq I^k$. We deduce that

$$\Delta_\alpha(h)h_m\Delta_{\alpha_1} \circ \ldots \circ \Delta_{\alpha_m}(f) + s_\alpha(h)\Delta_\alpha(h_m)\Delta_{\alpha_1} \circ \ldots \circ \Delta_{\alpha_m}(f) + s_\alpha(h)s_\alpha(h_m)\Delta_\alpha \circ \Delta_{\alpha_1} \circ \ldots \circ \Delta_{\alpha_m}(f)$$

is in I^k, consequently $s_\alpha(hh_m)\Delta_\alpha \circ \Delta_{\alpha_1} \circ \ldots \circ \Delta_{\alpha_m}(f) \in I^k$. Since any $h_{m+1} \in \mathcal{S}^{m+1}$ is a linear combination of polynomials of the form $s_\alpha(hh_m)$, the claim is proved.

We deduce that for any $v \in W$ with $l(v) = k$, and any $h_k \in \mathcal{S}^k$ we have that

$$h_k \Delta_v(f) \in I^k.$$

Fix now $w \in W$ with $l(w) = N - k$ and take $v := w_0 w^{-1}$. Then $\Delta_v(f) = \lambda_w$ by (1), hence $\lambda_w h_k \in I^k$, for any $h_k \in \mathcal{S}^k$. But then λ_w must vanish, since $I^k \neq \mathcal{S}^k$ (if they were equal, from $k \leq N$ we would deduce $I^N = S^N$, which is false). We conclude that $f = 0$, which is a contradiction. This finishes the proof. □

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