EQUIVARIANT COHOMOLOGY OF COHOMOGENEITY ONE ACTIONS

OLIVER GOERTSCHES AND AUGUSTIN-LIVIU MARE

ABSTRACT. We show that if $G \times M \to M$ is a cohomogeneity one action of a compact connected Lie group G on a compact connected manifold M then $H^*_G(M)$ is a Cohen-Macaulay module over $H^*(BG)$. Moreover, this module is free if and only if the rank of at least one isotropy group is equal to $\operatorname{rank} G$. We deduce as corollaries several results concerning the usual (de Rham) cohomology of M, such as a new proof of the following obstruction to the existence of a cohomogeneity one action: if M admits a cohomogeneity one action, then $\chi(M) > 0$ if and only if $H^{\rm odd}(M) = \{0\}$.

1. INTRODUCTION

Let G be a compact connected Lie group which acts on a compact connected manifold M, the cohomogeneity of the action being equal to one; this means that there exists a G-orbit whose codimension in M is equal to one. For such group actions, we investigate the corresponding equivariant cohomology $H^*_G(M)$ (the coefficient ring will always be $\mathbb{R}$). We are especially interested in the natural $H^*(BG)$-module structure of this space. The first natural question concerning this module is whether it is free, in other words, whether the G-action is equivariantly formal. One can easily find examples which show that the answer is in general negative. Instead of being free, we may also wonder whether the above-mentioned module satisfies the (weaker) requirement of being Cohen-Macaulay. It turns out that the answer is in our context always positive: this is the main result of our paper. Before stating it, we mention that the relevance of the Cohen-Macaulay condition in equivariant cohomology was for the first time noticed by Bredon [9], inspired by Atiyah [6], who had previously used this notion in equivariant K-theory. It has also attracted attention in the theory of equivariant cohomology of finite group actions, see e.g. [13]. More recently, group actions whose equivariant cohomology satisfies this requirement have been investigated in [16], [19], and [18]. We adopt the terminology already used in those papers: if a group G acts on a space M in such a way that $H^*_G(M)$ is a Cohen-Macaulay $H^*(BG)$-module, we simply say that the G-action is Cohen-Macaulay.

Theorem 1.1. *Any cohomogeneity one action of a compact connected Lie group on a compact connected manifold is Cohen-Macaulay.*

Concretely, if the group action is $G \times M \to M$, then the (Krull) dimension and the depth of $H^*_G(M)$ over $H^*(BG)$ are equal. In fact, we can say exactly what the value of these two numbers is: the highest rank of a G-isotropy group.

To put our theorem into perspective, we mention the following result, which has been proved in [18]: an action of a compact connected Lie group on a compact manifold with the property that all isotropy groups have the same rank is Cohen-Macaulay. Consequently, if the G-action is transitive, then it is Cohen-Macaulay (see also Proposition 2.6 and Remark 2.8 below). We deduce:

Corollary 1.2. *Any action of a compact connected Lie group on a compact connected manifold whose cohomogeneity is zero or one is Cohen-Macaulay.*

We also note that actions of cohomogeneity two or larger are not necessarily Cohen-Macaulay: examples already appear in the classification of T^2-actions on 4-manifolds by Orlik and Raymond [39], see Example 4.3 in this paper.

In general, a group action is equivariantly formal if and only if it is Cohen-Macaulay and the rank of at least one isotropy group is maximal, i.e. equal to the rank of the acting group (cf. [18], see also Proposition 2.9 below). This immediately implies the following characterization of equivariant formality for cohomogeneity one actions:

Corollary 1.3. *A cohomogeneity one action of a compact connected Lie group on a compact connected manifold is equivariantly formal if and only if the rank of at least one isotropy group is maximal.*

Corollary 1.3 shows that the cohomogeneity one action $G \times M \to M$ is equivariantly formal whenever M satisfies the purely topological condition $\chi(M) > 0$ (indeed, it is known that this inequality implies the condition on the rank of the isotropy groups in Corollary 1.3). Extensive lists of cohomogeneity one actions on manifolds with positive Euler characteristic can be found for instance in [3] and [15]. The above observation will be used to obtain the following obstruction to the existence of a group action on M of cohomogeneity zero or one, which follows also from a result of Grove and Halperin [22] about the rational homotopy of cohomogeneity one actions, see Remark 5.4 below.

Corollary 1.4. *If a compact manifold M admits a cohomogeneity one action of a compact Lie group, then we have $\chi(M) > 0$ if and only if $H^{\mathrm{odd}}(M) = \{0\}$.*

This topic is addressed in Subsection 5.1.1 below. We also mention that, if M is as in Corollary 1.4, then $\chi(M) > 0$ implies that $\pi_1(M)$ is finite, see Lemma 5.5. By classical results of Hopf and Samelson [30], respectively Borel [7], the fact that $\chi(M) > 0$ implies both $H^{\mathrm{odd}}(M) = \{0\}$ and the finiteness of $\pi_1(M)$, holds true also in the case when M admits an action of a compact Lie group which is transitive, i.e., of cohomogeneity equal to zero. This shows, for example, that there is no compact connected Lie group action with cohomogeneity zero or one on a compact manifold with the rational homology type of the connected sum $(S^1 \times S^3)\#(S^2 \times S^2)$. However, the 4-manifold $R(1,0)$ of Orlik and Raymond [39] mentioned in Example 4.3 is homeomorphic to this connected sum and has a T^2-action of cohomogeneity two. Thus, the equivalence of $\chi(M) > 0$ and $H^{\mathrm{odd}}(M) = \{0\}$ holds no longer for group actions with cohomogeneity greater than one.

The situation when M is odd-dimensional is discussed in Subsection 5.2. In this case, equivariant formality is equivalent to $\operatorname{rank} H = \operatorname{rank} G$, where H denotes a regular isotropy of the G-action. We obtain a relation involving $\dim H^*(M)$, the Euler characteristic of G/H and the Weyl group W of the cohomogeneity one action: see Corollary 5.13. This will enable us to obtain some results for cohomogeneity one actions on odd-dimensional rational homology spheres.

Finally, in Subsection 5.3 we show that for a cohomogeneity one action $G\times M \to M$, the $H^*(BG)$-module $H^*_G(M)$ is torsion free if and only if it is free. We note that the latter equivalence is in general not true for arbitrary group actions: this topic is investigated in [1] and [17].

Acknowledgement. We would like to thank Wolfgang Ziller for some valuable comments on a previous version of this paper.

2. Equivariantly formal and Cohen-Macaulay group actions

Let $G \times M \to M$ be a differentiable group action, where both the Lie group G and the manifold M are compact. The equivariant cohomology ring is defined in the usual way, by using the classifying principal G bundle $EG \to BG$, as follows: $H^*_G(M) = H^*(EG\times_G M)$. It has a canonical structure of $H^*(BG)$-algebra, induced by the ring homomorphism $\pi^* : H^*(BG) \to H^*_G(M)$, where $\pi : EG \times_G M \to BG$ is the canonical map (cf. e.g. [31, Ch. III] or [24, Appendix C]). We say that the group action is *equivariantly formal* if $H^*_G(M)$ regarded as an $H^*(BG)$-module is free.

There is also a relative notion of equivariant cohomology: if N is a G-invariant submanifold of M, then we define $H^*_G(M,N) = H^*(EG \times_G M, EG \times_G N)$. This cohomology carries again an $H^*(BG)$-module structure, this time induced by the map $H^*(BG) \to H^*_G(M)$ and the cup product $H^*_G(M) \times H^*_G(M,N) \to H^*_G(M,N)$; see [12, p. 178].

2.1. Criteria for equivariant formality.

The following result is known. We state it for future reference and sketch a proof for the reader's convenience. Here T denotes a maximal torus of G.

Proposition 2.1. *If a compact connected Lie group G acts on a compact manifold M, then the following statements are equivalent:*

(a) The G-action on M is equivariantly formal.

(b) We have $H^*_G(M) \simeq H^*(M) \otimes H^*(BG)$ by an isomorphism of $H^*(BG)$-modules.

(c) The homomorphism $i^* : H^*_G(M) \to H^*(M)$ induced by the canonical inclusion $i : M \to EG \times_G M$ is surjective. In this case, it automatically descends to a ring isomorphism $\mathbb{R} \otimes_{H^*(BG)} H^*_G(M) \to H^*(M)$.

(d) The T-action on M induced by restriction is equivariantly formal.

(e) $\dim H^*(M) = \dim H^*(M^T)$, where M^T is the T-fixed point set.

Proof. A key ingredient of the proof is the Leray-Serre spectral sequence of the bundle $M \xrightarrow{i} EG \times_G M \xrightarrow{\pi} BG$, whose E_2-term is $H^*(M) \otimes H^*(BG)$. Both (b) and (c) are clearly equivalent to the fact that this spectral sequence collapses at E_2. The equivalence to (a) is more involved, see for instance [2, Corollary (4.2.3)]. The equivalence to (d) is the content of [24, Proposition C.26]. For the equivalence to (e), see [31, p. 46]. □

We also mention the following useful criterion for equivariant formality:

Proposition 2.2. *An action of a compact connected Lie group G on a compact manifold M with $H^{\mathrm{odd}}(M) = \{0\}$ is automatically equivariantly formal. The converse implication is true provided that the T-fixed point set M^T is finite.*

Proof. If $H^{\mathrm{odd}}(M) = \{0\}$ then the Leray-Serre spectral sequence of the bundle $ET \times_T M \to BT$ collapses at E_2, which is equal to $H^*(M) \otimes H^*(BT)$. To prove the second assertion we note that the Borel localization theorem implies that the kernel of the natural map $H^*_T(M) \to H^*_T(M^T)$ equals the torsion submodule of $H^*_T(M)$, which vanishes in the case of an equivariantly formal action. Because M^T is assumed to be finite, $H^*_T(M^T)$ is concentrated only in even degrees. Consequently, the same holds true for $H^*_T(M)$, as well as for $H^*(M)$, due to the isomorphism $H^*_T(M) \simeq H^*(M) \otimes H^*(BT)$. □

2.2. **Cohen-Macaulay modules and group actions.** If R is a *local Noetherian graded ring, one can associate to any finitely generated graded R-module A its Krull dimension, respectively depth (see e.g. [10, Section 1.5]). We always have that $\operatorname{depth} A \leq \dim A$, and if dimension and depth coincide, then we say that the finitely-generated graded R-module A is a *Cohen-Macaulay module* over R. The following result is an effective tool frequently used in this paper:

Lemma 2.3. *If $0 \to A \to B \to C \to 0$ is a short exact sequence of finitely generated R-modules, then we have:*

(i) $\dim B = \max\{\dim A, \dim C\}$;
(ii) $\operatorname{depth} A \geq \min\{\operatorname{depth} B, \operatorname{depth} C + 1\}$;
(iii) $\operatorname{depth} B \geq \min\{\operatorname{depth} A, \operatorname{depth} C\}$;
(iv) $\operatorname{depth} C \geq \min\{\operatorname{depth} A - 1, \operatorname{depth} B\}$.

Proof. For (i) we refer to [34, Section 12] and for (ii)–(iv) to [10, Proposition 1.2.9]. □

Corollary 2.4. *If A and B are Cohen-Macaulay modules over R of the same dimension d, then $A \oplus B$ is again Cohen-Macaulay of dimension d.*

Proof. This follows from Lemma 2.3, applied to the exact sequence $0 \to A \to A \oplus B \to B \to 0$. □

Yet another useful tool will be for us the following lemma, which is the graded version of [44, Proposition 12, Section IV.B].

Lemma 2.5. *Let R and S be two Noetherian graded *local rings and let $\varphi : R \to S$ be a homomorphism that makes S into an R-module which is finitely generated. If A is a finitely generated S-module, then we have:*

$$\operatorname{depth}_R A = \operatorname{depth}_S A \quad \text{and} \quad \dim_R A = \dim_S A.$$

In particular, A is Cohen-Macaulay as R-module if and only if it is Cohen-Macaulay as S-module.

We say that the group action $G \times M \to M$ is *Cohen-Macaulay* if $H^*_G(M)$ regarded as $H^*(BG)$-module is Cohen-Macaulay. The relevance of this notion for the theory of equivariant cohomology was for the first time noticed by Bredon in [9]. Other references are [16], [18], and [19].

The following result gives an example of a Cohen-Macaulay action, which is important for this paper. We first note that if G is a compact Lie group and $K \subset G$ a subgroup, then there is a canonical map $BK \to BG$ induced by the presentations $BG = EG/G$ and $BK = EG/K$. The ring homomorphism $H^*(BG) \to H^*(BK)$ makes $H^*(BK)$ into an $H^*(BG)$-module.

Proposition 2.6. *Let G be a compact connected Lie group and $K \subset G$ a Lie subgroup, possibly non-connected. Then $H^*_G(G/K) = H^*(BK)$ is a Cohen-Macaulay module over $H^*(BG)$ of dimension equal to* $\operatorname{rank} K$.

Proof. The fact that $H^*_G(G/K) = H^*(BK)$ is Cohen-Macaulay is a very special case of [18, Corollary 4.3] because all isotropy groups of the natural G-action on G/K have the same rank. To find its dimension, we consider the identity component K_0 of K and note that also $H^*(BK_0)$ is a Cohen-Macaulay module over $H^*(BG)$; by Lemma 2.5, the dimension of $H^*(BK_0)$ over $H^*(BG)$ is equal to the (Krull) dimension of the (polynomial) ring $H^*(BK_0)$, which is $\operatorname{rank} K_0$. Let us now observe that $H^*(BK) = H^*(BK_0)^{K/K_0}$ is a nonzero $H^*(BG)$-submodule of the Cohen-Macaulay module $H^*(BK_0)$, hence by [19, Lemma 5.4] (the graded version of [16, Lemma 4.3]), $\dim_{H^*(BG)} H^*(BK) = \dim_{H^*(BG)} H^*(BK_0) = \operatorname{rank} K_0$. □

As a byproduct, we can now easily deduce the following result.

Corollary 2.7. *If G is a (possibly non-connected) compact Lie group, then $H^*(BG)$ is a Cohen-Macaulay ring, i.e., a Cohen-Macaulay module over itself.*

Proof. There exists a unitary group $U(n)$ which contains G as a subgroup. By Proposition 2.6, $H^*(BG)$ is a Cohen-Macaulay module over $H^*(BU(n))$. Because $H^*(BG)$ is Noetherian by a result of Venkov [46], we can apply Lemma 2.5 and conclude that $H^*(BG)$ is a Cohen-Macaulay ring. □

Remark 2.8. The assertions in Proposition 2.6 and Corollary 2.7 are not new and can also be justified as follows. Denote by $\mathfrak{g}$ the Lie algebra of a compact Lie group G and by $\mathfrak{t}$ the Lie algebra of a maximal torus T in G, then $H^*(BG) = S(\mathfrak{g}^*)^G = S(\mathfrak{t}^*)^{W(G)}$, where $W(G) = N_G(T)/Z_G(T)$ is the Weyl group of G. (Here we have used the Chern-Weil isomorphism and the Chevalley restriction theorem, see e.g. [35, p. 311], respectively [38, Theorem 4.12]). The fact that the ring $S(\mathfrak{t}^*)^{W(G)}$ is Cohen-Macaulay follows from [27, Proposition 13] (see also [10, Corollary 6.4.6] or [33, Theorem B, p. 176]). Proposition 2.6 is now a direct consequence of Lemma 2.5 and the well-known fact that the G-equivariant cohomology of a compact manifold is a finitely generated $H^*(BG)$-module, see e.g. [42].

In general, if a group action $G\times M \to M$ is equivariantly formal, then it is Cohen-Macaulay. The next result, which is actually Proposition 2.5 in [18], establishes a more precise relationship between these two notions. It also involves

$$M_{\max} := \{p \in M \,:\, \operatorname{rank} G_p = \operatorname{rank} G\}.$$

Proposition 2.9. ([18]) *A G-action on M is equivariantly formal if and only if it is Cohen-Macaulay and $M_{\max} \neq \emptyset$.*

Note, in particular, that the G-action on G/K mentioned in Proposition 2.6 is equivariantly formal if and only if $\operatorname{rank} K = \operatorname{rank} G$.

3. Topology of transitive group actions on spheres

The following proposition will be needed in the proof of the main result. It collects results that, in a slightly more particular situation, were obtained by Samelson in [43].

Proposition 3.1. *Let K be a compact Lie group, possibly non-connected, which acts transitively on the sphere S^m, $m \geq 0$, and let $H \subset K$ be an isotropy subgroup.*

(a) If m is even, then $\operatorname{rank} H = \operatorname{rank} K$ and the canonical homomorphism $H^*(BK) \to H^*(BH)$ is injective.

(b) If m is odd, then $\operatorname{rank} H = \operatorname{rank} K - 1$ and the canonical homomorphism $H^*(BK) \to H^*(BH)$ is surjective.

Proof. In the case when both K and H are connected, the result follows from [43, Satz IV] along with [7, Section 21, Corollaire] and [7, Proposition 28.2]. From now on, K or H may be non-connected. We distinguish the following three situations.

Case 1: $m \geq 2$. Let K_0 be the identity component of K. The induced action of K_0 on S^m is also transitive, because its orbits are open and closed in the K-orbits. We deduce that we can identify $K_0/(K_0 \cap H)$ with S^m. Since the latter sphere is simply connected, the long exact homotopy sequence of the bundle $K_0 \cap H \to K_0 \to S^m$ shows that $K_0 \cap H$ is connected. This implies that $K_0 \cap H$ is equal to H_0, the identity component of H, and we have the identification $K_0/H_0 = S^m$. This implies the assertions concerning the ranks.

Let us now consider the long exact homotopy sequence of the bundle $H \to K \to S^m$ and deduce from it that the map $\pi_0(H) \to \pi_0(K)$ is a bijection. This means that H and K have the same number of connected components, thus we may set $\Gamma := K/K_0 = H/H_0$. The (free) actions of Γ on BK_0 and BH_0 induce the identifications $BK = BK_0/\Gamma$, respectively $BH = BH_0/\Gamma$ (see e.g. [31, Ch. III, Section 1]). Moreover, the map $\pi : BH_0 \to BK_0$ is Γ-equivariant. The induced homomorphism $\pi^* : H^*(BK_0) \to H^*(BH_0)$ is then Γ-equivariant as well, hence it maps Γ-invariant elements to Γ-invariant elements. We have $H^*(BH) = H^*(BH_0)^\Gamma$ and $H^*(BK) = H^*(BK_0)^\Gamma$ which shows that if m is even, then the map $\pi^*|_{H^*(BH_0)^\Gamma} : H^*(BH_0)^\Gamma \to H^*(BK_0)^\Gamma$ is injective. We will now show that if m is odd, then the latter map is surjective. Indeed, since π^* is surjective, for any $b \in H^*(BK_0)^\Gamma$ there exists $a \in H^*(BH_0)$ with $\pi^*(a) = b$. But then $a' := \frac{1}{|\Gamma|}\sum_{\gamma\in\Gamma} \gamma a$ is in $H^*(BH_0)^\Gamma$ and satisfies $\pi^*(a') = b$.

Case 2: $m = 1$. We clearly have $\operatorname{rank} H = \operatorname{rank} K - 1$ in this case. We only need to show that the map $H^*(BK) \to H^*(BH)$ is surjective. Equivalently, using the identification with the rings of invariant polynomials [35, p. 311], we will show that the restriction map $S(\mathfrak{k}^*)^K \to S(\mathfrak{h}^*)^H$ is surjective. Choosing an Ad_K-invariant scalar product on $\mathfrak{k}$, we obtain an orthogonal decomposition $\mathfrak{k} = \mathfrak{h} \oplus \mathbb{R}v$, where $v \in \mathfrak{h}^\perp$. The Ad-invariance implies that $\mathfrak{h}$ is an ideal in $\mathfrak{k}$ and v a central element. Given $f \in S(\mathfrak{h}^*)^H$, we define $g : \mathfrak{k} = \mathfrak{h} \oplus \mathbb{R}v \to \mathbb{R}$ by $g(X + tv) = f(X)$. Clearly, g is a polynomial on $\mathfrak{k}$ that restricts to f on $\mathfrak{h}$; for the desired surjectivity we therefore only need to show that g is K-invariant. As $K/H \cong S^1$ is connected, K is generated by its identity component K_0 and H. Note that both the H- and the K_0-action respect the decomposition $\mathfrak{k} = \mathfrak{h} \oplus \mathbb{R}v$. The H-invariance of f therefore implies the H-invariance of g. Also, the adjoint action of $\mathfrak{k}$ on $\mathfrak{h}$ is the same as the adjoint action of $\mathfrak{h}$ (the $\mathbb{R}v$-summand acts trivially), so f is K_0-invariant, which implies that g is K_0-invariant.

Case 3: $m = 0$. Since $\operatorname{rank} H = \operatorname{rank} K$, a maximal torus $T \subset H$ is also maximal in K. The injectivity of $H^*(BK) \to H^*(BH)$ follows from the identifications $H^*(BK) = S(\mathfrak{t}^*)^{W(K)}$, $H^*(BH) = S(\mathfrak{t}^*)^{W(H)}$.

□

4. Cohomogeneity one actions are Cohen-Macaulay

In this section we prove Theorem 1.1.

There are two possibilities for the orbit space M/G: it can be diffeomorphic to the circle S^1 or to the interval $[0,1]$. If $M/G = S^1$, Theorem 1.1 follows readily from [18, Corollary 4.3] and the fact that all isotropy groups of the G-action are conjugate to each other.

From now on we assume that $M/G = [0,1]$. We start with some well-known considerations which hold true in this case. One can choose a G-invariant Riemannian metric on M and a geodesic γ perpendicular to the orbits such that $G\gamma(0)$ and $G\gamma(1)$ are the two nonregular orbits, and such that $\gamma(t)$ is regular for all $t \in (0,1)$. Let $K^- := G_{\gamma(0)}$, $K^+ := G_{\gamma(1)}$, and H be the regular isotropy G_γ on γ. We have $H \subset K^\pm$. The group diagram $G \supset K^-, K^+ \supset H$ determines the equivariant diffeomorphism type of the G-manifold M. More precisely, by the slice theorem, the boundaries of the unit disks $D_\pm$ in the normal spaces $\nu_{\gamma(0)}G\gamma(0)$, respectively $\nu_{\gamma(1)}G\gamma(1)$ are spheres $K^\pm/H = S^{\ell_\pm}$. The space M can be realized by gluing the tubular neighborhoods $G/K^\pm \times_{K^\pm} D^\pm$ along their common boundary $G/K^\pm \times_{K^\pm} K^\pm/H = G/H$ (see e.g. [8, Theorem IV.8.2], [23, Section 1] or [28, Section 1.1]). We are in a position to prove the main result of the paper in the remaining case:

Proof of Theorem 1.1. in the case when $M/G = [0,1]$. We may assume that the rank of any G-isotropy group is at most equal to $b := \operatorname{rank} K^-$, i.e. we have

$$\operatorname{rank} H \le \operatorname{rank} K^+ \le b. \tag{1}$$

By Proposition 3.1, we have $\operatorname{rank} K^- - \operatorname{rank} H \le 1$ and consequently $\operatorname{rank} H \in \{b-1, b\}$ (alternatively, we can use [40, Lemma 1.1]). If $\operatorname{rank} H = b$, then all isotropy groups of the G-action have the same rank and Theorem 1.1 follows from [18, Corollary 4.3]. From now on we will assume that

$$\operatorname{rank} H = b - 1. \tag{2}$$

This implies that the quotient K^-/H is odd-dimensional, that is, ℓ_- is an odd integer. By Proposition 3.1 (b), the homomorphism $H^*(BK^-) \to H^*(BH)$ is surjective. On the other hand, the equivariant Mayer-Vietoris sequence of the covering of M with two tubular neighborhoods around the singular orbits can be expressed as follows:

$$\ldots \longrightarrow H^*_G(M) \longrightarrow H^*_G(G/K^-) \oplus H^*_G(G/K^+) \longrightarrow H^*_G(G/H) \longrightarrow \ldots$$

But $H^*_G(G/K^\pm) = H^*(BK^\pm)$ and $H^*_G(G/H) = H^*(BH)$, hence the last map in the above sequence is surjective. Thus the equivariant Mayer-Vietoris sequence splits into short exact sequences of the form:

$$0 \longrightarrow H^*_G(M) \longrightarrow H^*_G(G/K^-) \oplus H^*_G(G/K^+) \longrightarrow H^*_G(G/H) \longrightarrow 0. \tag{3}$$

We analyze separately the two situations imposed by equations (1) and (2).

Case 1: $\operatorname{rank} K^+ = b$. By Proposition 2.6, both $H^*_G(G/K^-)$ and $H^*_G(G/K^+)$ are Cohen-Macaulay modules over $H^*(BG)$ of dimension b, so by Corollary 2.4, the middle term $H^*_G(G/K^-) \oplus H^*_G(G/K^+)$ in the sequence (3) is also Cohen-Macaulay of dimension b. Because $H^*_G(G/H)$ is Cohen-Macaulay of dimension $b-1$, we can apply Lemma 2.3 to the sequence (3) to deduce that $b \le \operatorname{depth} H^*_G(M) \le \dim H^*_G(M) = b$, which implies that $H^*_G(M)$ is Cohen-Macaulay of dimension b.

Case 2: $\operatorname{rank} K^+ = b-1$. In this case, $H \subset K^+$ are Lie groups of equal rank and therefore, by Proposition 3.1 (a), the canonical map $H^*(BK^+) \to H^*(BH)$ is injective. Exactness of the sequence (3) thus implies that the restriction map $H_G^*(M) \to H_G^*(G/K^-)$ is injective. We deduce that the long exact sequence of the pair $(M, G/K^-)$ splits into short exact sequences, i.e. the following sequence is exact:

$$0 \longrightarrow H_G^*(M) \longrightarrow H_G^*(G/K^-) \longrightarrow H_G^*(M, G/K^-) \longrightarrow 0. \tag{4}$$

We aim to show that $H_G^*(M, G/K^-)$ is a Cohen-Macaulay module over $H^*(BG)$ of dimension $b-1$; once we have established this, we can apply Lemma 2.3 to the sequence (4) to deduce that $H_G^*(M)$ is Cohen-Macaulay of dimension b, in exactly the same way as we used the sequence (3) in Case 1 above.

By excision, $H_G^*(M, G/K^-) = H_G^*(G \times_{K^+} D^+, G/H)$, where $G \times_{K^+} D^+$ is a tubular neighborhood of G/K^+ and $G/H = G \times_{K^+} S^+$ is its boundary (i.e. S^+ is the boundary of D^+). We have the isomorphism

$$\begin{aligned} H_G^*(G \times_{K^+} D^+, G/H) &= H^*(EG \times_G (G \times_{K^+} D^+), EG \times_G (G \times_{K^+} S^+)) \\ &= H^*(EG \times_{K^+} D^+, EG \times_{K^+} S^+) \\ &= H_{K^+}^*(D^+, S^+), \end{aligned}$$

which is $H^*(BG)$-linear, where the $H^*(BG)$-module structure on the right hand side is the restriction of the $H^*(BK^+)$-module structure to $H^*(BG)$. Because $H^*(BK^+)$ is a finitely generated module over $H^*(BG)$, it is sufficient to show that $H_{K^+}^*(D^+, S^+)$ is a Cohen-Macaulay module over $H^*(BK^+)$ of dimension $b-1$. Then Lemma 2.5 implies the claim, because the ring $H^*(BK^+)$ is Noetherian and *local (by [10, Example 1.5.14 (b)]).

Denote by K_0^+ the connected component of K^+. There is a spectral sequence converging to $H_{K_0^+}^*(D^+, S^+)$ whose E_2-term equals $H^*(BK_0^+) \otimes H^*(D^+, S^+)$; because $H^*(D^+, S^+)$ is concentrated only in degree $\ell^+ + 1$, this spectral sequence collapses at E_2, and it follows that $H_{K_0^+}^*(D^+, S^+)$ is a free module over $H^*(BK_0^+)$ (cf. e.g. [24, Lemma C.24]). In particular, it is a Cohen-Macaulay module over $H^*(BK_0^+)$ of dimension $b-1$. Because $H^*(BK_0^+)$ is finitely generated over $H^*(BK^+)$, Lemma 2.5 implies that $H_{K_0^+}^*(D^+, S^+)$ is also Cohen-Macaulay over $H^*(BK^+)$ of dimension $b-1$.

A standard averaging argument (see for example [18, Lemma 2.7]) shows that the $H^*(BK^+)$-module $H_{K^+}^*(D^+, S^+) = H_{K_0^+}^*(D^+, S^+)^{K^+/K_0^+}$ is a direct summand of $H_{K_0^+}^*(D^+, S^+)$: we have $H_{K_0^+}^*(D^+, S^+) = H_{K^+}^*(D^+, S^+) \oplus \ker a$, where $a : H_{K_0^+}^*(D^+, S^+) \to H_{K_0^+}^*(D^+, S^+)$ is given by averaging over the K^+/K_0^+-action. It follows that $H_{K^+}^*(D^+, S^+)$ is Cohen-Macaulay over $H^*(BK^+)$ of dimension $b-1$. □

Combining this theorem with Proposition 2.9, we also immediately obtain Corollary 1.3.

The following statement was shown in the above proof; we formulate it as a separate proposition because we will use it again later.

Proposition 4.1. *Whenever* $M/G = [0,1]$ *and* $\operatorname{rank} H = \operatorname{rank} K^- - 1$*, the sequence* (3) *is exact.*

An immediate corollary to this proposition is the following Goresky-Kottwitz-MacPherson [20] type description of the ring $H^*_G(M)$ (we denote by $\mathfrak{k}^{\pm}$ and $\mathfrak{h}$ the Lie algebras of $K^{\pm}$, respectively H).

Corollary 4.2. *If $M/G = [0,1]$ and $\operatorname{rank} H = \operatorname{rank} K^- - 1$ then $H^*_G(M)$ is isomorphic to the $S(\mathfrak{g}^*)^G$-subalgebra of $S((\mathfrak{k}^-)^*)^{K^-} \oplus S((\mathfrak{k}^+)^*)^{K^+}$ consisting of all pairs (f,g) with the property that $f|_{\mathfrak{h}} = g|_{\mathfrak{h}}$.*

We end this section with an example which concerns Corollary 1.2. It shows that if the cohomogeneity of the group action is no longer smaller than two, then the action is not necessarily Cohen-Macaulay.

Example 4.3. The T^2-manifold $R(1,0)$ appears in the classification of T^2-actions on 4-manifolds by Orlik and Raymond [39]. It is a compact connected 4-manifold, homeomorphic to the connected sum $(S^1 \times S^3)\#(S^2 \times S^2)$. The T^2-action on $R(1,0)$ is effective, therefore of cohomogeneity two, and has exactly two fixed points. The action is not Cohen-Macaulay, because otherwise it would be equivariantly formal (as it has fixed points); as the fixed point set is finite, this would imply using Proposition 2.2 that $H^{\rm odd}(R(1,0)) = \{0\}$, contradicting $H^1(R(1,0)) = \mathbb{R}$.

5. Equivariantly formal actions of cohomogeneity one

In this section we present some extra results in the situation when $M/G = [0,1]$ and the action $G \times M \to M$ is equivariantly formal. By Corollary 1.3, this is equivalent to the fact that the rank of at least one of K^- and K^+ is equal to the rank of G.

5.1. The case when M is even-dimensional.

5.1.1. *Cohomogeneity-one manifolds with positive Euler characteristic.* A discussion concerning the Euler characteristic of a compact manifold (of arbitrary dimension) admitting a cohomogeneity one action of a compact connected Lie group can be found in [3, Section 1.2] (see also [15, Section 1.3]). By Proposition 1.2.1 therein we have

$$\chi(M) = \chi(G/K^-) + \chi(G/K^+) - \chi(G/H). \tag{5}$$

The Euler characteristic $\chi(M)$ is always nonnegative, and one has $\chi(M) > 0$ if and only if $M/G = [0,1]$, M is even-dimensional, and the rank of at least one of K^- and K^+ is equal to $\operatorname{rank} G$, see [3, Corollary 1.2.2]. What we can show with our methods is that $\chi(M) > 0$ implies that $H^{\rm odd}(M) = \{0\}$.

Proposition 5.1. *Assume that a compact connected manifold M admits a cohomogeneity one action of a compact connected Lie group. Then the following conditions are equivalent:*

(i) $\chi(M) > 0$,
(ii) $H^{\rm odd}(M) = \{0\}$,
(iii) *M is even-dimensional, $M/G = [0,1]$, and the G-action is equivariantly formal.*

These conditions imply that

$$\dim H^*(M) = \chi(M) = \chi(G/K^-) + \chi(G/K^+). \tag{6}$$

Proof. The equivalence of (i) and (iii) follows from the previous considerations and Corollary 1.3. Let us now assume that $\chi(M) > 0$. Then (iii) holds: consequently, $\operatorname{rank} H = \operatorname{rank} G - 1$ and the space $M_{\max}$ is either G/K^-, G/K^+ or their union $G/K^- \cup G/K^+$. A maximal torus $T \subset G$ therefore acts on M with fixed point set M^T equal to $(G/K^-)^T$, $(G/K^+)^T$ or $(G/K^-)^T \cup (G/K^+)^T$, depending on the rank of K^- and K^+. As M^T is in particular finite, the equivariant formality of the G-action is the same as the condition $H^{\mathrm{odd}}(M) = \{0\}$ (see Proposition 2.2), and (ii) follows. The last assertion in the corollary follows readily from Equation (5). □

Remark 5.2. Note that by the classical result of Hopf and Samelson [30] concerning the Euler-Poincaré characteristic of a compact homogeneous space, along with the formula of Borel given by Equation (7) below, the topological obstruction given by the equivalence of (i) and (ii) holds true also for homogeneity, i.e., the existence of a transitive action of a compact Lie group.

We remark that the equivalence of (i) and (ii) is no longer true in the case when M only admits an action of cohomogeneity two. Consider for example the cohomogeneity two T^2-manifold $R(1,0)$ mentioned in Example 4.3: since it is homeomorphic to $(S^1 \times S^3)\#(S^2 \times S^2)$, its Euler characteristic is equal to 2, and the first cohomology group is $\mathbb{R}$.

Remark 5.3. The Euler characteristics of the nonregular orbits appearing in Equation (6) can also be expressed in terms of the occurring Weyl groups: we have $\chi(G/K^\pm) \neq 0$ if and only if $\operatorname{rank} K^\pm = \operatorname{rank} G$, and in this case $\chi(G/K^\pm) = \frac{|W(G)|}{|W(K^\pm)|}$.

Remark 5.4. By a theorem of Grove and Halperin [22], if M admits a cohomogeneity-one action, then the rational homotopy $\pi_*(M) \otimes \mathbb{Q}$ is finite-dimensional. This result also implies the equivalence between $\chi(M) > 0$ and $H^{\mathrm{odd}}(M) = 0$, as follows: If $\chi(M) > 0$, then one can show using the Seifert-van Kampen theorem that $\pi_1(M)$ is finite, see Lemma 5.5 below. Therefore the universal cover of M, call it $\widetilde{M}$, is also compact, and carries a cohomogeneity one action with orbit space $[0,1]$. The theorem of Grove and Halperin then states that $\widetilde{M}$ is rationally elliptic. Since the covering $\widetilde{M} \to M$ is finite, we have $\chi(\widetilde{M}) > 0$, and by [26, Corollary 1] (or [14, Proposition 32.16]), this implies $H^{\mathrm{odd}}(\widetilde{M}) = \{0\}$. But M is the quotient of $\widetilde{M}$ by a finite group, say Γ, hence $H^{\mathrm{odd}}(M) = H^{\mathrm{odd}}(\widetilde{M})^\Gamma = \{0\}$.

Note that by Proposition 2.2 this line of argument also implies that the G-action is equivariantly formal, and hence provides an alternative proof of Corollary 1.3 in the case of an even-dimensional cohomogeneity-one manifold M with $M/G = [0,1]$.

Lemma 5.5. *Assume that a compact connected manifold M admits a cohomogeneity one action of a compact connected Lie group. If $\chi(M) > 0$ then $\pi_1(M)$ is finite.*

Proof. We may assume that $\operatorname{rank} K^- = \operatorname{rank} G$. Consider the tubular neighborhoods of the two non-regular orbits G/K^+, respectively G/K^- mentioned before: their union is the whole M and their intersection is G-homotopic to a principal orbit G/H. The inclusions of the intersection into each of the two neighborhoods induce between the first homotopy groups the same maps as those induced by the canonical projections $\rho^+ : G/H \to G/K^+$, respectively $\rho^- : G/H \to G/K^-$

(see [28, Section 1.1]). These are the maps $\rho_*^+ : \pi_1(G/H) \to \pi_1(G/K^+)$ and $\rho_*^- : \pi_1(G/H) \to \pi_1(G/K^-)$. Consider the bundle $G/H \to G/K^-$ whose fiber is $K^-/H = S^{\ell_-}$ of odd dimension $\ell_- = \dim K^- - \dim H \geq 1$. The long exact homotopy sequence of this bundle implies readily that the map ρ_*^- is surjective. From the Seifert-van Kampen theorem we deduce that $\pi_1(M)$ is isomorphic to $\pi_1(G/K^+)/A$, where A is the smallest normal subgroup of $\pi_1(G/K^+)$ which contains $\rho_*^+(\ker \rho_*^-)$ (see e.g. [37, Exercise 2, p. 433]).
Case 1: $\dim K^+ - \dim H \geq 1$. As above, this implies that ρ_*^+ is surjective. Consequently, the map $\pi_1(G/K^-) = \pi_1(G/H)/\ker \rho_*^- \to \pi_1(G/K^+)/A$ induced by ρ_*^+ is surjective as well. On the other hand, $\operatorname{rank} G = \operatorname{rank} K^-$ implies that $\pi_1(G/K^-)$ is a finite group. Thus, $\pi_1(G/K^+)/A$ is a finite group as well.
Case 2: $\dim K^+ = \dim H$. We have $K^+/H = S^0$, which consists of two points, thus ρ^+ is a double covering. This implies that ρ_*^+ is injective and its image, $\rho_*^+(\pi_1(G/H))$, is a subgroup of index two in $\pi_1(G/K^+)$. The index of $\rho_*^+(\ker \rho_*^-)$ in $\rho_*^+(\pi_1(G/H))$ is equal to the index of $\ker \rho_*^-$ in $\pi_1(G/H)$, which is finite (being equal to the cardinality of $\pi_1(G/K^-)$). Thus, the quotient $\pi_1(G/K^+)/\rho_*^+(\ker \rho_*^-)$ is a finite set. Finally, we only need to take into account that the canonical projection map $\pi_1(G/K^+)/\rho_*^+(\ker \rho_*^-) \to \pi_1(G/K^+)/A$ is surjective. □

Remark 5.6. Assume again that $M/G = [0,1]$ and M is even-dimensional. If M admits a metric of positive sectional curvature, then at least one of K^- and K^+ has maximal rank, by the so-called "Rank Lemma" (see [21, Lemma 2.5] or [23, Lemma 2.1]). By Corollary 1.3 and Proposition 5.1, the G-action is equivariantly formal and $H^{\mathrm{odd}}(M) = \{0\}$. This however is not a new result as by Verdiani [47], M is already covered by a rank one symmetric space.

5.1.2. *Cohomology.* Consider again the situation that M is a compact even-dimensional manifold admitting a cohomogeneity one action of a compact connected Lie group G such that $M/G = [0,1]$ and that at least one isotropy rank equals the rank of G. In this section we will give a complete description of the Poincaré polynomial of M, purely in terms of G and the occurring isotropy groups, and eventually even of the ring $H^*(M)$.

Proposition 5.7. *If M is even-dimensional, $M/G = [0,1]$, and the rank of at least one of K^- and K^+ equals the rank of G, then the Poincaré polynomial of M is given by*

$$P_t(M) = \frac{P_t(BK^-) + P_t(BK^+) - P_t(BH)}{P_t(BG)}.$$

In particular, $P_t(M)$ only depends on the abstract Lie groups $G, K^\pm, H$, and not on the whole group diagram.

Proof. Assume that $\operatorname{rank} K^- = \operatorname{rank} G$. Since the pincipal orbit G/H is odd-dimensional, and $\operatorname{rank} H \in \{\operatorname{rank} K^-, \operatorname{rank} K^- - 1\}$ (see Proposition 3.1), we actually have $\operatorname{rank} H = \operatorname{rank} K^- - 1$. Hence by Proposition 4.1, the equivariant Mayer-Vietoris sequence (3) is exact, which implies that the G-equivariant Poincaré series of M is

$$P_t^G(M) = P_t^G(G/K^-) + P_t^G(G/K^+) - P_t^G(G/H) = P_t(BK^-) + P_t(BK^+) - P_t(BH).$$

We only need to observe that, since the G-action on M is equivariantly formal, Proposition 2.1 (b) implies that $P_t^G(M) = P_t(M) \cdot P_t(BG)$. □

Remark 5.8. In case H is connected, the above description of $P_t(M)$ can be simplified as follows. Assume that $\operatorname{rank} K^- = \operatorname{rank} G$. Then $\operatorname{rank} H = \operatorname{rank} K^- - 1$, and we have that $K^-/H = S^{\ell_-}$ is an odd-dimensional sphere. Consequently, the Gysin sequence of the spherical bundle $K^-/H \to BH \to BK^-$ splits into short exact sequences, which implies readily that $P_t(BH) = (1 - t^{\ell_-+1})P_t(BK^-)$. Similarly, if also the rank of K^+ is equal to the rank of G, then $P_t(BH) = (1 - t^{\ell_++1})P_t(BK^+)$ and we obtain the following formula:

$$P_t(M) = \left(\frac{1}{1-t^{\ell_-+1}} + \frac{1}{1-t^{\ell_++1}} - 1\right) \cdot \frac{P_t(BH)}{P_t(BG)}.$$

Numerous examples of cohomogeneity one actions which satisfy the hypotheses of Proposition 5.7 can be found in [3] and [15] (since the condition on the isotropy ranks in Proposition 5.7 is equivalent to $\chi(M) > 0$, see Proposition 5.1). Proposition 5.7 allows us to calculate the cohomology groups of M in all these examples. We will do in detail one such example:

Example 5.9. ([3, Table 4.2, line 5]) We have $G = SO(2n+1)$, $K^- = SO(2n)$, $K^+ = SO(2n-1) \times SO(2)$, and $H = SO(2n-1)$. The following can be found in [36, Ch. III, Theorem 3.19]:

$$P_t(BSO(2n+1)) = \frac{1}{(1-t^4)(1-t^8)\cdots(1-t^{4n})}.$$

We have $\ell_- = 2n-1$ and $\ell_+ = 1$, and consequently, using Remark 5.8, we obtain the following description of the Poincaré series of the corresponding manifold M:

$$\begin{aligned} P_t(M) &= \left(\frac{1}{1-t^{2n}} + \frac{1}{1-t^2} - 1\right) \cdot (1-t^{4n}) \\ &= 1 + t^2 + t^4 + \ldots + t^{2n-2} + 2t^{2n} + t^{2n+2} + \ldots + t^{4n}. \end{aligned}$$

Let us now use equivariant cohomology to determine the ring structure of $H^*(M)$ in the case at hand. In Corollary 4.2 we determined the $S(\mathfrak{g}^*)^G$-algebra structure of $H^*_G(M)$, and because of Proposition 2.1 (c), the ring structure of $H^*(M)$ is encoded in the $S(\mathfrak{g}^*)^G$-algebra structure. The following proposition follows immediately.

Proposition 5.10. *If M is even-dimensional, $M/G = [0,1]$, and the rank of at least one of K^- and K^+ equals the rank of G, then we have a ring isomorphism*

$$H^*(M) \simeq \mathbb{R} \otimes_{S(\mathfrak{g}^*)^G} A,$$

where A is the $S(\mathfrak{g}^*)^G$-subalgebra of $S((\mathfrak{k}^-)^*)^{K^-} \oplus S((\mathfrak{k}^+)^*)^{K^+}$ consisting of all pairs (f, g) with the property that $f|_{\mathfrak{h}} = g|_{\mathfrak{h}}$.

Remark 5.11. Recall that the cohomology ring of a homogeneous space G/K where G, K are compact and connected, such that $\operatorname{rank} G = \operatorname{rank} K$, is described by Borel's formula [7] as follows:

$$H^*(G/K) \simeq \mathbb{R} \otimes_{S(\mathfrak{g}^*)^G} S(\mathfrak{k}^*)^K. \tag{7}$$

The above description of the ring $H^*(M)$ can be considered as a version of Borel's formula for cohomogeneity one manifolds.

Equation (7) is particularly simple in the case when $K = T$, a maximal torus in G. Namely, if $\mathfrak{t}$ is the Lie algebra of T and $W(G)$ the Weyl group of G, then

$$H^*(G/T) \simeq S(\mathfrak{t}^*)/\langle S(\mathfrak{t}^*)^{W(G)}_{>0}\rangle,$$

where $\langle S(\mathfrak{t}^*)^{W(G)}_{>0}\rangle$ is the ideal of $S(\mathfrak{t}^*)$ generated by the non-constant $W(G)$-invariant polynomials. The following example describes the cohomology ring of a space that can be considered the cohomogeneity one analogue of G/T. It also shows that the ring $H^*(M)$ depends on the group diagram, not only on the isomorphism types of G, $K^{\pm}$, and H, like the Poincaré series, see Proposition 5.7 above.

Example 5.12. Let G be a compact connected Lie group, T a maximal torus in G, and $H \subset T$ a codimension one subtorus. The cohomogeneity one manifold corresponding to G, $K^- = K^+ := T$, and H is $M = G \times_T S^2$, where the action of T on S^2 is determined by the fact that H acts trivially and T/H acts in the standard way, via rotation about a diameter of S^2: indeed, the latter T-action has the orbit space equal to $[0,1]$, the singular isotropy groups both equal to T, and the regular isotropy group equal to H; one uses [28, Proposition 1.5]. Let $\mathfrak{h}$ be the Lie algebra of H and pick $v \in \mathfrak{t}$ such that $\mathfrak{t} = \mathfrak{h} \oplus \mathbb{R}v$. Consider the linear function $\alpha : \mathfrak{t} \to \mathbb{R}$ along with the action of $\mathbb{Z}_2 = \{1, -1\}$ on $\mathfrak{t}$ given by

$$\alpha(w + rv) = r, \quad (-1).(w + rv) = w - rv,$$

for all $w \in \mathfrak{h}$ and $r \in \mathbb{R}$. We denote the induced $\mathbb{Z}_2$-action on $S(\mathfrak{t}^*)$ by $(-1).f =: \tilde{f}$, for all $f \in S(\mathfrak{t}^*)$. Corollary 4.2 induces the $H^*(BG) = S(\mathfrak{t}^*)^{W(G)}$-algebra isomorphism

$$H^*_G(M) \simeq \{(f, g) \in S(\mathfrak{t}^*) \oplus S(\mathfrak{t}^*) \, : \, \alpha \text{ divides } f - g\},$$

and the right hand side is, as an $S(\mathfrak{t}^*)$-algebra, isomorphic to $H^*_T(S^2)$, where the T-action on S^2 is the one described above. Note that T/H can be embedded as a maximal torus in $SO(3)$, in such a way that the latter group acts canonically on S^2 and induces the identification $S^2 = (H \times SO(3))/T$. We apply [25, Theorem 2.6] for this homogeneous space and deduce that the map $S(\mathfrak{t}^*) \otimes_{S(\mathfrak{t}^*)^{\mathbb{Z}_2}} S(\mathfrak{t}^*) \to H^*_G(M)$ given by $f_1 \otimes f_2 \mapsto (f_1 f_2, f_1 \tilde{f}_2)$ is an isomorphism of $S(\mathfrak{t}^*)^{W(G)}$-algebras, where the structure of $S(\mathfrak{t}^*)^{W(G)}$-algebra on $S(\mathfrak{t}^*) \otimes_{S(\mathfrak{t}^*)^{\mathbb{Z}_2}} S(\mathfrak{t}^*)$ is given by inclusion into the first factor. By Corollary 4.2 (b), we have the ring isomorphism

$$H^*(M) \simeq \mathbb{R} \otimes_{S(\mathfrak{t}^*)^{W(G)}} (S(\mathfrak{t}^*) \otimes_{S(\mathfrak{t}^*)^{\mathbb{Z}_2}} S(\mathfrak{t}^*)) = \left(S(\mathfrak{t}^*)/\langle S(\mathfrak{t}^*)^{W(G)}_{>0}\rangle\right) \otimes_{S(\mathfrak{t}^*)^{\mathbb{Z}_2}} S(\mathfrak{t}^*).$$

To obtain descriptions in terms of generators and relations we need an extra variable u with $\deg u = 2$ and also a set of Chevalley generators [11] of $S(\mathfrak{t}^*)^W$, call them $f_1, \ldots, f_k$, where $k := \operatorname{rank} G$. We have:

$$H^*_G(M) = S(\mathfrak{t}^*) \otimes \mathbb{R}[u]/\langle \alpha^2 - u^2 \rangle, \quad H^*(M) = S(\mathfrak{t}^*) \otimes \mathbb{R}[u]/\langle \alpha^2 - u^2, f_1, \ldots, f_k \rangle.$$

Let us now consider two concrete situations. For both of them we have $G = U(3)$ and T is the space of all diagonal matrices in $U(3)$; the role of H is played by the subgroup $\{\mathrm{Diag}(1, z_2, z_3) \, : \, |z_2| = |z_3| = 1\}$ in the first situation, respectively $\{\mathrm{Diag}(z_1, z_2, z_3) \, : \, |z_1| = |z_2| = |z_3| = 1, z_1 z_2 z_3 = 1\}$ in the second. If we denote the corresponding manifolds by M_1 and M_2, then

$$H^*(M_1) \simeq \mathbb{R}[x_1, x_2, x_3, u]/\langle x_1^2 - u^2, x_1 + x_2 + x_3, x_1x_2 + x_1x_3 + x_2x_3, x_1x_2x_3 \rangle$$

and

$$H^*(M_2) \simeq \mathbb{R}[x_1, x_2, x_3, u]/\langle u^2, x_1 + x_2 + x_3, x_1x_2 + x_1x_3 + x_2x_3, x_1x_2x_3 \rangle.$$

We observe that, even though $H^*(M_1)$ and $H^*(M_2)$ are isomorphic as groups, by Proposition 5.7, they are not isomorphic as rings. Indeed, we write

$$H^*(M_1) \simeq \mathbb{R}[x_1, x_2, u]/\langle x_1^2 - u^2, x_1^2 + x_2^2 + x_1x_2, x_1x_2(x_1 + x_2)\rangle$$

and

$$H^*(M_2) \simeq \mathbb{R}[x_1, x_2, u]/\langle u^2, x_1^2 + x_2^2 + x_1x_2, x_1x_2(x_1 + x_2)\rangle$$

and note that $H^*(M_1)$ does not contain an element of degree 2 and order 2.

5.2. **The case when M is odd-dimensional.** Assume we are given a cohomogeneity one action of a compact connected Lie group G on a compact connected odd-dimensional manifold M such that $M/G = [0, 1]$, with at least one isotropy group of maximal rank, such that the action is equivariantly formal by Corollary 1.3. This time the principal orbit G/H is even-dimensional, hence the ranks of G and H are congruent modulo 2. By Proposition 3.1, the rank of H differs from the singular isotropy ranks by only at most one; consequently, $\operatorname{rank} H = \operatorname{rank} G$, i.e. all isotropy groups of the G-action have maximal rank. (Note also that conversely, $\operatorname{rank} H = \operatorname{rank} G$ implies that M is odd-dimensional.)

The applications we will present here concern the Weyl group associated to the cohomogeneity one action of G. To define it, we first pick a G-invariant metric on M. The corresponding Weyl group W is defined as the G-stabilizer of a G-transversal geodesic γ modulo H. We have that W is a dihedral group generated by the symmetries of γ at $\gamma(0)$ and $\gamma(1)$. It is finite if and only if γ is closed. (For more on this notion, see [23, Section 1], [48, Section 1], and the references therein.) In the case addressed in this section we have the following relation between the dimension of the cohomology of M and the order of the occuring Weyl groups:

Corollary 5.13. *If $M/G = [0, 1]$ and $\operatorname{rank} H = \operatorname{rank} G$, then W is finite and*

$$\dim H^*(M) = 2 \cdot \frac{\chi(G/H)}{|W|}.$$

Proof. Let T be a maximal torus in H, which then is automatically a maximal torus in K^+, K^-, and G. We wish to understand the submanifold M^T consisting of the T-fixed points. To this end, note that for each $p \in M^T$, the T-action on the orbit Gp has isolated fixed points, hence the T-action on the tangent space $T_p(Gp)$ has no fixed vectors; if p is regular, then $\nu_p(Gp)$ is one-dimensional, hence T must act trivially on it. Therefore, M^T is a finite union of closed one-dimensional totally geodesic submanifolds of M, more precisely: a finite union of closed normal geodesics. The geodesic γ is among them and therefore closed; hence, W is finite.

The order $|W|$ of the Weyl group has the following geometric interpretation, see again [23, Section 1] or [48, Section 1]: the image of γ intersects the regular part of M (i.e., $M \setminus (G/K^- \cup G/K^+)$) in a finite number of geodesic segments; this number equals $|W|$. Note that each such geodesic segment intersects the regular orbit G/H in exactly one point, which is obviously in $(G/H)^T$. Consequently, $W(G)$ acts transitively on the components of M^T, hence each of the components of M^T meets G/H equally often. This yields a bijective correspondence between $(G/H)^T$ and the components of $M^T \setminus (G/K^- \cup G/K^+)$. The cardinality of $(G/H)^T$ is equal to $\chi(G/H)$ and we therefore have

$$\chi(G/H) = |W| \cdot (\text{number of components of } M^T). \tag{8}$$

As each component of M^T is a circle and therefore contributes with 2 to $\dim H^*(M^T)$, we obtain from Proposition 2.1 (e) that

$$\begin{aligned}\dim H^*(M) &= \dim H^*(M^T)\\ &= 2\cdot(\text{number of components of } M^T)\\ &= 2\cdot\frac{\chi(G/H)}{|W|}.\end{aligned}$$

□

Remark 5.14. If $G\times M\to M$ is a cohomogeneity one action, then different choices of G-invariant metrics on M generally induce different Weyl groups (see for instance [23, Section 1]). However, if the action satisfies the extra hypothesis in Corollary 5.13, then the Weyl group depends only on the rational cohomology type of M and the principal orbit type of the action. In particular, W is independent on the choice of the metric, but this is merely due to the fact that even the normal geodesics do not depend on the chosen G-invariant metric (being just the components of M^T, as the proof of Corollary 5.13 above shows).

Remark 5.15. In the special situation when M has the rational cohomology of a product of pairwise different spheres, the result stated in Corollary 5.13 has been proved by Püttmann, see [41, Corollary 2].

Let us now consider the case when M is a rational homology sphere. Examples of cohomogeneity one actions on such spaces can be found in [23] (see particularly Table E, for linear actions, and Table A for actions on the Berger space $B^7 = SO(5)/SO(3)$ and the seven-dimensional spaces P_k); the Brieskorn manifold $W^{2n-1}(d)$ with d odd and the $SO(2)\times SO(n)$ action defined in [32] is also an example; finally, several of the 7-dimensional $\mathbb{Z}_2$-homology spheres that appear in the classification of cohomogeneity one actions on $\mathbb{Z}_2$-homology spheres by Asoh [4, 5] are also rational homology spheres. As usual in this section, we assume that $\operatorname{rank} H = \operatorname{rank} G$. Under these hypotheses, Corollary 5.13 implies that $|W| = \chi(G/H)$. In fact, the latter equation holds under the (seemingly) weaker assumption that the codimensions of both singular orbits are odd, as it has been observed in [41, Section 1]. Combining this result with Corollary 5.13 we have:

Corollary 5.16. *Let G act on M with cohomogeneity one, such that $M/G = [0,1]$ and let H be a principal isotropy group. The following statements are equivalent:*

(i) M is a rational homology sphere and $\operatorname{rank} H = \operatorname{rank} G$.

(ii) M is a rational homology sphere and the codimensions of both singular orbits are odd.

(iii) $|W| = \chi(G/H)$.

In any of these cases, the dimension of M is odd and the G-action is equivariantly formal.

Let us now observe that for a general cohomogeneity one action with $M/G = [0,1]$, the Weyl group is contained in $N(H)/H$. The previous results allow us to make some considerations concerning whether the extra assumption $\operatorname{rank} G = \operatorname{rank} H$ implies that $W = N(H)/H$. The following example shows that this is not always the case.

Example 5.17. Let us consider the cohomogeneity one action determined by $G = SU(3)$, $K^- = K^+ = S(U(2)\times U(1))$, and $H = T$, a maximal torus in $K^\pm$.

Hoelscher [29] denoted this manifold by N^7_G and showed that it has the same integral homology as $\mathbb{C}P^2 \times S^3$. This implies that $\dim H^*(N^7_G) = 6$. On the other hand, $\chi(G/H) = \chi(SU(3)/T) = 6$, hence by Corollary 5.13, $|W| = 2$. Since $N(H)/H = N(T)/T = W(SU(3))$ has six elements, we have $W \neq N(H)/H$.

However, it has been observed in [23, Section 5] that for linear cohomogeneity one actions on spheres, the condition $\operatorname{rank} G = \operatorname{rank} H$ does imply that $W = N(H)/H$. The following more general result is an observation made by Püttmann, see [41, Footnote p. 226]. We found it appropriate to include a proof of it, which is based on arguments already presented here.

Proposition 5.18. (Püttmann) *Let G act on M with cohomogeneity one, such that $M/G = [0,1]$ and let H be a principal isotropy group. If $\chi(G/H) = |W|$ then $W = N(H)/H$.*

Proof. We only need to show that $N(H)/H$ is contained in W. The hypothesis $\chi(G/H) > 0$ implies that $\operatorname{rank} G = \operatorname{rank} H$, hence G/H is even-dimensional and M is odd-dimensional. Let T be a maximal torus in H. By Equation (8), M^T consists of a single closed geodesic γ. The group H fixes the geodesic γ pointwise. Consequently, any element of $N(H)$ leaves M^T invariant, hence its coset modulo H is an element of W. □

The following example shows that the condition $\chi(G/H) = |W|$ is stronger than $W = N(H)/H$.

Example 5.19. In general, $\operatorname{rank} G = \operatorname{rank} H$ and $W = N(H)/H$ do not imply that $\chi(G/H) = |W|$. To see this we consider the cohomogeneity one manifold determined by $G = Sp(2)$, $K^- = K^+ = Sp(1) \times Sp(1)$, $H = Sp(1) \times SO(2)$. Its integer homology has been determined in [29, Section 2.9]: the manifold, denoted there by N^7_I, has the same homology as the product $S^3 \times S^4$. Consequently, $\dim H^*(N^7_I) = 4$. An easy calculation shows that $\chi(G/H) = |W(Sp(2))|/(|W(Sp(1))| \cdot |W(SO(2))|) = 4$. By Corollary 5.13, $|W| = 2$, hence $\chi(G/H) \neq |W|$. On the other hand, the group $N(H)$ can be determined explicitly as follows. We regard $Sp(2)$ as the set of all 2×2 quaternionic matrices A with the property that $A \cdot A^* = I_2$ and $H = Sp(1) \times SO(2)$ as the subset consisting of all diagonal matrices $\operatorname{Diag}(q, z)$ with $q \in \mathbb{H}$, $z \in \mathbb{C}$, $|q| = |z| = 1$. Then $N(H)$ is the set of all diagonal matrices $\operatorname{Diag}(q, r)$, where $q, r \in \mathbb{H}$, $|q| = |r| = 1$, $r \in N_{Sp(1)}(SO(2))$. This implies that $N(H)/H \simeq N_{Sp(1)}(SO(2))/SO(2) = W(Sp(1)) = \mathbb{Z}_2$, hence $N(H)/H = W$.

5.3. **Torsion in the equivariant cohomology of cohomogeneity one actions.** It is known [17] that if $G \times M \to M$ is an arbitrary group action, then the $H^*(BG)$-module $H^*_G(M)$ may be torsion-free without being free. (But note that this cannot happen if G is either the circle S^1 or the two-dimensional torus T^2, see [1].) The goal of this subsection is to show that this phenomenon also cannot occur if the cohomogeneity of the action is equal to one. This is a consequence of the following lemma.

Lemma 5.20. *Assume that the action of the compact connected Lie group G on the compact connected manifold M is Cohen-Macaulay. Then the $H^*(BG)$-module $H^*_G(M)$ is free if and only if it is torsion-free.*

Proof. If $H^*_G(M)$ is a torsion-free $H^*(BG)$-module, then, by [18, Theorem 3.9 (2)], the space $M_{\max}$ is non-empty. Since the G-action is Cohen-Macaulay, it must be equivariantly formal by Proposition 2.9. □

From Theorem 1.1 we deduce:

Corollary 5.21. *Assume that the action $G \times M \to M$ has cohomogeneity one. Then the $H^*(BG)$-module $H^*_G(M)$ is free if and only if it is torsion-free.*

References

[1] C. Allday, *A family of unusual torus group actions*, Group Actions on Manifolds, Contemp. Math., Vol. 36, Amer. Math. Soc., Providence, RI, 1985, pp. 107-111

[2] C. Allday and V. Puppe, *Cohomological Methods in Transformation Groups*, Cambridge Studies in Advanced Mathematics, Vol. 32, Cambridge University Press, Cambridge 1993

[3] D. V. Alekseevsky and F. Podestà, *Compact cohomogeneity one Riemannian manifolds of positive Euler characteristic and quaternionic Kähler manifolds*, Geometry, topology and physics (Campinas, 1996), 1-33, de Gruyter, Berlin, 1997

[4] T. Asoh, *Compact transformation groups on $\mathbb{Z}_2$-cohomology spheres with orbit of codimension 1*, Hiroshima Math. J. **11** (1981), 571-616

[5] T. Asoh, *Supplement to: "Compact transformation groups on $\mathbb{Z}_2$-cohomology spheres with orbit of codimension 1"*, Hiroshima Math. J. **11** (1981), 571-616

[6] M. Atiyah, *Elliptic Operators and Compact Groups*, Lecture Notes in Mathematics, Vol. 401, Springer-Verlag, Berlin, 1974

[7] A. Borel, *Sur la cohomologie des espaces fibrés principaux et des espaces homogènes des groupes de Lie compacts*, Ann. of Math. (2) **57** (1953), 115-207

[8] G. E. Bredon, *Introduction to Compact Transformation Groups*, Academic Press, New-York, 1972

[9] G. E. Bredon, *The free part of a torus action and related numerical equalities*, Duke Math. J. **41** (1974), 843-854

[10] W. Bruns and J. Herzog, *Cohen-Macaulay Rings*, Cambridge Studies in Advanced Mathematics, Vol. 39, Cambridge University Press, Cambridge 1993

[11] C. Chevalley, *Invariants of finite groups generated by reflections*, Amer. J. Math. **77**, 778-782

[12] T. tom Dieck, *Transformation Groups*, de Gruyter Studies in Mathematics, 8, Walter de Gruyter & Co., Berlin, 1987

[13] J. Duflot, *Depth and equivariant cohomology*, Comment. Math. Helv. **56** (1981), 627-637

[14] Y. Felix, S. Halperin, and J.-C. Thomas, *Rational Homotopy Theory*, Graduate Texts in Mathematics, Vol. 205, Springer-Verlag, New-York, 2001

[15] P. Frank, *Cohomogeneity one manifolds with positive euler characteristic*, Doctoral Dissertation, Universität Münster, 2011

[16] M. Franz and V. Puppe, *Exact sequences for equivariantly formal spaces*, C. R. Math. Acad. Sci. Soc. R. Can. **33** (2011), 1-10

[17] M. Franz and V. Puppe, Freeness of equivariant cohomology and mutants of compactified representations, Toric Topology (Proc. Intl. Conf. on Toric Topology, Osaka 2006), Contemp. Math., Vol. 460, Amer. Math. Soc., Providence, RI, 2008, pp. 87-98

[18] O. Goertsches and S. Rollenske, *Torsion in equivariant cohomology and Cohen-Macaulay actions*, Transform. Groups **16** (2011), no. 4, 1063-1080

[19] O. Goertsches and D. Töben, *Torus actions whose equivariant cohomology is Cohen-Macaulay*, J. of Topology **3** (2010), 1-28

[20] M. Goresky, R. Kottwitz, and R. MacPherson, *Equivariant cohomology, Koszul duality, and the localization theorem*, Invent. Math. **131** (1998), no. 1, 25–83

[21] K. Grove, *Geometry of, and via, symmetry*, Conformal, Riemannian and Lagrangian geometry (Knoxville, TN, 2000), 31–53, Univ. Lecture Ser., Vol. 27, Amer. Math. Soc., Providence, RI, 2002

[22] K. Grove and S. Halperin, *Dupin hypersurfaces, group actions and the double mapping cylinder*, J. Differential Geom. **26** (1987), 429-459

[23] K. Grove, B. Wilking, and W. Ziller, *Positively curved cohomogeneity one manifolds and 3-Sasakian geometry*, J. Diff. Geom. **78** (2008), 33–111

[24] V. W. Guillemin, V. L. Ginzburg, and Y. Karshon, *Moment maps, cobordisms, and Hamiltonian group actions*, Mathematical Surveys and Monographs, Vol. 96, Amer. Math. Soc., Providence, RI, 2002

[25] V. W. Guillemin, T. Holm, and C. Zara, *A GKM description of the equivariant cohomology ring of a homogeneous space*, J. Alg. Combin. **23** (2006), 21-41

[26] S. Halperin, *Finiteness in the minimal models of Sullivan*, Trans. Amer. Math. Soc. **230** (1977), 173-199

[27] M. Hochster and J. A. Eagon, *Cohen-Macaulay rings, invariant theory, and the generic perfection of determinantal loci*, Amer. J. Math. **93** (1971), 1020-1058

[28] C. A. Hoelscher, *Classification of cohomogeneity one manifolds in low dimensions*, Pacific J. Math. **246** (2010), no. 1, 129-185

[29] C. A. Hoelscher, *On the homology of low dimensional cohomogeneity one manifolds*, preprint, `arXiv:0907.2449`

[30] H. Hopf and H. Samelson, *Ein Satz über die Wirkungsräume geschlossener Liescher Gruppen*, Comment. Math. Helv. **13** (1941), 240-251

[31] W.-Y. Hsiang, *Cohomology Theory of Topological Transformation Groups*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Vol. 85, Springer-Verlag, New-York 1975

[32] W.-C. Hsiang and W.-Y. Hsiang, *On compact subgroups of the diffeomorphism groups of Kervaire spheres*, Ann. of Math. **85** (1967), 359-369

[33] R. Kane, *Reflection Groups and Invariant Theory*, CMS Books in Mathematics, Vol. 5, Springer-Verlag, New York, 2001

[34] H. Matsumura, *Commutative Algebra*, Mathematics Lecture Notes Series, The Benjamin/Cummings Publishing Company, Inc., Advanced Book Program, Reading, Massachusets, 1980

[35] J. W. Milnor and J. D. Stasheff, *Characteristic Classes*, Annals of Mathematical Studies, Vol. 76, Princeton University Press, Princeton, 1964

[36] M. Mimura and H. Toda, *Topology of Lie Groups*, Translations of Mathematical Monographs, Vol. 91, Amer. Math. Soc., Provodence, Rhode Island, 1991

[37] J. Munkres, *Topology, a First Course*, 2nd edition, Prentice-Hall, 1999

[38] R. Palais and C.-L. Terng, *A general theory of canonical forms*, Trans. Amer. Math. Soc. **300** (1987), no. 2, 771-789

[39] P. Orlik and F. Raymond, *Actions of the torus on a 4-manifold - II*, Topology **13** (1974), 89-112

[40] T. Püttmann, *Homogeneity rank and atoms of actions*, Ann. Global Anal. Geom. (4) **22** (2002), 375-399.

[41] T. Püttmann, *Cohomogeneity one manifolds and selfmaps of nontrivial degree*, Transform. Groups **14** (2009), 225-247

[42] D. Quillen, *The spectrum of an equivariant cohomology ring: I*, Ann. Math. **94** (1971), 549-572

[43] H. Samelson, *Beiträge zur Topologie der Gruppen-Mannigfaltigkeiten*, Ann. of Math. **42** (1941), 1091-1137

[44] J.-P. Serre, *Algèbre locale. Multiplicités, 3 éd.*, Lecture Notes in Mathematics, Vol. 11, Springer-Verlag, Berlin 1975

[45] J.-P. Serre, *Groupes d'homotopie et classes de groupes abéliens*, Ann. of Math. **58** (1953), 258-294

[46] B. B. Venkov, *Cohomology algebras for some classifying spaces* (Russian), Dokl. Akad. Nauk. SSSR **127** (1959), 943-944

[47] L. Verdiani, *Cohomogeneity one manifolds of even dimension with strictly positive sectional curvature*, J. Differential Geom. **68** (2004), no. 1, 31-72

[48] W. Ziller, *On the geometry of cohomogeneity one manifolds with positive curvature*, Riemannian topology and geometric structures on manifolds, 23–262, Progr. Math., Vol. 271, Birkhäuser Boston, Boston, MA, 2009

(O. Goertsches) Fachbereich Mathematik, Universität Hamburg, Germany

(A.-L. Mare) Department of Mathematics and Statistics, University of Regina, Canada
E-mail address: `oliver.goertsches@math.uni-hamburg.de`
E-mail address: `mareal@math.uregina.ca`