# Math 101

## Introductory Finite Mathematics 1

## 8th WNCP Edition

Department of Mathematics and Statistics
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# Course Content

# Introduction to the 8th WNCP Edition

The goal of this *Introduction to Finite Mathematics I* text is, as it has been with previous editions, to provide a textbook for a course in mathematics concepts and skills at a level suitable for mathematics teachers in elementary (Grade K-8) schools in Canada. Its heavy emphasis on problem-solving and numeration make it also highly suitable as a text for critical thinking courses required by students in most Liberal Arts programs.

The necessity to revise the previous 7th Edition of *Introduction to Finite Mathematics I* directly resulted from the 2008 decision of Western and Northern Canadian ministries of education to move to a K-12 mathematics curriculum based on new educational philosophies originating with the 2000 National Council of Teachers of Mathematics (NCTM) Standards in the United States. While the previous 1989 NCTM Standards primarily emphasized problem solving as the main goal, the new curriculum structure has a much broader base of learning objectives it seeks to emphasize:

It has four general ***aims and goals***:
*Number sense, Logical thinking, Spatial sense, and Mathematics as a human endeavour;*

every K-9 concept falls into one of four ***curriculum strands***:
*Numbers, Patterns and relations, Shape and space, and Statistics and probability;*

teaching and learning outcomes are arranged around ***seven mathematical processes*** to be emphasized in the classroom:
*Communication, Connections, Mental mathematics and estimation, Problem solving, Reasoning, Visualization, and Technology;* and

teachers are expected to encourage students to represent concepts using several types of ***models***:
*Concrete, Physical, Pictorial, Oral, and Symbolic.*

Along with this broad base of emphasis, the style of teaching mathematics in the new curriculum has moved further toward discovery-learning approaches, and away from memorization and drill. The teaching of arithmetic in the early grades has especially been affected, as traditional (right-to-left) methods for addition, subtraction, and multiplication are discouraged in favour of left-to-right methods more suitable for efficient estimation and mental calculation.

By 2012, confusion surrounding the new mathematics curriculum led to public debates between parents and the general public perceiving a decline in the arithmetic abilities of school-age children on the one hand, and defenders of the new curriculum in the education system on the other. An official provincial government consultation in early 2012 collected feedback from public concerning the effectiveness of the new curriculum in Saskatchewan. At the University of Regina, the consensus opinion of Mathematics and Mathematics Education Faculty coming out of this debate was that an improvement in the preparation of early- and middle-years mathematics teachers is needed. The new curriculum and educational approaches mean that mathematics teachers need to be familiar with several different methods for doing arithmetic calculations, be able to recognize the same concepts presented in many different models, and have more familiarity with the concept of probability than before.

As a result, the revisions for this edition are quite extensive, especially regarding the approach to arithmetic. An effort has been made to integrate the new language of the 2008 WNCP Curriculum into the text. We also have incorporated prevalent models for arithmetic and probability that are necessary for teaching the WNCP curriculum. However, our goal has remained on providing a sensible foundation for the mathematical content and skills at the level necessary to teach at the Elementary and Middle School levels, rather than a complete shift to the latest philosophies in mathematics education. Rather than try to meet all of the current curriculum's strands, this text's content focuses on the first and second strands, with content from the third and fourth strands touched upon only briefly.

Unit 1 remains focused on the development of problem-solving strategies and skills, with an added section on logical reasoning skills adapted from the previous logic unit. The solutions to the problems in this section require a variety of techniques and approaches that may not be evident on a first reading. Throughout the course, we will practice strategies such as examining what the question is asking, writing down our ideas, organizing the given information, and analyzing our solutions. The basic problem-solving-for-learning 4-step approach **Think → Plan → Carry it out → Look Back** is consistently reinforced throughout the text.

Unit II on arithmetic has been completely rewritten to introduce several new calculation methods (and tricks!) that future teachers will see and use in their classrooms. Methods most suitable to estimation or efficient mental calculation are emphasized. Calculation tricks for special situations are demonstrated, along with applications like systematic counting. The final sections provide a brief introduction to arithmetic in bases other than 10, and the Roman and Mayan numerals. As with previous editions, the first two units can be done in either order.

Unit III covers modular and calendar arithmetic, a topic whose emphasis was increased in the WNCP curriculum.

In Unit III on Number Theory, we develop the topics of primes, divisibility rules, prime factorization, greatest common divisor, and least common multiple, working our way up to an understanding of arithmetic with rational numbers. Unit IV concludes with decimal and irrational numbers, and introduces methods for dealing with calculations involving roots.

Unit IV on Ratios, Proportion, Percent, and Probability focuses on situations where calculations with rational and decimal numbers are encountered. The problem-solving skills we learned in the earlier chapters will be reinforced. The main revision here is the inclusion of a new chapter on probability.

A. H., November 1, 2012

For a snapshot of what is in store for readers of this book, here are a few problems to get you warmed up:

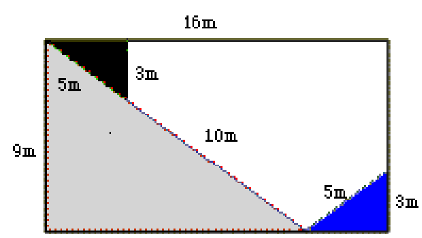
## Exercises: Introduction

1. Suppose that Farmer Fred has some pigs and chickens. He counts 70 heads and 200 legs all together. How many chickens and how many pigs are there?

2. Chicken McNuggets® come in packs of 6, 9, and 20. Is it possible to order exactly 217 Chicken McNuggets®?

3. Chris is training Hoppity, her pet rabbit, to climb stairs. Hoppity can hop up one or two stairs at a time. If a flight of stairs has ten steps, in how many different ways can Hoppity hop up the flight of stairs?

4. A $16 \times 9$ meter rectangle is cut as shown. The pieces can be rearranged to form a square. What is the perimeter of the square?



# Unit 1
# Strategies for Problem Solving

Mathematics is like a big tool box. When you open it up, many of the tools look strange and unfamiliar. You don't know what their purpose is or how to use them. A few, like the hammer, have a purpose that is easier to understand, that you can start to use right away. The purpose of a tool is to fix things. The purpose of mathematics is to solve problems. Problems that require mathematical tools. And so we begin the book with problem solving, to become familiar with the basic mathematical tools and problem solving strategies that are needed most often.

## 1.1 Getting Ready to Solve Problems

In order to get started (and build some confidence), we will try the following exercises involving sequences. The exercises start off with sequences that are called *arithmetic progressions*. If you remember what those are, that's a hint!

The problems are arranged so the methods or ideas used to solve the first problems can be generalized in some way to solve the next ones. It is important to start with easy problems, learn from them, and work up to harder problems. The most important part of problem solving is selecting and using a method. If we understand the solution method, we can adapt it to solve any problem to which the method applies. ***One method will provide solutions to many problems!***

### Exercises: Section 1.1

For exercises #1 to #4, find the next number most likely to occur in each sequence.

1. 5, 10, 15, 20, 25, __
2. 1, 4, 7, 10, 13, __
3. 23, 30, 37, 44, __
4. 12, 47, 82, 117, __
5. What is the common feature of all the sequences from #1 to #4?

6. An **arithmetic progression** is a sequence where the difference between each term and the next is always the same. The number that is added to each term is called the **common difference**. For each sequence in exercises #1 to #4, find the common difference.

7. Find the next two numbers in the sequence.

1, 5, 4, 10, 7, 15, 10, __, __

8. Find the next term in the following sequence. (Hint: What sequence is formed by the differences between each term and the next?)

1, 2, 6, 13, 23, 36, __

For exercises #10 to #14, find the next term in each of these sequences.

9. 0, 2, 8, 21, 44, 80, __

10. 1, 2, 10, 31, 75, 156, __

11. 2, 8, 21, 41, 68, 102, __

12. 3, 5, 13, 34, 75, 143, 245, __

13. 1, 4, 9, 16, 25, 36, __

For exercises #14 to #18, find the next term in each of these sequences.

14. 6, 18, 54, 162, __

15. 2, 4, 8, 16, 32, __

16. 21, 147, 1 029, 7 203, __

17. 56, 28, 14, 7, __

18. 0.09, 0.000 9, 0.000 009, 0.000 000 09, __

19. What is the common feature of all the sequences from exercises #14 to #18?

20. A **geometric progression** is a sequence where each term is multiplied by the same number in order to obtain the next term. The number that each term is multiplied by is called the **common ratio**. For each of the geometric progressions in #15 to #19, find the common ratio.

21. In an arithmetic progression, the first term is 1 and the fifth term is 21. Find the common difference.

22. In a geometric progression, the second term is 12 and the fifth term is $^{81}/_{2}$. Find the first term and the common ratio.

<u>Other Familiar Sequences</u>

For each of the familiar sequences in exercises #24 to #28, describe the rule used to obtain the next term.

23. 2, 3, 5, 7, 11, 13, __, ... (Can you guess the name of this sequence?)

24. 1, 4, 9, 16, 25, __, ... (Can you find two rules for this one?)

25. 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, __, ... (Fibonacci Sequence)

26. 1, 3, 4, 7, 11, 18, 29, 47, 76, __, ... (Lucas Sequence)

27. 1, 2, 6, 24, 120, 720, __, ... (Factorial Sequence)

28. In the sequence of numbers 2, 5, 3, …, each term from the third term on is equal to the term preceding it minus the term preceding that one. What is the sum of the first 100 terms of the sequence?

29. Find the next symbol in the sequence.

# 1.2 Polya's Four Steps to Problem Solving

George Pólya (1888 - 1985), the Hungarian author of the modern classic How to Solve It, listed four essential steps to problem solving:

- Step 1: Understand the Problem
- Step 2: Devise a Plan
- Step 3: Carry Out the Plan
- Step 4: Look Back

In order to solve even the simplest problems, you must first **understand the problem**, then think up a **plan** as to how you are going to go about solving it, then actually solve the problem by **carrying out the plan**. Step 4 is to **look back**, which is essential for learning from the problems that you have already solved. If you have taken the time to think about how you solved a problem and how you might have solved the problem in a different or perhaps interesting way, then you can improve on the way you solve a similar problem the next time you encounter it. When you have developed the habit of looking back, you can really improve on your problem solving skills. It is also important to look back and check to see if the way you answered the problem was reasonable. Here are some tips to carrying out each of the four steps.

## Step 1: Understand the Problem.

- Identify the information given in the problem.

- Realize what a solution requires.

- Organize the information in your own words, your own diagrams, or your own pictures. Check to see that there is enough information given to solve the problem.

In order to completely understand a problem, you should be able to translate all of the given information and describe the requirements of a correct solution in a language that you understand.

## Step 2: Devise a Plan.

The "plan", or strategy, used to solve the problem depends on the particular problem itself. Elementary problem-solving strategies can be described as follows:

- **Solution Method 1:** Use something familiar.

  If the problem can be solved using the same method as a problem you have solved before, then you can use the same method again. It may be that the method has to be used in a slightly different way than before. When this happens, we say that the method was <u>generalized</u> to solve the new problem. This strategy is called <u>using an analogy</u>.

- **Solution Method 2:** Guess and check.

  Guess at the answer. Check that your guess satisfies all of the requirements of an answer. If it does, you are finished. If it does not, then try to make a better guess using the experience gained from your previous incorrect guess. Eventually, by making better and better guesses one can come upon a correct solution. Another name for this method is **trial and error**. Guessing and checking can work well if you know the answer is limited, such as belonging to a particular range of numbers, but it will not be effective all the time.

- **Solution Method 3:** Look for a pattern.

  Patterns can come in many forms. Sometimes you can **solve a series of simpler problems**, which makes it possible to tell what the answer is to a more complicated problem. This type of reasoning is a form of **inductive reasoning**. Sometimes you can visualize a pattern by **drawing a diagram or picture** or **constructing a model** that describes the problem. Or perhaps the problem can be modeled with **an algebraic equation** whose solution gives the answer, as are the problems one often encounters high school mathematics.

- **Solution Method 4:** Invent something new, be creative.

  Everybody gets stuck. Some problems require you to change the way you usually think in order to solve them. They require a new idea; an "Aha!" With experience you can learn to open your mind to new possibilities when you are stuck. Inventiveness and imagination do improve with practice.

- **Solution Method 5:** Deductive reasoning.

  Logical reasoning will always be needed to some extent in solving a problem. Deductive reasoning is the basic technique used in most mathematical proofs, but its basic premises reduce to just a few simple ideas that form the basis of what is often called **using common sense**.

## Step 3: Carry Out the Plan.

Once you decide which strategy might work for a particular problem, you have to carry it out to reach a potential answer or solution. If the method was chosen wisely and no mistakes were made in carrying it out, a correct solution should be reached. However, the answer does need to be checked to see if it satisfies all of the criteria to be correct. Once this is done, you can say that you have solved the problem.

So, the only things to keep in mind are:

- Work toward a solution.

  If you are confident that you will get a reasonable solution, then keep working. If you do not think that your plan is working, then you should change your strategy.

- Check that the solution you get makes sense.

  You could have made mistakes in working out a solution. If you find these mistakes, correct them.

## Step 4: Look Back.

At this point, you may question the way the problem was solved. You can ask yourself the following questions:

- Can this problem be solved using a different (faster, easier, more interesting, etc.) method?

- Is there a shortcut?

You can (and should) identify other problems that you can solve using the same method.

In the next section we will work through problems that will be solved using elementary strategies, and work through all four steps for each problem. When doing this, pay special attention to the benefits of looking back.

# 1.3 Problem Solving Examples

The following eight problems are in no particular order. They are all classical problems and involve elementary solution methods described in the previous section. None of them require algebraic methods but algebra can be used in some cases. The elementary solution may entail trial calculations, drawing diagrams, using manipulatives (physically or mentally), or any other techniques you find helpful.

Try each problem before looking at our solution.

## 1.3.1 Heads and Feet Problems

**Problem:** At a horse show there are 50 heads and 130 feet. How many horses and how many riders are there?

**Step 1: Understanding the Problem:**

We are given that there are 50 heads and 130 feet. We want to know how many riders (people) and how many horses there are at the horse show. Of course, we will need to use the common knowledge that people have two feet (and one head) and horses have four feet (and one head).

**Step 2: Devise a Plan:**

There are a number of possible methods to solve this problem. Since the answer will be a pair of numbers that add up to 50 (the total number of heads involved), ***Guessing and Checking*** can work well here. More sophisticated students might be attempted to go for an ***Algebraic solution***, or use ***Deductive reasoning***, and less sophisticated ones might want to construct a ***Concrete model*** of the situation and bring out the playdough!

**Step 3: Carry out the Plan:**

***<u>Guess and Check Solution</u>***

Guess: 30 people, 100 horses
Check:

* 30 + 100 = 130 heads.
* Too Many Heads!

So: the number of people plus the number of horses should be 50, and that will always be the number of heads.

Guess: 30 people, 20 horses
Check:

* $30 \times 2 = 60$ people's feet, and
* $20 \times 4 = 80$ horses' feet, so 140 feet.
* Too many feet

So: guess more people and fewer horses

Guess: 40 people, 10 horses
Check:

* $40 \times 2 = 80$ people's feet, and
* $10 \times 4 = 40$ horses' feet, so 120 feet.
* Not enough feet

So: Guess fewer people than 40, but more than 30

Guess: 35 people, 15 horses
Check:

* $35 \times 2 = 70$ people's feet, and
* $15 \times 4 = 60$ horses' feet, so 130 feet.
* That's right!

Answer: There are 35 people and 15 horses at the horse show.

## ***Algebraic Solution***

Let's let $r$ stand for the number of riders and $h$ for the number of horses. Then translating the facts that there are 50 heads and 130 feet we must have that:

$$r + h = 50$$
$$2r + 4h = 130.$$

We need to solve these (simultaneous) equations for $r$ and $h$. Multiply the first equation by 2. Then we must solve:

$$2r + 2h = 100$$
$$2r + 4h = 130.$$

Now (subtracting the first of these equations from the second) we see that $2h = 30$, i.e. $h = 15$. Since $r + h = 50$, we must have $r = 35$.

## ***Deductive Reasoning Solution***

For simplicity, we will say a rider has a head and two front feet while a horse has a head, two front feet and two back feet. Therefore:

If there are 50 heads, then there are 100 front feet.
If there are 100 front feet, then there are 130 - 100 = 30 back feet.
If there are 30 back feet, then there are 30 - 15 = 15 horses.
If there are 15 horses, then there are 50 - 15 = 35 riders.

***Constructive Model Solution***

Imagine that we have some children's toys like 50 playdough heads and 130 toothpicks for feet. We are to make simple people and horses. A head with 2 feet is a person and a head with 4 feet is a horse.

We take the 50 heads and put 2 feet in each to make 50 people. We have 30 feet left over, which is 15 pairs of feet left over. So we put these 15 pairs of feet into 15 of our people to turn them into horses. Now we have 15 horses and 50 – 15 = 35 people.

**Step 4: Looking Back:**

In all of the solutions we came up with, the answer we needed was in the form of two numbers that satisfied two simple criteria:

- They added together to get 50
- 2 times one plus 4 times the other was 130.

It was fairly easy to check our guesses against these criteria, so the guess and check method worked quite well, and was easy to use. This is not always the case in more complicated problems or problems with larger numbers.

Readers may want to try problems #1 to #3 in the Exercises for Section 1.3 at this point.

# 1.3.2 Last Digit Problems

**Problem:** Find the last digit of $2^{97}$.

**Step 1: Understanding the Problem:**

$2^{97}$ is the number obtained by multiplying 97 twos together. It will be a huge number (about 30 digits), so certainly one should not expect to have to compute it. There must be a shortcut to find the last digit (the digit in the one's place) of this huge number.

**Step 2: Devise a Plan:**

Guess and Check will not work here because we have no way to check our answer. (You would have to know the last digit of $2^{97}$ already.) We will have to ***Look for a Pattern***.

Start with smaller problems where we know what the last digit is:

* The last digit of $2^1$ is 2
* The last digit of $2^2$ is 4
* The last digit of $2^3$ is 8
* The last digit of $2^4$ is 6 (since $8 \times 2 = 16$ has last digit 6)
* The last digit of $2^5$ is 2 (since $6 \times 2 = 12$ has last digit 2)
* The last digit of $2^6$ is 4 (since $2 \times 2 = 4$)
* The last digit of $2^7$ is 8 (since $4 \times 2 = 8$)
* The last digit of $2^8$ is 6 (since $8 \times 2 = 16$ has last digit 6)

$\vdots \quad \vdots$

There is a pattern forming! Can you describe it?

**Step 3: Carry out the Plan:**

We can see that when 2 is raised to a power that is a multiple of 4, 6 is always the last digit. We also see that the last digits of consecutive powers of 2 occur in the repeating pattern 2, 4, 8, and then 6. This pattern will be repeated because the last digit of the next power of 2 is determined by 4 times the last digit of the previous power of 2. Since $97 = (24 \times 4) + 1$, 97 is the next number after a multiple of 4, so the last digit of $2^{97}$ has to be a 2.

<u>Answer</u>: The last digit of $2^{97}$ is 2.

**Step 4: Look Back:**

Some nice features of this method are that it is easy to use and the pattern guarantees that we found the correct last digit of $2^{97}$ without having to compute the huge product. The hard part of using this method is recognizing the pattern correctly. We found the correct pattern by working out enough powers of 2 so that the last digit pattern repeated itself twice, and the last digits were resulting from the same calculation as before. We actually could have recognized the pattern after $2^5$ instead of going all the way to $2^8$, but you should do as many as it takes to see for sure what the correct pattern is. Describing the pattern in your own words is also challenging at a first try.

A good problem-solving method can be generalized to solve lots of other similar problems. This especially applies to methods based on a pattern. Observe that we can predict the last digit of $2^n$ for any positive exponent $n$:

For example:

The last digit of $2^{777}$ is 2 since 777 = (194× 4)+ 1.
The last digit of $2^{32387}$ is 8 since 32387 = (8096×4)+ 3.
The last digit of $2^{1\,000\,000}$ is 6 since 1 000 000 ≡ (250 000 × 4).

In fact, the same method can be used to determine the last digit of any number ending in 2 to any power! You should think about what the last digit of $12^{97}$ is. What about $52^{97}$? It can also be applied to powers ending in other digits. Just find the pattern of last digits for the base, and find how the given power fits in that pattern.

For example, to find the last digit of $3^{371}$, we first find the repeating pattern for the last digits of powers of 3: $3^1 = 3$, $3^2 = 9$, $3^3$ ends in 7, $3^4$ ends in 1, and $3^5$ ends in 3. So, the repeating pattern will be 3, 9, 7, and 1, with the 1's occurring at powers that are multiples of 4. Since 371 = (92 ×4) + 3, we can conclude that the last digit of $3^{371}$ is a 7.

Readers may want to try problems #4 to #13 in the Exercises for Section 1.3 at this point.

# 1.3.3 The River Crossing Problem

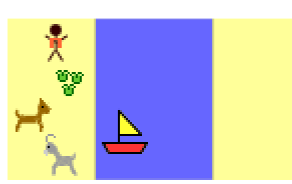
**Problem:** A man must take a wolf, a goat, and some cabbage heads across a river. He has only one rowboat with room for himself and one of the wolf, the goat, or the cabbage heads. If the man leaves the wolf and the goat alone on the same side of the river, the wolf will eat the goat. If the man leaves the goat alone with the cabbage, the goat will eat the cabbage. How can the man get everything across the river safely?

**Step 1: Understanding the Problem:**

The solution to this problem will be a description of steps where the man rows across the river with one of the three things, then comes back. In the correct solution, the wolf and the goat, and the goat and the cabbage are never left alone on either side of the river.

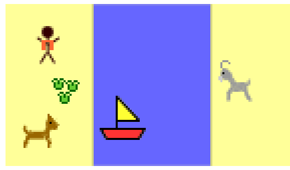
**Step 2: Devise a Plan:**

It is clear that the problem can be modeled with a picture, which will capture the situation better than describing everything in words. . So let's **draw a picture** and use it to look for a pattern.

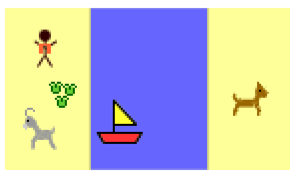


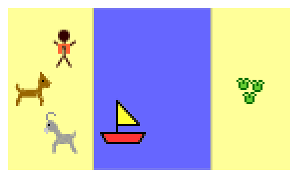
**Step 3: Carry out the Plan:**

It is quite clear that this problem has an easy first step: The man has to first cross the river with the goat, leaving the wolf and the cabbage behind. The man then leaves the goat alone on the other side, and then comes back. The new picture is:

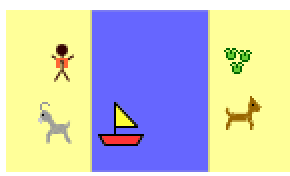


For the man's next crossing, he will take either the wolf or the cabbage across the river. He cannot come back alone because he would then be leaving either the wolf-goat pair or the goat-cabbage pair alone on the other side. He has to come back with the goat or else he would be right back where he started this step. Thus, he crosses with either the wolf or the cabbage and returns with the goat. This gives two possible pictures.





He should not cross with the goat in the next crossing (as he just came back with the goat) so he crosses with the cabbage (or the wolf) and returns alone. In either case, the new picture is:



Now the man crosses with the goat, and everything has been safely moved across the river.

Paraphrasing these steps, we have

<u>Answer</u>:

1st crossing: The man crosses with the goat and returns alone.
2nd crossing: The man crosses with the wolf (or the cabbage) and returns with the goat.
3rd crossing: The man crosses with the cabbage (or the wolf) and returns alone.
4th crossing: The man crosses with the goat, and is finished.

**Step 4: Look Back:**

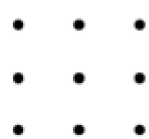
We start by finding a possible first move and trying it. We analyze what has happened and look for the next thing that can be done. We try this possibility and continue the pattern. If we are lucky, we will reach a solution to the original problem after a series of smaller steps.

We must remember to check the criteria as we go along, and to never undo something that we just did. Having a picture to manipulate was essential to keep track of what had already been done. When a problem involves moving objects, it is always a good idea to draw a picture to help you understand what is happening.

Readers may want to try problems #14 to #16 in the Exercises for Section 1.3 now.

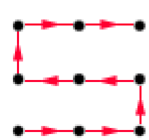
## 1.3.4 The Nine Dots Problem

**Problem:** Connect these dots using only four straight lines, without lifting your pencil off the paper or retracing any of the lines.

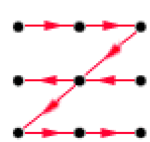


**Step 1: Understanding the Problem:**

"Connecting the dots using four lines" means that all of the dots are on some line that you draw, just like the dot-to-dot drawings that you did as a child. It would not be a difficult problem if we were allowed to use <u>five</u> lines instead of four. For example, a solution would be



If we were allowed to lift our pencil off the paper a solution would be



So the challenging part of the problem is to connect the dots using four lines while not lifting our pencil.

**Step 2: Devise a Plan:**

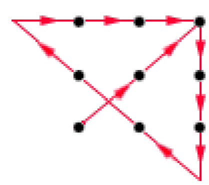
Really, the only thing we can do is use trial and error. After a few tries at finding an easy solution to a problem, usually one gets stuck. We feel as if we have tried everything possible and we have failed to find a solution. The first thing to do at this point is to be sure that the problem is stated correctly so that we are working on a problem that has a solution. Clarification is always helpful. ***Be Creative!***

**Step 3: Carry out the Plan:**

We are stuck and want to devise a way out of being stuck. We have to **get creative and invent something new**. Here are some pointers for getting "unstuck":

* Remove self-imposed barriers. Often we get stuck because we impose rules on ourselves that really are not part of the problem.
* Try something really different. Sometimes we are stuck because our first efforts led us down the wrong path.
* Learn more first. Get more information about the type of problem you are trying to solve. Do some easier problems of the same type to give you new ideas. (Of course, you do not have the liberty of being able to look things up on an exam, but in the real world this tip is useful.)

For the Nine Dots Problem, the self-imposed barrier that we have to overcome is thinking that the lines cannot go beyond the nine dots. Since this is not stated, the lines can go beyond the dots. Once we start to draw lines that can extend beyond the nine dots, a correct answer should not be too long in coming:



**Step 4: Look Back:**

The Nine Dots Problem is a problem with an unusual solution that requires one to overcome a self-imposed barrier and to be creative. Are there other problems that can be solved with this method? Certainly. We have to use this method whenever we get stuck. In the exercises you will find similar problems. They are easy once you know the answers, but they are sure to get you stuck for a little while if you have not seen them before.
Brave readers may want to try problems #17 to #23 in the Exercises for Section 1.3 at this point. Warning: You WILL get stuck!

# 1.3.5 The Camel Problem

**Problem:** One fourth of a herd of camels was seen in the forest; twice the square root of that herd had gone to the mountain slopes; and 3 times 5 camels remained on the riverbank. How many camels are in this herd?

**Step 1: Understanding the Problem:**

This is a classic type of problem where some information is given, with one piece of information missing. We must find the missing piece of information. Some people have trouble with the wording of the problem and are mistakenly led to believe that more than one herd of camels is being referred to. This would make the problem unsolvable, so one must interpret that there is only one herd of camels.

We organize the information as to what is given and what we want to find.

We want to find:

* The number of camels in the herd.

We are given:

* One fourth of them are in the forest.
* Twice the square root of the number of camels are in the mountains.
* Three times five (=15) are on the riverbank.
* There are no other camels (since the word "remained" is used).

**Step 2: Devise a Plan:**

We will look at this problem in two ways. There is a simple answer (one number) and a means of checking the answer by adding up the number of camels in each of the three distinct groups making up the herd. A sophisticated problem-solver could come up with an **algebraic model.** However, if your polynomial factoring skills are a bit rusty, the answer is a positive integer with some limitations, so a **Guess and Check** strategy should be available.

**Step 3: Carry out the Plan:**

***<u>Guess and Check Solution</u>***

For those who prefer to avoid algebra, one can also use the guess-and-check method. Before making wild guesses, we can save a great deal of time by applying some <u>common</u>

sense and experience to make good guesses right away. This is often referred to as making an **educated guess**. In this problem, one fourth of the number of camels in the herd is the number of camels that went into the forest. The number of camels in the forest has to be a positive integer. Hence, the number of camels in the herd is a multiple of four. So only guess multiples of four.

Also, twice the square root of the number of camels in the herd is the number of camels that had gone to the mountain slopes, so this is another positive integer. Thus, our guess should have the property that its square root is a positive integer.

GUESS: 100. (100 is a multiple of 4 whose square root is 10)
CHECK:

* $100/4 = 25$ camels went to the forest.
* $2 \times \sqrt{100} = 2 \times 10 = 20$ camels went to the mountain slopes.
* and 15 camels remained on the riverbank, for a total of 25 + 20 + 15 = 60 camels.

But we guessed 100 camels, not 60, so this is wrong.

We should use our experience with the incorrect guess of 100 to try to decide whether the next guess should be higher or lower. It's not immediately clear whether to guess higher or lower, so let's try higher. The next perfect square multiple of 4 is 144.

If we guess 144, we predict that 144/4=36 camels went to the forest, $2 \times \sqrt{144} = 2 \times 12 = 24$ camels went to the mountain slopes, and 15 camels remained on the riverbank, for a total of 36 + 24 + 15 = 75 camels. Unfortunately the gap between our guess of 144 and the predicted number of camels 75 has grown larger. So it looks like we should be guessing less than 100.

Guessing 64 camels makes the predicted number of camels 16+16+15 = 47. Much closer, but still wrong.

Guessing 36 camels makes the prediction 9+12+15=36. So 36 checks out as a correct answer.

## ***Algebraic Solution***

Let $h$ = the size of the herd. Then "one fourth of the herd of camels was seen in the forest; twice the square root of that herd had gone to the mountain slopes" tells us that $h/4$ is a positive integer and that $h$ is the square of a positive integer. Thus $h = 4k$ and $h = q^2$ for some positive integers $k$ and $q$. Since $h$ is a square, $k$ must also be a square, say $k = m^2$. Thus $h = 4m^2$ for some positive integer $m$. Hence one-fourth the number of camels in the forest ($h/4 = m^2$), plus the number of camels on the mountain slope ($2\sqrt{h} = 2\sqrt{4m^2} = 2(\sqrt{4} \times \sqrt{m^2}) = 4m$), plus the camels by the riverbank ($3 \times 5 = 15$) is the total number of camels ($h = 4m^2$).

Therefore: $m^2 + 4m + 15 = 4m^2$

Equivalently: $3m^2 - 4m - 15 = 0$

which can be factored as: $(3m+5)(m-3) = 0$

(Multiply it out and check it.) Since the product of these two factors is 0, at least one of them must be 0. i.e.:

$$3m + 5 = 0 \quad \text{or} \quad m - 3 = 0.$$

Therefore, since we know $m$ is an integer, $m = 3$ and $h = 36$.

**Step 4: Look Back:**

Although the guess and check strategy is time consuming, it allowed us to do this problem without using advanced mathematical skills, like introducing variables or solving equations involving square roots. The use of common sense, in that certain calculated numbers had to be positive integers, helped us make reasonable guesses and also avoided problems with trying to calculate square roots that were not positive integers. A good tip to take from this problem is to be aware that sometimes certain quantities in a problem have to be positive integers. It really helps to make educated guesses rather than wild ones.

Readers may wish to try problems #24 to #27 in the Exercises for Section 1.3 now.
Warning: You may find #27 confusing. We do.

## 1.3.6 A Not-as-easy-as-it-looks Problem

**Problem:** A bottle full of pop costs \$2.20. If the pop costs \$2.00 more than the bottle, how much does the bottle cost?

**Step 1: Understanding the Problem:**

Some problems are quite easy to understand but not so easy to solve. In this problem, we are again given information and asked to find a certain quantity. We need to find:

* How much the bottle alone costs.

We are given:

* The bottle and the pop together cost \$2.20.
* The pop costs \$2.00 more than the bottle.

**Step 2: Devise a Plan:**

The problem looks easy. The cost of the pop minus the cost of the bottle equals $2.00.

**Step 3: Carry out the Plan:**

One place where the problem solver can make a mistake is to miss the word more and thus conclude that the pop costs $2.00 and the bottle costs $0.20. But if we subtract the cost of the pop from the cost of the bottle, we get a difference of $1.80. This is not what the problem states.

These problems are known to fool almost everyone at first, though they are usually quite simple. If the problem looks too easy, read it very carefully until you understand exactly what it is asking.

***Guess and Check Solution***

The first guess of the pop costing $2.00 and the bottle $0.20 gives a difference of $1.80 which is too small. Thus the pop must cost more than $2.00. Try $2.10 for the pop. This gives $0.10 for the bottle and thus the pop costs $2.10 – $0.10 = $2.00 more than the bottle.

Answer: The pop costs $2.10 and the bottle costs $0.10.

***Algebraic Solution***

If $b$ is the cost of the bottle, then $(2.00 + b)$ is the cost of the pop (because the pop costs $2.00 more than the bottle). If the cost of the pop is $\$2.00 + b$, then:

$$(\$2.00 + b) + b = \$2.20.$$
$$\$2.00 + 2b = \$2.20$$
$$2b = \$0.20$$
$$b = \$0.10$$

Thus, the bottle costs $0.10 and the pop costs $2.10.

**Step 4: Look Back:**

Upon reading this problem the first time through, one immediately wants to jump to the conclusion that the pop costs $2.00. It is good to keep in mind that even though a problem looks easy, it might not be as easy as it looks! By reading it carefully and doing the steps carefully, we were able to avoid getting fooled by the problem.

Readers may want to try problems #28 to #33 in the Exercises for Section 1.3 at this point. Then look at their answers. Hope you don't get fooled!

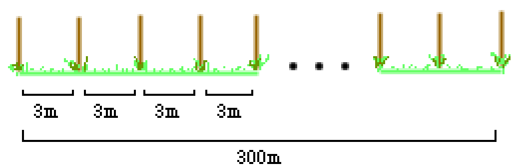
# 1.3.7 The Fence Post Problem

**Problem:** A man wants to build a 300m straight fence using fence posts exactly 3m apart. How many fence posts does he need?

**Step 1: Understanding the Problem:**

This is another classic problem. We are told that the fence is to be 300m long with posts 3m apart. We should assume that no extra posts are required to build gates or braces (although this is something you would need clarified if you were actually going to build the fence). Also, we will assume that 3m apart means 3m from the centre of one post to the centre of the next. (Again, in an actual situation this would need to be clarified.) We need to know the number of posts required to build the fence.

Before jumping to the conclusion that the answer is 300 ÷ 3 = 100 (which is tempting), one should pause and think: "Is it really that easy?" This problem can be represented by a picture, which is the best way of translating it into our own words.




How many posts?

**Step 2: Devise a Plan:**

In the picture, we could just count the number of posts. That would take too long, though, so think about how you would build the fence instead.

***Constructive Solution***

**Step 3: Carry out the Plan:**

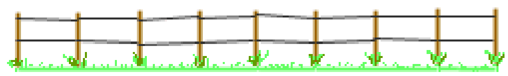
Drive the first post. Move 3 meters and drive the second post. Put up the fencing material. You now have a 3m fence and it required 2 posts. Move three meters, drive the next post and construct the next section of fence. You now have a 6m fence and it required 3 posts. Continue.

***Look for a pattern***

The pattern here is that you need one more post then the number of 3m sections of fence. Thus for a 300m fence there are 300/3 = 100 sections and hence you require 101 posts.

**Step 4: Look Back:**

The actual answer, 101, is one more than the answer of 100, which is what we originally expected. The reason is evident from the picture of the fence:



There will always be **one more fence post than spaces between the posts**. This is because there has to be a fence post at the beginning of each space and one more post at the end of the last space. The same reasoning can be used in a wide variety of problems.

Readers may want to try problems #34 to #40 in the Exercises for Section 1.3 now.

# 1.3.8 Summing Arithmetic Progressions

Find the sum:

$$1 + 2 + 3 + 4 + . . . + 4\,998 + 4\,999 + 5\,000$$

**Step 1: Understanding the Problem:**

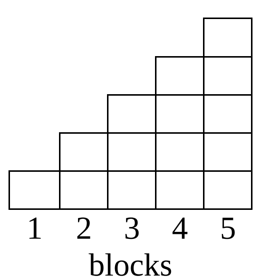
This problem is stated quite simply as an arithmetic problem and we are to add together all the whole numbers from 1 to 5 000. One way of doing this problem is to add the numbers one at a time, which would take 4 999 steps. Obviously, we are not expected to do this much work (and our human nature should tell us to avoid this much work), so we have to look for a shortcut.

**Step 2: Devise a Plan:**

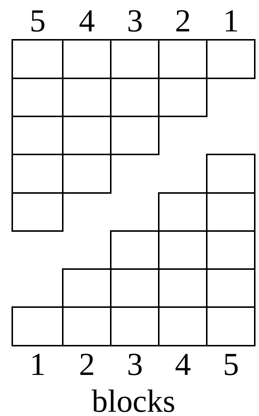
To look for the shortcut, we can start with smaller progressions where we can easily find the answer, and look for pattern. Or we can try to invent a new and creative approach.

**Step 3: Carry out the Plan:**

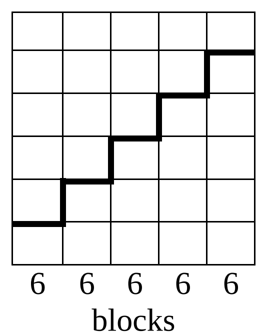
First, let's model the the sum 1 + 2 + 3 + 4 + 5 using blocks.



If we count the blocks in the array, we will see that the sum is 15. Let's be creative and find a new way to deduce this answer without counting the blocks. If we have two copies of the block array above, we can flip one over and stack them to create a rectangle




This rectangle is 5 blocks wide and 6 blocks high.




So the number of blocks in the rectangular array is

$$(1 + 2 + 3 + 4 + 5) + (5 + 4 + 3 + 2 + 1) = 5 \times 6,$$

which is equivalent to

$$2(1 + 2 + 3 + 4 + 5) = 30$$

so

$$1 + 2 + 3 + 4 + 5 = {}^{30}/_{2} = 15.$$

The rectangular array is five columns of 6 blocks each. The same structure can be seen in the numeric array

$$\begin{array}{r} 1 + 2 + 3 + 4 + 5 \\ \underline{+\ 5 + 4 + 3 + 2 + 1} \\ 6 + 6 + 6 + 6 + 6 \end{array}$$

where there are 5 columns of numbers and the sum of each column is 6.

Now, how can we use this idea on 1 + 2 + ... + 5 000?

Write the sum on one line:

1 + 2 + 3 + . . . + 4 998 + 4 999 + 5 000,

then write the same sum in reverse order on the line below:

5 000 + 4 999 + 4 998 + . . . + 3 + 2 + 1.

Add the numbers that lie above and below one another to get a total that is twice the original sum:

```
       1 +     2 +     3 +   .   .   . + 4 998 + 4 999 + 5 000
 + 5 000 + 4 999 + 4 998 +   .   .   . +     3 +     2 +     1
 -------------------------------------------------------------
   5 001 + 5 001 + 5 001 +   .   .   . + 5 001 + 5 001 + 5 001
```

The numbers that appear in the last sum are all the same, 5 001. Since there are exactly 5 000 terms in the sequence 1, 2, 3, . . . 5 000, there are also 5 000 of these 5 001's.

That is, (1 + 2 + ... + 4 999 + 5 000) + (5 000 + 4 999 + ... + 2 + 1) = 5 000 × 5 001.

Since 5 000 × 5 001 is twice the sum we want, we must divide that by two.

Therefore,

$$\begin{aligned} 1 + 2 + 3 + \ldots + 5\,000 &= 5\,000 \times 5\,001 \div 2 \\ &= 25\,005\,000 \div 2 \\ &= 12\,502\,500 \end{aligned}$$

The creative, new approach is a technique is called the **Gauss sum trick**, named after the famous German mathematician Carl Friedrich Gauss (1777 – 1855). Gauss amazed his teachers in elementary school by seeing quickly how to sum the integers from 1 to 100. You can read more about Gauss at:
www-history.mcs.st-andrews.ac.uk/history/Mathematicians/Gauss.html

**Step 4: Look Back:**

Once you understand this method, it is unlikely that you will ever forget it. This method is easy to use, does not require any sophisticated mathematics (like variables or a formula), and applies to a wide variety of other sums.

If we change 5 000 to any other number, like 137 for instance, the same approach works:

```
       1  +  2  +  3  +  .    .    .  + 135  + 136 + 137
 +   137 + 136 + 135 +   .    .    .  +  3   +  2  +  1
 -------------------------------------------------------
     138 + 138 + 138 +   .    .    .  + 138  + 138 + 138
             (There are 137 of these 138's)
```

$= 137 \times 138$ (twice the sum we want);

$$\begin{aligned} \text{so, } 1 + 2 + 3 + \ldots + 137 &= 137 \times 138 \div 2 \\ &= 18\,906 \div 2 \\ &= 9\,453 \end{aligned}$$

Another type of problem we can try is to change the starting number from 1 to something else, like 74.

```
     74  +  75 +  76 +   .   .   .  + 135 + 136 + 137
+   137 + 136 + 135 +   .   .   .  +  76 +  75 +  74
    211 + 211 + 211 +   .   .   .  + 211 + 211 + 211

             = (twice the sum we want)
```

How many of these 211's are there? This will be the same as the number of terms in the sequence 74, 75,..., 137. This can be considered as a sub-problem, which is quite similar to the fence-post problem we just looked at!

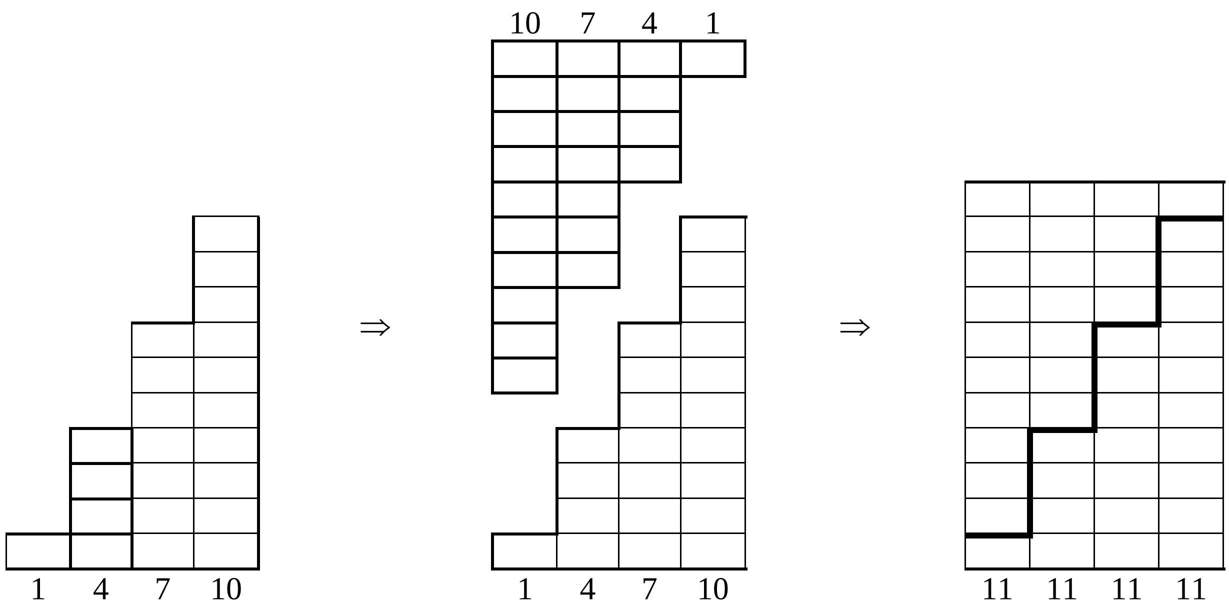
To get from 74 to 137, in steps of 1, we need 137 - 74 = 63 steps. Therefore, there are 63 + 1 = 64 terms. Thus,

$$(74 + 75 + 76 + \ldots + 137) = 64 \times 211 \div 2$$

So,

$$\begin{aligned} 74 + 75 + 76 + \ldots + 137 &= 32 \times 211 \\ &= 6\,752 \end{aligned}$$

Another problem is to find the sum of terms of a progression whose common difference is a number other than one. For example: How about 1 + 4 + 7 + 10?



If we duplicate the blocks and flip the second copy over, they fit together perfectly! This gives us

$$2 \times (1 + 4 + 7 + 10) = 4 \times 11 = 44, \text{ or } 1 + 4 + 7 + 10 = 22$$

Try to find the sum: 17 + 21 + 25 + . . . + 401.

Perform Gauss' method to get

```
     17  +  21  +  25  +   .   .   .    + 397  + 401
+   401  + 397  + 393  +   .   .   .    +  21  +  17
---------------------------------------------------
    418  + 418  + 418  +   .   .   .    + 418  + 418

             = (twice the sum we want).
```

There are as many 418's as there are terms in the sequence 17, 21, 25, . . . 397, 401. Finding the number of terms in this sum is one of the problem we learned how to solve using the method for the fence post problem.

```
|     |     |    .    .    .    |     |
 17    21    25                397   401
   \   /  \   /                   \   /
    +4     +4                      +4
```

To find the number of terms, we have to find how many times, $k$, we add 4 to 17 to get 401. If $17 + 4k = 401$, then $k = 96$. Since the first term is 17, there are 97 terms in all.

$$\begin{aligned}
\text{Therefore, } 17 + 21 + 25 + \ldots + 401 &= 97 \times 418 \div 2 \\
&= 40\,546 \div 2 \\
&= 20\,273
\end{aligned}$$

Readers may want to try problems #42 to #47 in the Exercises for Section 1.3 now.

## Exercises: Section 1.3

Solve exercises #1 to #3 using more than one method.

1. There are 18 heads and 46 feet. How many cows and ducks are there?

2. There are 296 wheels on the 100 vehicles that are parked outside. How many are cars and how many are bikes?

3. Susan is making gift bags to give to the 50 guests at her birthday party. She has 90 pieces of fruit and 27 chocolate bars. Some bags have only a chocolate bar, some bags have 2 pieces of fruit and no chocolate bar, and some bags have a chocolate bar and 4 pieces of fruit. How many bags of each type are there?

In exercises #4 to #9, find the last digit of each number.

| | | |
|---|---|---|
| 4. $6^{904}$ | 5. $7^{905}$ | 6. $8^{906}$ |
| 7. $9^{907}$ | 8. $23^{93}$ | 9. $899^{899}$ |

10. Find the single digit from 0 to 9 that is the remainder of $2^{8111} \div 10$.

11. Find the remainder of $3^{777} \div 5$.

For exercises #12 and #13, find the last two digits of the given number.

12. $25^{395}$ 13. $11^{4063}$

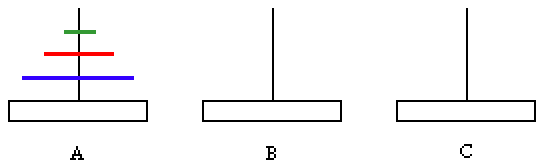
14. Three teenage couples want to go out to a movie, but their only means of transportation is a scooter that can only carry two of them at a time. Everyone can drive the scooter, but none of the girls want to be left alone with a guy unless her date is present. How can they get to the movie?

15. Three mathematicians and three physicists want to cross a river using a small boat that only holds, at most, two of them at a time. The mathematicians never want to be outnumbered by the physicists on either shore. Find a way they can safely cross the river.

16. Tower of Hanoi

a) Move the tower of rings from stand A to stand C using only seven moves, moving only one ring at a time, and never placing a larger ring on top of a smaller ring. (If you want to physically move pieces, try using different sized coins instead of the rings and just imagine 3 stands.)

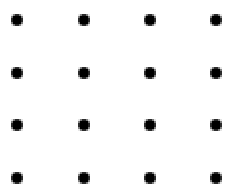
(Tower of Hanoi with 3 rings)



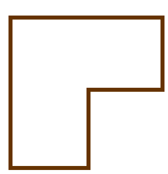

b) How many moves would be required for 4 rings?  5 rings?

c) How many moves would be required for 64 rings?

17. Connect the 16 dots using only 6 straight lines while not lifting your pencil off of the paper and without retracing any of the lines.



18. A farmer left a plot of land, shaped like the drawing below, to his four children. The plot was to be divided so that each child would receive a plot exactly the same size and shape as the others. How should it be divided?



19. There is a hole in the barn floor .5m wide and 3m long. How can it be covered with a board .75m wide and 2m long that can only be cut once into two pieces?

20. A room which is $9 \times 12$ is to be covered with carpet, but the carpet has been provided in one $8 \times 1$ piece and one $10 \times 10$ piece. The larger piece is to be cut into two pieces so that the room can be covered in carpet. How should the $10 \times 10$ piece be cut? (**Suggestions**: Put the 8 by 1 piece in the middle of the 9 by 12 area and make symmetric shapes with the cut. Try covering a 3 by 6 room with a 4 by 4 piece of carpet and a 1 by 2 piece of carpet where the 4 by 4 can only be cut once)

21. Put 10 coins on the table. See if you can arrange them in 5 rows of 4 coins each.

22. A man died leaving a square piece of land to his five children. The land was to be divided so that each child's portion of land bordered each of the other four children's portions at more than a corner. How could they divide the land?

23. Arrange six matchsticks to make exactly four equilateral triangles with no extra triangles or matchsticks left over.

24. Demochares lived one-fourth of his life as a boy, one-fifth as a youth, one-third as a man, and spent 13 years in his dotage (feeble-minded old age). How long did Demochares live?

25. A record store ordered a certain number of the latest hit CD's. $^2/_3$ of the CD's sold on the day they arrived. On the next day, they sold $^2/_3$ of what was left. On the day after that, they again sold $^2/_3$ of what was left. On the beginning of the fourth day, the three store employees bought the remaining 3 CD's. How many CD's did the store order?

26. On Christmas Day 1999, I saw Frank, who had celebrated his birthday on Christmas Eve. He told me that he will turn $x$ years old in the year $x^2$. How old was Frank on Christmas Day 1999?

27. When Gwen and Bill applied for their marriage licence, they were asked their ages. Bill, who was a bit short-tempered, said they were both in their twenties and that was all he was going to reveal to a bunch of bureaucrats. To smooth things over, Gwen added, "We both have the same birthday, and I am four times as old as Bill was when I was three times as old as Bill was when I was twice as old as Bill was." At hearing this, the clerk fainted, so Bill and Gwen snatched up their marriage licence and left. When the clerk came to, he realized that he would have to complete his records some way, so he began to do a little figuring. Before long, he had found out how old the two were. Can you?

28. There are 3 separate, equal-size boxes, and inside each box there are 2 separate small boxes, and inside each of the small boxes there are 4 even smaller boxes. How many boxes are there altogether?

29. Three empty cereal boxes each weigh 9 oz., and each box holds 11 oz. of cereal. How much do 2 full boxes of cereal weigh together?

30. If it takes 7 minutes to boil an egg, how long does it take to boil two eggs?

31. Ten full crates of oranges weigh 410 kg, while an empty crate weighs 10 kg. How much do the oranges alone weigh?

32. A stack of 100 new $5 bills is exactly 1 cm tall. How tall is a stack of one million $5 bills?

33. Jeanne buys a CD for $17.00. The CD itself is worth $15.00 more than the packaging. How much is the CD itself worth?

34. A square plot of land 4900$m^2$ in area is to be fenced. If the posts are to be 7m apart, how many posts are needed?

35. A telephone rings 5 times in 10 seconds. How long will it take to ring 10 times?

36. At 6:00 p.m., a grandfather clock takes 5 seconds to "bong" 6 times. At midnight, how long will it take to "bong" 12 times?

37. A person reads from page 52 to page 75 of a novel. How many pages did the person read?

For exercises #38 to #40, identify the number of terms in each of the sequences?

38. 1, 3, 5, 7, 9, ..., 41

39. 101, 201, 301, ..., 11 101

40. 17, 40, 63, 86, ..., 1 581

41. [The following puzzle is from Martin Gardner's *New Mathematical Diversions from Scientific American*, 1996.] Three high schools – Washington, Lincoln, and Roosevelt – completed in a track meet. Each school entered one man, and one only in each event. Susan, a student at Lincoln High, sat in the bleachers to cheer her boyfriend, the school's shot put champion. When Susan returned home later in the day, her father asked how her school had done.

"We won the shot put all right," she said, "but, Washington High won the track meet. They had a final score of 22. We finished with 9. So did Roosevelt High."

"How were the events scored?" her father asked.

"I don't remember exactly," Susan replied, "but there was a certain number of points for the winner of each event, a smaller number for second place, and a still smaller number for third place. The numbers were the same for all events."

"How many events were there all together?"

"Gosh, I don't know, Dad. All I watched was the shot put."

"Was there a high jump?" asked Susan's brother.

Susan nodded.

"Who won it?"

Susan didn't know.

Incredible as it might seem, this last question can be answered with only the information given. Which school won the high jump?

Find the sum of the following sequences:

42. 1 + 2 + 3 + . . . + 90

43. -5 + -4 + -3 + . . . + 34 + 35

44. 6 + 16 + 26 + . . . + 286

45. 15 + 18 + 21 + . . . + 297

46. Show that Gauss' method should not be used with sequences whose successive terms do not differ by a fixed amount. For example, what happens when you try to apply Gauss' method to:

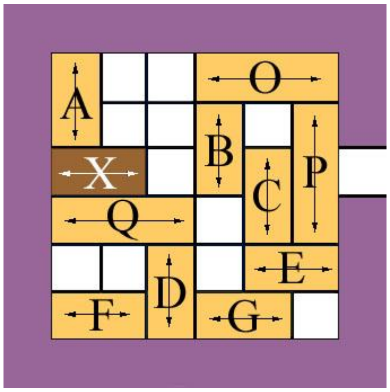
$$1 + 2 + 4 + 8 + 16 + 32 + 64 + 128?$$

47. Find the sum of:

$$(1\,000 - 200) + (995 - 198) + (990 - 196) + . . . + (505 - 2)$$

48. A standard deck of 52 cards (26 red, 26 black) is separated into two piles, not necessarily equal in size. The first pile contains seven times as many black cards as red cards. The second pile contains a number of red cards that is not a multiple of the number of black cards in that pile. How many red cards are in the first pile?

49. The following puzzle is from Binary Arts' Rush Hour. The goal is to slide block X out the exit on the right side of the grid. To accomplish this goal, slide any of the blocks lengthwise, as indicated by their arrows, to create a path for block X. How can block X exit the grid in 8 slides?




For more information on any Binary Arts games, or Rush Hour, go to www.puzzles.com, or www.puzzles.com/products/rushhour.

# 1.4 Deductive Reasoning Patterns

**Deductive logic** is the art of constructing valid arguments to deduce that desired conclusion is true given certain initial assumptions. Logic and mathematics go together very well because most simple mathematical statements are definite: they have to be either true or false but are never both true and false at the same time.

**Simple logic** is a collection of rules that predicts the truth or falsehood of compound sentences built up from initial statements using connecting words or phrases, such as "and", "or", and "not", or "if". The initial statements need to be definite in the above sense, and the effect of the connecting words "and", "or", "not", and "if" on the truth or falsehood can be calculated directly from their meaning. This is usually done formally by means of **truth tables**. Different combinations of connecting words often result in compound logical statements whose truth values agree for every possible combination of truth values of the initial statements. In this case, we say that the compound statements are **logically equivalent.** Since this can get a bit too formal, we have deferred a detailed treatment of truth tables and logical equivalence to Appendix A of this section.

## 1.4.1 Valid Arguments

### Conditional Statements

The most important logical construction for making arguments is the conditional statement **if p then q**, which we denote by $\mathbf{p \rightarrow q}$. Sometimes it is good to think of this statement as being "**p implies q**". The truth table for $\mathbf{p \rightarrow q}$ is

| p | q | p → q |
|---|---|---|
| T | T | T |
| T | F | F |
| F | T | T |
| F | F | T |

The easiest way to understand the truth table for $\mathbf{p \rightarrow q}$ is to think of **p** as being an *assumption* and **q** as being the *conclusion*. The <u>only way</u> for "assumption implies conclusion" to be a false statement overall is to have true assumption when you have a false conclusion. So the only F in the truth table of $\mathbf{p \rightarrow q}$ occurs when **p** is T and **q** is F.

A consequence of this is that it automatically makes $\mathbf{p \rightarrow q}$ true when **p** is false, and that can seem confusing at first. Some examples should clarify why this should be the case.

If I had a million dollars, then I'd buy you a green dress.

Observe that the speaker does not unconditionally declare that he will buy a green dress. The speaker simply says what would happen if he or she had a million dollars. If the speaker both has a million dollars and buys the dress, then the declaration is true. It is likewise clear that if the speaker has a million dollars but fails to buy the dress, then the statement would be false. If the speaker did not have a million dollars, then he or she could promise anything without consequence. So any conclusion following "If I had a million dollars, then..." would result in a true implication for the speaker, since the assumption is not in effect.

We actually use $\mathbf{p} \rightarrow \mathbf{q}$ statements in our personal lives by making clever promises, where we count on **p** being false. A parent might tell a child who repeatedly begs for a hamster but cannot swim:

If you swim a length of the pool, then I'll buy you a hamster.

As long as the child cannot make the antecedent true, the parent has no obligation to deliver the consequent. If the parent gives in and buys the hamster for some other reason, the parent has not gone against what was promised.

**Arguments**

There are two patterns of reasoning: **deductive and inductive**.

**Inductive Reasoning** is the act of making a conclusion based on observing a pattern that occurs in previous examples, as we saw in some of the problems in section 1.3. For instance,

The last four signs on the road were orange.
Therefore, the next sign will be orange.

One plus one is two,
Two plus two is four,
Three plus three is six,
Therefore, any number plus itself is always less than ten.

The above examples are a little exaggerated to make a point. That point is that <u>the conclusion reached using inductive reasoning may not always be correct</u>. The "conclusion" of both arguments (the statement after the word, "therefore") will not necessarily be true under the given assumptions (the statements coming before the word "therefore"). Such a conclusion is called **invalid**. In fact, we know the conclusion of the second argument is false, although we can see where it came from. Inductive reasoning can be used to detect a pattern, but a proof that the pattern repeats beyond the instances observed requires additional justification.

**Deductive Reasoning** is the act of making conclusions based on assumptions that are combined according to the laws of logic. This is done using **valid arguments**, which are statements of the form

[assumption 1 and assumption 2 and etc.] $\rightarrow$ conclusion

that are logically equivalent to a statement that is *always true*. It may seem surprising that a complicated conditional statement like this could be true no matter what, but in fact we encounter valid arguments often in everyday situations, and they become part of the common sense ideas we use to reason in the course of our everyday lives. Valid arguments are also called **proofs**. A conclusion to a valid argument whose assumptions are known to be true is a statement that has been proven to be true beyond *any* doubt.

Consider the simple argument:

If it is raining then I am going shopping.
It is raining.
Therefore, I am going shopping.

Examine the logical structure of this argument as one big compound conditional statement. Let **p** be the simple statement "it is raining" and **q** be "I am going shopping". The two assumptions of the argument are **p → q** and **p**. The conclusion is **q**. The whole argument is logically equivalent to **[(p → q) and p] → q**.

The truth table for the above argument **[(p → q) and p] → q** is:

| p | q | p → q | (p → q) and p | [(p → q) ∧ p] → q |
|---|---|---|---|---|
| T | T | T | T | T |
| T | F | F | F | T |
| F | T | T | F | T |
| F | F | T | F | T |

We see that the logical structure of this argument is a **tautology** -- it is true for all truth-values of the initial statements **p** and **q**. Arguments that are logical tautologies are called valid arguments.

Conclusions reached using a logical pattern in the form of a valid argument <u>must be true</u> as long as the assumptions are true. When we believe all of the assumptions, and we know that the argument is a conditional statement that is always true, it forces the conclusion of the argument to be true. (It could not be false since that would make the conditional statement false.)

The basic argument **[(p → q) and p] → q** above is the most common form of a valid argument, known as the **direct argument** (or, in Latin, *modus ponens*). The other important valid argument often used is the **indirect argument** (*modus tollens*):

**[(p→ q) and (not-q)] → (not-p).**

The indirect form of argument is a bit harder to understand, but it is quite effective because it allows one to reach conclusions in situations where the easier arguments do not work. Here is a simple example:

If $96 \div 12 = 9$, then $12 \times 9 = 96$.
$12 \times 9 \neq 96$.
Therefore, $96 \div 12 \neq 9$.

Some of the most important mathematical ideas we encounter later in the book, such as the set of prime numbers being infinite and the irrationality of the square root of 2, will be justified with an indirect argument. One way to understand indirect arguments is to think of the following: *If your assumptions lead to false conclusions, then at least one of your assumptions must be false.*

We will encounter three more common valid argument forms, those known as *syllogisms*, in the exercises. Most complicated valid arguments are just these easier ones strung together.

## Exercises: Section 1.4.1

1. Show that the indirect argument is a tautology.

2. Show that *hypothetical syllogism* **[(p → q) and (q → r)] → (p→ r)** is a valid argument. (Hint: The truth table for this statement will require 8 rows.)

3. Show that *disjunctive syllogism* **[(p or q) and (not-p)] → q** is a valid argument.

4. Show that the simple arguments **(p and q) → p** and **p → (p or q)** are valid.

For exercises #5 to #9, select the statement that is a valid conclusion for the given assumptions.

5. If you build it, he will come.
   If he comes then you will see him.

   i) If you build it, then you will see him.
   ii) He will come.

6. If Danny sells enough Tupperware, he will get a bonus.
   Danny sold enough Tupperware.

   i) Hence, he made lots of money.
   ii) He got a bonus.

7. If R, then not S.
   If not S, then T.

   i) Therefore, if S, then not T.
   ii) Thus, if R, then T.

8. You will fail the test if you do not study.
   You do not fail the test.

   i) You studied.
   ii) If you failed the test, you did not study.

9. If 1 + 1 = 3, then 1 + 3 = 5.
   1 + 3 = 4.

   i) Therefore, 1 + 1 = 2.
   ii) So $1 + 1 \neq 3$

For exercises #10 to #13, write conclusions making the arguments valid that use all the given assumptions.

10. If the Tragically Hip comes to town, then I will go to the concert.
    If I go to the concert, then I will be broke until payday.

11. Randy Ferbey curls or Wayne Gretzky is a singer.
    Wayne Gretzky is not a singer.

12. If beggars were choosers, then I could ask for it.
    I cannot ask for it.

13. Warren's hobby is collecting rocks.
    If his wife likes to knit, then Warren's hobby is not collecting rocks.
    If his wife does not like to knit, then Joel likes cartoons.

14. Everyone who is sane can do logic. Nobody who is insane is fit to serve on a jury. None of your sons can do logic.

15. Nobody who really appreciates Beethoven fails to keep silent while Moonlight Sonata is being played. Guinea pigs are hopelessly ignorant of music. No beings that are hopelessly ignorant of music ever keep silent while the Moonlight Sonata is being played.

16. Promise-breakers are untrustworthy. Wine-drinkers are very communicative. A person who keeps a promise is honest. No teetotallers [opposite of wine-drinkers] are pawnbrokers. One can always trust a very communicative person.

17. The inhabitants of the remote island of Balilo are divided into two hereditary castes. To an outsider the members of these two castes look exactly alike. But those in one caste, the Arbus, always tell the truth; those in the other, the Bosnins, always tell the exact opposite of the truth. An explorer went into this island knowing their customs, but not knowing their language. He met three natives -- Fang, Gang and Hang. All three could understand English; Fang could not speak English, but Gang and Hang could.

> "Of what caste are you?" the explorer asked Fang.
> "U*-1^%I!@," said Fang.
> "What did he say?" asked the explorer of Gang and Hang.
> "He say he Arbu," said Gang.
> "He say he Bosnin," said Hang.

Explain clearly just how the explorer was now able to tell to which castes Gang and Hang belonged.

# 1.4.2 Arguments involving Quantified Statements

The conditional statement

If it is a mouse then it eats cheese.

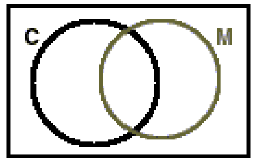
has the same meaning as the *quantified* statement

All mice eat cheese.

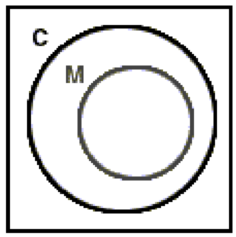
The phrase "mice eat cheese" is a bit vague, but when the word "all" is added at the beginning, it sets the context so that we can decide if what is being said is true or false but not both true and false. Vague phrases like "mice eat cheese" are called **predicates**, and words or phrases like "all" that provide the context to turn them into statements are called **quantifiers**.

The meaning of the "all" quantifier, or any of its equivalent forms ("for all," "every," "for every," etc.) is easily explained using sets. **Sets** are collections of things for which the question "is $x$ in the set $S$?" always has a definite answer yes or no for every thing $x$. We say "$x$ is an **element** of $S$" when the thing $x$ is in the set $S$, and write this as "$x \in S$".

Sets are easy to model with a special kind of diagram we call a **Venn diagram**. To illustrate the quantified statement "All mice eat cheese" with a Venn diagram, let M be the set of creatures that are mice, and C be the set of creatures that eat cheese. The typical Venn diagram for the two sets M and C of creatures looks like this:




But this general picture allows the possibility that a mouse, an element of M, does not belong to C. To illustrate "All mice eat cheese", we should modify it so that the set M is inside C:

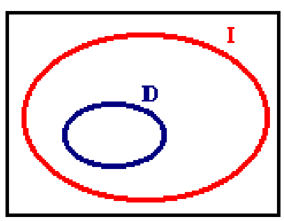



This picture indicates that all elements of the set M are elements of the set C. So, in set terminology, saying "All mice eat cheese" is just a way to say that "M is a **subset** of C", which we write this way: $M \subseteq C$.

Now consider the following statement that is quantified using "all".

"All drugs are illegal."

We can illustrate this statement using the Venn diagram:




in which D is the set of all drugs and I is the set of all illegal things. However, aspirin is a drug and aspirin is not illegal. We cannot place aspirin in the Venn diagram above since it must be inside D and outside I. Therefore, "All drugs are illegal" is actually a false statement. The negation of this statement is "Some drugs are not illegal," or to avoid a double negative, "Some drugs are legal."

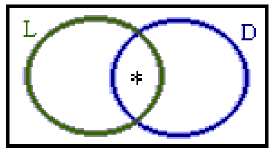
Notice that to negate a statement that is quantified using "all", we change the quantifier to "some" and negate the predicate.

not-(All $p$'s are $q$'s) $\equiv$ Some $p$'s are not $q$'s.

By applying a negative, we can also determine the negation of a statement quantified using "some".

not-(Some *p*'s are *q*'s) ≡ All *p*'s are not *q*'s.

The Venn diagram for a true quantified statement using the term "some" is a picture of overlapping sets with something in common to both sets. The Venn diagram for "Some drugs are legal" looks like this:




where L is the set of legal things, and D is the set of drugs. The overlap of two sets is their **intersection**. The * in the intersection of L and D is used to indicate that that there is an element in that intersection. So, in set notation, saying "Some drugs are legal" is another way to say that "the intersection of the sets L and D is not equal to the empty set", which we write this way: $L \cap D \neq \varnothing$.

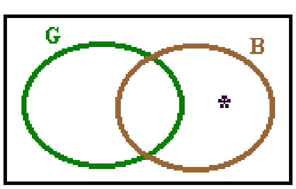
The meanings of the "all" and "some" quantifiers can also be described without using sets:

| | | |
|---|---|---|
| All *p*'s are *q*'s | is TRUE | if every p is also a q, |
| | is FALSE | if there is at least one p that is not a q. |
| Some *u*'s are *v*'s | is TRUE | if there is at least one u that is also a v, |
| | is FALSE | if every u is not a v. |

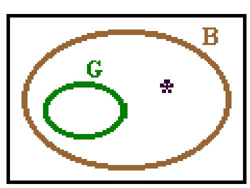
**Deductive reasoning** can also be done with quantified statements. Validity of arguments involving quantified statements can be checked using Venn diagrams. Consider the argument:

Some beans are not green.
All green things are edible.
Therefore, all beans are edible.

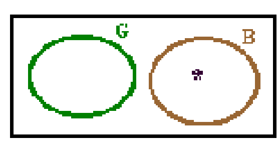
Let B be the set of beans, G the set of green things, and E be the set of edible things. A Venn diagram representing "Some beans are not green" could be:




or

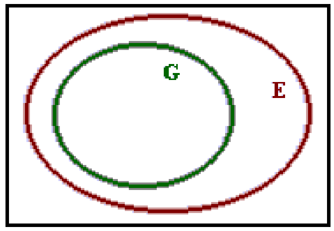



or

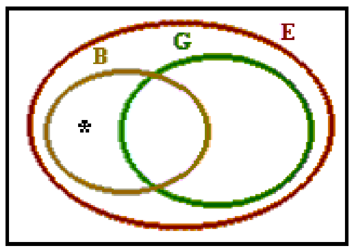



What is required is that there is something in a part of B that is not in G. The first of these diagrams is the most general in that it allows for green things that are not beans and for beans that are green. When analyzing an argument using Venn diagrams we should always use the most general diagram possible.

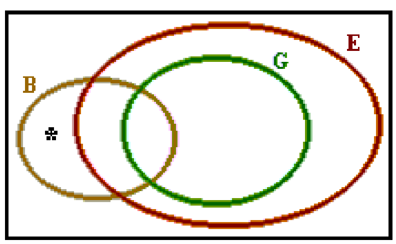
The diagram for “All green things are edible” is




Both of the assumptions “Some beans are not green” and “All green things are edible” are true for each of the following diagrams.




and

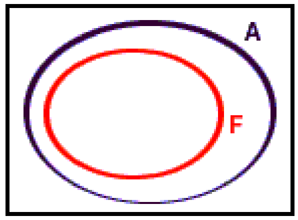



In the first diagram, the conclusion “all beans are edible” would be true. In the second, “all beans are edible” is false. This second diagram shows that all of the assumptions of the argument can be true at the same time that the conclusion is false. Therefore, this argument is invalid. (It may be true that “all beans are edible”, but this does not follow from these assumptions.) We could also check the validity of this argument using the truth table method.

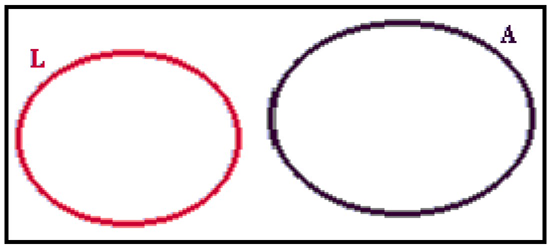
The word “none” is often used to mean “not some...” or equivalently “all are not”. Here is an example of an argument involving “none”.

> All fast runners are athletes.
> None of the people with long legs are athletes.
> Therefore, none of the people with long legs are fast runners.

To check the validity of this argument using Venn diagrams, let F be the set of people who are fast runners, A be the set of athletes, and L be the set of people with long legs. The Venn diagram for the statement "All fast runners are athletes" is

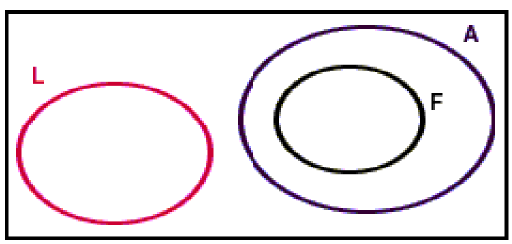



The diagram for "None of the people with long legs are athletes" is




("None" means "not some". "None" of the elements of L can be elements of A, so the two sets do not overlap, and $L \cap A = \varnothing$.)

There is only one way to draw a Venn diagram containing all of the three sets F, A, and L because we have to put the set F inside the set A.

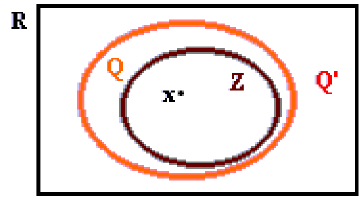



In this diagram, the set L and the set F do not overlap, so "none of the people with long legs are fast runners" is true. Therefore, this is a valid conclusion, and the argument is **valid**.

Simple arguments can be accommodated within this Venn diagram technique. For example, consider the argument

*x* is an integer.
All integers are rational.
No rational is an irrational.
All rationals are reals.
All irrationals are reals.
Therefore, *x* is real.

A diagram for this argument might be: with **Z**, **Q**, **Q**', and **R** representing their usual number sets and with R representing the universal set.




(Note that in the diagram, "*x* is an integer" is indicated by a point for *x* inside the integer set.) Since the point for *x* lies inside the universal set, R, the conclusion "*x* is real" is **valid**.

Other valid conclusions can also be drawn from these assumptions. For instance, from the Venn diagram we find that

*x* is rational
*x* is not irrational
No integer is irrational
All integers are reals

are all valid conclusions from the given assumptions. Thus, valid conclusions are not always unique.

## Exercises: Section 1.4.2

For exercises #1 to #10, decide whether the quantified statement is true or false.

1. Every rational number is a real number.

2. Every even integer is divisible by two.

3. Every integer is a whole number.

4. There exists a real number that is not a rational number.

5. All irrational numbers are real numbers.

6. All integers are rational numbers.

7. There are whole numbers that are not rational numbers.

8. Some irrational numbers are not integers.

9. Each non-positive number is a negative number.

10. Each rational number is an integer.

Write negations for each of the following quantified statements.

11. Everybody likes coffee.

12. Some zebras do not have stripes.

13. For some integer $x$, $49 - x \equiv -1$ and $49 - x = 0$.

14. For all real numbers $x$, there exists a real number $y$ such that $xy = 1$.

15. All dogs go to heaven.

For exercises #16 to #20, determine which arguments are valid and which arguments are invalid.

16. All men are mortal. Socrates is a man. Therefore, Socrates is mortal.

17. Carelessness always leads to accidents. Mr. Jones had an accident. Therefore, Mr. Jones is careless.

18. All men are created equal. All people who are created equal are women. Therefore, all men are women.

19. All cats are intelligent animals. Odie is an intelligent animal. Hence, Odie is a cat.

20. Some ems are blue. All ims are ems. Therefore, some ims are blue.

For exercises #21 to #26, write a conclusion making the argument valid that requires all of the assumptions.
Hint: It may help to convert statements to the "if…, then…" form.

21. Everyone who is sane can do logic. Nobody who is insane is fit to serve on a jury. None of your sons can do logic.

22. Nobody who really appreciates Beethoven fails to keep silent while Moonlight Sonata is being played. Guinea pigs are hopelessly ignorant of music. No beings that are hopelessly ignorant of music ever keep silent while the Moonlight Sonata is being played.

23. Promise-breakers are untrustworthy. Wine-drinkers are very communicative. A person who keeps a promise is honest. No teetotallers [opposite of wine-drinkers] are pawnbrokers. One can always trust a very communicative person.

24. No one who is going to a party ever fails to brush his hair. No one looks fascinating if he is untidy. Opium-eaters have no self-command. Everyone who has brushed his hair looks fascinating. No one ever wears white kid gloves unless he is going to a party. A man is always untidy if he has no self-command.

25. All the dated letters in this room are written on blue paper. None of them are black ink, except those that are written in the third person. I have not filed any of them that I can read. None of them that are written on one sheet of paper are undated. All of them that are not crossed are in black ink. All of them written by Brown begin with “Dear Sir”. All of them written on blue paper are filed. None of them written on more than one sheet are crossed. None of them that begin with “Dear Sir” are written in the third person.

# 1.4.3 Matching Problems

In this section we will practice using deductive reasoning to solve matching problems. We start with a relatively easy example to illustrate the use of the basic arguments. Indirect reasoning is especially useful for this type of problems.

**Problem**: Three people, Mr. Brown, Mr. White, and Mr. Green, own houses and cars exactly one of each being brown white, and green. Nobody’s house and car are of the same colour, and nobody owns a house or a car that is the same colour as his surname. If Mr. White correctly states to Mr. Brown “if your car is white, then so is your house”, determine the colour of each person’s house and car.

**Solution**:

To understand the problem, we first translate the clues into our own words. For example:
Clues:

1. Nobody’s house is the same colour as his car.
2. Nobody’s house is the colour of his surname.
3. Nobody’s car is the colour of his surname.
4. The houses are green, white, and brown (all used).
5. The cars are green, white, and brown (all used).
6. The statement “If Mr. Brown’s car is white, then Mr. Brown’s house is white.” is true.

These six statements (clues) are all assumed true. We need to fill in this chart:

| Person | Mr. White | Mr. Brown | Mr. Green |
|---|---|---|---|
| House Colour | | | |
| Car Color | | | |

None of the clues enable us to fill in these squares without having to think first. A good idea is to write in all possible correct answers in each blank square. By clues 2 to 5, each square has only two possible correct answers. Use G for green, W for white, and B for brown. (Why do we do that? There is only one reason. Mathematicians are lazy when it comes to writing things out! We always end up substituting letters or symbols for things, because we do not have the patience to write them out. You have probably already noticed this, but now we are admitting it.)

| Person | Mr. White | Mr. Brown | Mr. Green |
|---|---|---|---|
| House Colour | B G | G W | B W |
| Car Color | B G | G W | B W |

Now consider clue 6. If "Mr. Brown's car is white" is true, then clue 6 says that "Mr. Brown's house is white" is also true. So Mr. Brown's house and car would both be white. But clue 1 says Mr. Brown's house and car are not the same colour. So we are in the situation where both "If Mr. Brown's car is white, then Mr. Brown's house and car are the same colour" and "Mr. Brown's house and car are not the same colour" are both true. By indirect reasoning, it is not true that Mr. Brown's car is white.

So cross out the W in the box representing Mr. Brown's car.

| Person | Mr. White | Mr. Brown | Mr. Green |
|---|---|---|---|
| House Colour | ~~B~~ (G) | ~~G~~ (W) | (B) ~~W~~ |
| Car Color | (B) ~~G~~ | (G) ~~W~~ | ~~B~~ (W) |

So Mr. Brown's car is green. (Circle it.)
Now we can fill in the table using the clues and the rules of inference.
"Mr. Brown's car is green" and "clue 1" → "Mr. Brown's house it white"
"Mr. Brown's car is green" and "clue 5" → "Mr. White's car is not green"
"Mr. White's car is not green" → "Mr. White's car is brown"
"Mr. White's car is brown" → "Mr. Green's car is not brown"
"Mr. Green's car is not brown" → "Mr. Green's car is white"
"Mr. Brown's house is white" → "Mr. Green's house is not white"
"Mr. Green's house is not white" → "Mr. Green's house is brown"
"Mr. Green's house is brown" → "Mr. White's house is not brown"
"Mr. White's house is not brown" → "Mr. White's house is green"

We used indirect reasoning for the above problem. We assumed Mr. Brown's car is white. This led to a statement that was equivalent to one of the clues being false. So the assumption that Mr. Brown's car is white could not be true, and so we were able to conclude, "Mr. Brown's car is not white." This enables us to reach the conclusion that Mr. Brown's car is green and it enables us to obtain the entire solution. This is a standard strategy to use whenever you get stuck in a matching problem.

## Exercises: Section 1.4.3

1. There were 5 fine ladies of Carruther,
   who named their pets after each other.
   From the following clues,
   can you carefully choose,
   the pet which belongs to Sue's mother?

   Toni Taylor owns a hog,
   Belle Bradowski owns a frog,
   Janet Jackson owns a crow,
   the garter snake is owned by Jo.

   Sue's the name they call the frog,
   and "Here Jo, Here Jo" brings the hog.
   The name by which they call the pony,
   is the name of the woman whose pet is Toni.
   The final clue, which I'll now tell,
   is that Sue's mother's pet is Belle.

2. Dana, Sarah and Brad each have a truck.
   The trucks are a Dodge, a Chev and a Ford.
   The trucks are white, blue or red.

   Clues:

   i) Dana's truck is not a Chev and is not blue.
   ii) If the Dodge is not Dana's, then it is white.
   iii) If the blue truck is either the Chev or belongs to Sarah, then the Ford is white.
   iv) If the Dodge is either red or white, then Sarah does not own the Chev.

   Find the colour and owner of each truck.

# 1.4 Appendix A: Logic and Truth Tables

**Simple logic** is a collection of rules that predicts the truth or falsehood of compound sentences built up from initial statements using connecting words or phrases, such as "and", "or", and "not", or "if". Logical rules can accommodate arbitrary truth values of the initial statements. For example, if the initial statements are denoted by **p** and **q**, then the possible truth values of the compound statements **p and q, p or q,** and **not-p** are determined by the truth or falsehood of p and q individually as shown here:

| p | q | p and q |
|---|---|---|
| T | T | T |
| T | F | F |
| F | T | F |
| F | F | F |

| p | q | p or q |
|---|---|---|
| T | T | T |
| T | F | T |
| F | T | T |
| F | F | F |

| p | not-p |
|---|---|
| T | F |
| F | T |

This kind of table is called a **truth table**. The truth table for **p and q** illustrates that the statement **p and q** will only be true when the statements p is true at the same time as the statement q is true. This is in agreement with our everyday use of the word "and" in English. The statement **p or q** has a different truth table, sort of opposite to that of **p and q**, which reflects the fact that the only way p or q can be false is when statement p is false at the same time as statement q is false. The statement **not-p** is the logical opposite to the statement p, so it will be true when p is false and false when p is true.

The rules of logic are absolute. A pair of **logically equivalent** statements will always have the same truth value given the same initial conditions concerning the truth of underlying statements. For example, the **double negative** statement **not-(not-p)** always has the same truth value as the initial statement **p**. We will write **≡** for logical equivalence, so for the double negative statement we have the law:

**not-(not-p) ≡ p.**

Other basic laws of logic, that can also be established using a truth table, are the following:

De Morgan's laws: **not-(p and q) ≡ (not-p) or (not-q); not-(p or q) ≡ (not-p) and (not-q)**
Commutative laws: **p and q ≡ q and p; p or q ≡ q or p**
Associative laws: **(p and q) and r ≡ p and (q and r); (p or q) or r ≡ p or (q or r)**
Distributive laws: **p and (q or r) ≡ (p and r) or (p or r); p or (q and r) ≡ (p or q) and (p or r)**
Idempotent laws: **(p or p) ≡ p; (p and p) ≡ p**

Inverse laws: **(p or not-p)** ≡ T**; (p and not-p)** ≡ F

The truth table for the conditional statement **if p then q**, which we write as **p → q** is

| **p** | **q** | **p → q** |
|---|---|---|
| T | T | T |
| T | F | F |
| F | T | T |
| F | F | T |

The only F in the truth table of **p → q** occurs when **p** is T and **q** is F. So we can understand the truth values of **p → q** from its negation law:

**not-(p → q) ≡ (not-p) and q.**

We actually use **p → q** statements in our personal lives by making clever promises, where we count on **p** being false. A parent might tell a child who repeatedly begs for a hamster but cannot swim:

If you swim a length of the pool, then I'll buy you a hamster.

As long as the child cannot make the antecedent true, the parent has no obligation to deliver the consequent. If the parent gives in and buys the hamster for some other reason, the parent has not gone against what was promised.

Many other idioms in English produce the same meaning as "**if p then q**". We note the following:

| | | |
|---|---|---|
| **p** implies **q** | all **p**'s are **q**'s | **q** when **p** |
| **q** is implied by **p** | **p** only if **q** | **q** whenever **p** |
| if **p**, **q** | when **p** then **q** | **q** is necessary for **p** |
| **q** if **p** | **p,** therefore **q** | **p** is sufficient for **q** |
| | **q** unless **~p** | |

## Exercises: Section 1.4 Appendix A

For exercises #1 to #10,

let **f** be the statement "I will go to the football game."
let **s** be the statement "The sun is shining."
let **b** be the statement "I will get a sunburn."
let **w** be the statement "I will wear sunscreen."

Write the following statements symbolically.

1. I will go to the football game if and only if the sun is shining.

2. The sun is not shining and I will not get a sunburn.

3. The sun is not shining but I will go to the football game.

4. I will wear sunscreen if I go to the football game, and I will not get a sunburn.

5. I will wear sunscreen or I will get a sunburn.

6. I will get a sunburn if I go to the football game and do not wear sunscreen.

7. If I go to the football game, then I will wear sunscreen.

8. If the sun is shining then I will wear sunscreen, or if the sun is not shining then I will not wear sunscreen.

9. I get a sunburn whenever the sun is shining.

10. If the sun is shining then I will wear sunscreen, and I will not get a sunburn.

11. The scene is a courtroom. The prosecutor says, “If the defendant is guilty, then he had an accomplice.” “That’s not true!” shouted the defence attorney. The judge agreed and sentenced the defendant to one year in jail. Explain the judge’s action.

For exercises #12 to #19, construct the truth table for each statement.

12. **not-(p and q)**

13. **(not-p) or (not-q)**

14. **q → p**

15. **(not-q) or p**

16. Identify the pairs of statements above that are logically equivalent.
Is **q → p ≡ p → q** ? Explain.

# 1.5 Unit Review Exercises

Find the next two terms in each of the sequences.

1. 1, 2, 3, 6, 11, 20, ___, ___, …

2. 31, 32, 37, 62, 187, ____, ____, …

3. Cindy raises ducks and goats. When she is in her yard there are 18 heads and 50 feet. How many ducks and how many goats does she have?

Find the last digit (units digit) of the following:

4. $3^{45}$

5. $7^{90}$

Find the last two digits of the following:

6. $4^{2000}$

7. $11^{71}$

8. Several park rangers must cross a deep river at a point where there is no bridge. The rangers spot two Boy Scouts in a small rowboat. The rowboat can only hold two Boy Scouts or one ranger. All the rangers and Boy Scouts get across the river. How?

9. Four people are in a treacherous cave and need to get to the other side. They can travel at most two at a time, but need the sole flashlight to make their journey across the narrow path. They know the flashlight has only 12 hours of power. They are aware that each of them travels at different maximum speeds: one travels either way in one hour; another takes two hours; another takes four hours; and another takes five hours. How do the 4 people get to the other side before the flashlight dies?

10. Joe Johnson plays poker every Wednesday night. One week he tripled his money, but then lost $6. He took his money back the next week, doubled it, but then lost $20. The following week he tried again, taking his money back with him. He quadrupled it and took that much home with him, a total of $40. How much did he start with the first week?

11. Brooke and Brittany just received their allowance. Brooke tells Brittany, "If you give me one dollar, then we will have the same amount of money." Brittany then replies, "Brooke, if you give me one dollar, I will have double the amount of money you are left with." How much is each girl's allowance?

12. A triangle of pennies is made as in Figure a) below. What is the least number of pennies that can to be moved to turn the triangle pattern upside down as in Figure b)?

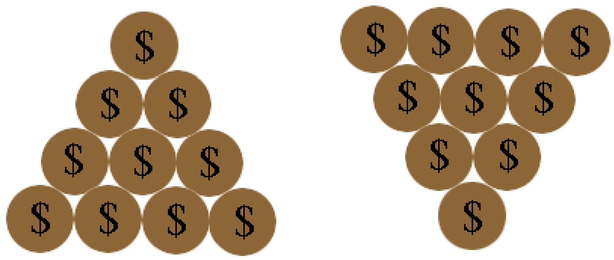
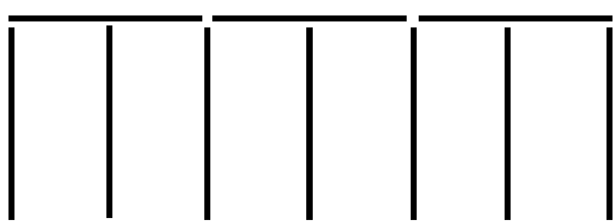


Figure a) Figure b)

13. The drawing below shows how a farmer used thirteen toothpicks to make six identical sheep pens. Unfortunately, one of the toothpicks was damaged. Use twelve toothpicks to show how the farmer can still make six identical pens.



14. The mathematician Augustus De Morgan lived in the nineteenth century. He once made the statement: "I was $x$ years old in the year $x^2$. " In what year was De Morgan born?

15. If you take 7 bowling pins from 10 bowling pins, how many bowling pins do you have?

16. The junior group at daycare is standing in a circular arrangement. They are evenly spaced and marked in numerical order. The fourth child is standing directly opposite the twelfth child. How many children are there in the junior group?

17. If you were to construct a fence 60 metres long with the posts spaced 3 metres apart and with no gates, how many posts would you need?

18. What is the sum of all the numbers from 79 to 178?

19. In a 30-term arithmetic progression, the second term is 9 and the eleventh term is 40.5. Find the common difference and the sum of the progression.

20. In a geometric progression, the third term is 81 and the seventh term is 16. Find the common ratio and the first 8 terms in the sequence.

21. The sum of seven consecutive integers always satisfies which of the following?
    a) odd
    b) a multiple of 7
    c) even
    d) a multiple of 4
    e) a multiple of 3

22. A girl has as many brothers as sisters; her brother has twice as many sisters as brothers. How many offspring are there?

23. Judi agreed to work for 1 year. At the end of that time she was to receive $2 400 and one horse. After 7 months she quit the job to become a math professor, but still received the horse and $1 000. What was the value of the horse?

24. Two runners start at the same point and run in opposite directions. One runs at 6 miles per hour and the other runs at 8 miles per hour. In how many hours will they be 21 miles apart?

25. A census-taker knocks on a door. A woman answers, and the census-taker asks her for the number of children living in the household and their ages. The woman, annoyed at this waste of her time, says "There are three children, the square root of the product of their ages is 6, their average age is $4\frac{1}{3}$, and the oldest likes ice cream." Then she slams the door. The census-taker, quite satisfied, goes on to the next house. What are the ages of the three children?

26. If one hen costs 30 cents, how much will 8 eggs cost at 2 cents an egg?

27. What is the smallest amount of change that cannot be produced using 10 or fewer coins (allowing 1¢, 5¢, 10¢, 25¢, 50¢, \$1, and \$2 coins)?

28. Smith and Jones between them raised 120 cantaloupes. They each took 60 and sold them around the neighbourhood. Smith sold half of his at the rate of 2 for \$1, and the other half at the rate of 3 for \$1. Jones sold all of his at the rate of 5 for \$2. How much more did Smith make than Jones?

29. If $(1 + 3 + 5 \ldots + p) + (1 + 3 + 5 + \ldots + q) = (1 + 3 + 5 + \ldots + 19)$ with $p$ and $q$ both odd numbers, evaluate $p + q$.

30. Find a 10-digit number whose first digit gives the number of zeroes, the second gives the number of ones, the third gives the number of twos, …, and the tenth gives the number of nines.

31. Why study problem-solving at university? That's easy! So you can win the top prize money on some reality-based TV program. Here is a problem that was featured in Episode 4 of *The Mole 2* (the series on ABC television a few years back).

    Bob went out to buy some fishing equipment. He spent half of what he had plus \$5 at the first store. At the second store, he spent half of what was left plus \$4, and at the third store, he spent half of the remainder plus \$3. He then had \$5 to put aside for bait. How much did he start with?
    (Warning: Don't mix up what he spent with what was is left.)

32. A square floor is tiled with standard square tiles; all tiles are white except for the 101 tiles of the two main diagonals (from corner to corner), which are black. How many tiles are there in all?

33. The five binomials $2x + 1$, $2x - 3$, $x + 2$, $x + 5$, and $x - 3$ can be arranged in a different order so that the first three have the sum $4x + 3$ and the last three have the sum $4x + 4$. What must the middle term be?

34. A man walks to his friend's house at two km/h. He spends an hour eating lunch, and then he rides home on his friend's bicycle, five times faster than he walked. If the distance to his friend's house is ten km, at what time must he leave home in order to return by 4 p.m.?

35. A snail is climbing out of a well. The well is 20 meters deep. Every day, the snail climbs up three meters, and every night, it slips back two meters. How many days will it take to get out of the well?

36. Weighing a baby at the clinic was a problem, as the baby would not keep still causing the scales to wobble, so I held the baby and stood on the scale. The nurse read off 79 kg. Then the nurse held the baby while I read off 69 kg. Finally I held the nurse while the baby read off 137 kg. What is the combined weight of all three?

37. Three men enter a hotel and rent a room for $30. After they are taken to their room, the manager discovers that he had overcharged them; the room actually rents for only $25. He thereupon sends a bellhop upstairs with the $5 refund. The dishonest bellhop decides to keep $2 and returns only $3 to the men. Now, the room originally cost $30, but the men had $3 returned to them. This means that they paid $27 for the room. The bellhop kept $2. 27 + 2 = 29. What happened to the extra dollar? Inflation?

38. We have 12 rods, each 13cm long, that are to be cut into pieces measuring 3, 4, and 5 cm and the resulting pieces assembled into 13 triangles with side lengths of 3, 4, and 5cm. How long should they be cut?

39. Jones owed Smith one dollar. When Smith asked him for it, Jones pulled a two-dollar coin out of his pocket, and asked Smith if he could take it out of that.

“Sorry,” said Smith, “I can’t.”
“Sorry,” said Jones not sounding very sorry, “that’s all I have, except a fifty dollar bill.”
“Oh, I can take it out of that very easily,” said Smith and proceeded to do so.

How did he do it?

40. There are five people at a family party consisting of two fathers, two mothers, one grandfather, one grandmother, two sons, and one grandson. How is this possible?

41. You have 15878 equilateral triangles whose sides are 1 cm long. You want to put these triangles together to make a large mosaic in the shape of an equilateral triangle.

    i. What is the side-length of the largest possible mosaic you can make with your triangles?

    ii. How many pieces will be left over?

Decide whether the given arguments in exercises #40 to #46 are valid or invalid.

42. Some teachers are absent-minded.
Mrs. Milne is a teacher.
___
Mrs. Milne is absent-minded.

43. All frogs are green.
Kermit is a frog.
___
Kermit is green.

44. If I do not study, I'll get a bad grade.
I study
___
I'll get a good grade.

45. All vehicle-owners have a mechanic.
Kyle has a mechanic.
___
Kyle owns a vehicle.

46. All birds fly.
All planes fly.
___
A bird is not a plane.

47. All zigs are zags.
All zags are zoops.
___
All zigs are zoops.

48. All kicks are wicks.
Some wicks are knicks.
___
Some kicks are knicks.

49. Translate the following into symbolic notation and prove that "I play golf and I go skiing" is the valid conclusion.

If I don't play golf (**p**) or I don't go skiing (**s**), then I have had a rotten year (**r**).
If I have had a rotten year, then I'm grumpy (**g**) or I make up hard exams (**h**).
I'm not grumpy and I'm not old (**o**).
If I'm not old, then I do not make hard exams.

50. Every statement made by a fortune-teller on Monday through Friday is false. Every statement she makes on a weekend is true.

a) On which days of the week can she make the statement, "I lied yesterday"?

b) On which days of the week can she make the statement, "I will be lying tomorrow"?

c) On which days of the week can she make the statement, “I lied yesterday and I will be lying tomorrow”?

51. There was a blind beggar who had a brother, but this brother had no brothers. Can you explain how this is possible?

52. Suppose that you ask someone for the time and you get the following response:

“If I tell you the time, then we’ll start chatting. If we start chatting, then you’ll want to meet me at a truck stop. If we meet at a truck stop, then we’ll discuss my family. If we discuss my family, then you will find out that my daughter is available for marriage. If you find out that she is available for marriage, then you’ll want to marry her. If you marry her, then my life will be miserable since I don’t want my daughter to marry someone who can’t afford a $10 watch.”

Use deductive reasoning to draw a valid conclusion from what the man said.

53. Roger Bacon (1214-1292), one of the most important names in medieval science, was once called upon to answer the question whether plants feel. The question was easy for him because of the central scientific principle of his time: *An object exists in order to serve some real purpose in the total economy of nature*. “The purpose served by feeling,” he replied, “is to enable one to move either toward or away from an object exciting it. A plant is stationary and cannot move. Therefore feeling would be unnecessary to a plant, and nature would have given feeling to it in vain. But nature does nothing in vain. Therefore the plant cannot feel.”

a) Formulate Bacon’s argument letting **e** stand for the statement “nature has given existence to feeling,” **s** stand for “feeling serves a purpose,” and **m** stand for “the purpose served by feeling is movement.”

b) Explain why the argument is valid. Do you find the argument convincing?

# Unit 2
# Arithmetic

## 2.1 Numbers and Counting

All of us learned how to count in early childhood by learning the names of the **counting numbers**…. zero, one, two, three, four, five, six, seven, eight, nine, ten, eleven, twelve,... Our system for writing numbers is a **base-10** system. We normally use an alphabet of ten symbols, the **single digits** 0,1,2,3,4,5,6,7,8,9 to denote the counting numbers from zero to nine, and use a place value positioning system to count by multiples of 10 beyond these.

In our first experiences, numbers were always used to count things, like the number of fingers someone was holding up, a number of sticks, or toy blocks. We learned the words for counting numbers first, and learned the symbols later. Most school textbooks use the following base 10 block representations for representing counting numbers:

The single basic item is the small block, sometimes called a *cubie*:  which is also represented as a single dot: ●

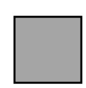
Counting numbers less than 10 are represented by rows of dots:

● for 1, ●● for 2, ●●● for 3, ●●●● for 4, and so on, up to ●●●●● ●●●● for 9.

It is convenient to gather sets of 5 together because it helps us count faster and more accurately.

Ten dots are represented by a single *rod*: | represents 10**.**

Counting more than one rod is counting by tens. So ||||| ||| ●●●●● ●● is 8 tens and 7 dots, so 87**.**

Once we have 10 rods, we have one hundred dots. We represent 100 with a square: 

This reinforces the idea that 100 is “10 squared”, which we write as $100 = 10^2$. It is also useful for connecting the concepts of multiplication with area calculations. Ten sets of ten is also the meaning of the multiplication $10 \times 10$, which we can represent as the power $10^2$, and say “10 squared”. Looking back, 10 is represented by the power $10^1$, and

1 by the power $10^0$. So there is a relationship between the number of digits being used and the power of 10 involved. This continues with the higher powers of 10:

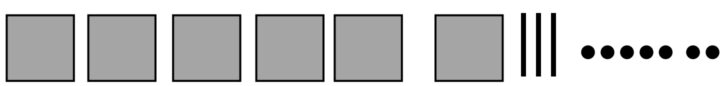
1 000 = $10^3$ (one thousand),
10 000 = $10^4$ (ten thousand),
100 000 = $10^5$ (one hundred thousand),
1 000 000 = $10^6$ (one million)...

(Gathering digits in groups of three like this makes it easier to recognize the size of the number. This has become the standard in recent years.) In general,

**the number represented by a 1 followed by *n* zeroes is $10^n$.**

We do have a few more names commonly used for very large numbers (billion, trillion, etc.), but we run out of new names for large numbers eventually.

Base 10 block representations for numbers in the 100's consist of collections of squares, rods, and dots. So, for example, the simplest way to represent 637 would use 6 squares, 3 rods, and 7 dots:

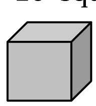


Our base-10 place value system for writing numbers helps us think of numbers in terms of their relationship to the powers of 10. Each of the digits has a place: the 7 is in the **ones** (or **units)** place, the 3 is in the **tens** place, and the 6 is in the **hundreds** place.

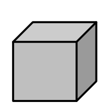
That is,
$$637 = 600 + 30 + 7$$
$$= (6 \times 100) + (3 \times 10) + (7 \times 1)$$
$$= (6 \times 10^2) + (3 \times 10^1) + (7 \times 10^0)$$

Writing a number in base 10 like this is called writing it in **expanded notation.**

Once we get to 10 squares, we have one thousand, which we represent with a larger block, or cube:

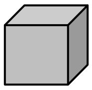
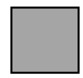


Again, this reinforces the connection between volume calculation and the product of three numbers. We also write 1 000 as $10^3$ and say "10 cubed". The extra space in the 1 000 not only helps us to read the number easier, it also connects with the base 10 block representations, since both the units place and the thousands place are represented by the same basic shapes, cubes. If we want to appreciate what larger numbers represent, we can think of zooming out from the basic size to a new perspective. For example, we can think of one million, 1 000 000, as being represented by a large block if we zoom out so that 1 000 is represented by the small block:



If 1 000 =  then 1 000 000 = .

This pattern for place values can also be used to represent decimal place values, for which we think of by zooming in:

If 1 =  then one one thousandth, 0.001, is represented by  or a ●, one one hundredth, 0.01, by , and one tenth, 0.1, by a .

## Exercises: Section 2.1

1. Simplify the expression, writing it as a single number.
   $( 9 \times 10^7 ) + (6 \times 10^4 ) + (3 \times 10^3 ) + (4 \times 10) + 1$

2. Expand each number, writing it in terms of its sum of multiples of powers of 10.

   a) 12 345　　　b) 1 900 803　　　c) 5 007 082 000

3. Count the dots. Give the simplest base 10 block representation of your answer.

   a)
   ●●●●● ●●●●● ●●●●● ●●●●● ●●●●● ●●●●●
   ●●●●● ●●●●● ●●●●● ●●●●● ●●●●● ●●●

   b)
   ●●●●● ●●●●● ●●●●● ●●●●● ●●●●● ●●●●● ●●●●● ●●●●●
   ●●●●● ●●●●● ●●●●● ●●●●● ●●●●● ●●●●● ●●●●● ●●●●●
   ●●●●● ●●●●● ●●●●● ●●●●● ●●●●● ●●●●● ●●●●● ●●●●●
   ●●●●● ●●●●● ●●●●● ●●●●● ●●●●● ●●●●● ●●●●● ●●●●●
   ●●●●● ●●●●● ●●●●● ●●●●● ●●●●● ●●●●● ●●●●● ●●●●●
   ●●

   c)
   ●●●●● ●●●●● ●●●●● ●●●●● ●●●●● ●●●●● ●●●●● ●●●●●
   ●●●●● ●●●●● ●●●●● ●●●●● ●●●●● ●●●●● ●●●●● ●●●●●
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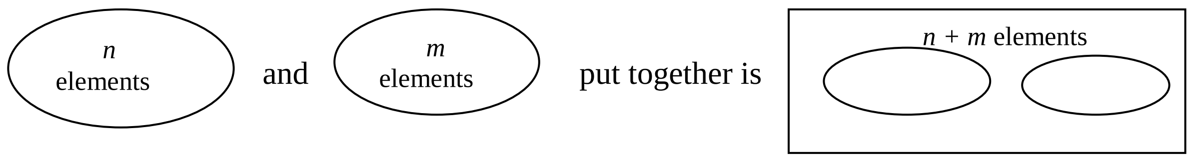
# 2.2 Addition

**Addition** is the most basic of number operations. It is a simple generalization of counting, since counting is just repeatedly adding 1 to the previous number. The meaning of the problem adding two counting numbers *n* and *m* is simply to find the size of the set you would have if you combine a set of size *n* and a set of size *m* into one set.




For example, if you had 26 sticks and then found 17 more, in order to find out how many sticks you would have now, you would need to count all your sticks:

||||| |||||   ||||| |||||   ||||| |   ← 26 sticks, and

||||| |||||   ||||| ||   ← 17 sticks.

To count all of these, it is best to first look for as many sets of ten as we could find and then count what is left over. This is the concept behind the **left-to-right algorithm** for addition. Since we see 4 sets of ten sticks and 3 more, we have 43 sticks. Symbolically, we write this solved addition problem as the equation $26 + 17 = 43$.

As obvious as this looks, it probably is not the way most adults would approach the addition problem 26+17. In the **traditional or standard (right-to-left) algorithm** for addition most current adults learned in school, we first add the digits in the ones place: 6+7=13, then carry the 1 ten in this 13 and add it to the sum of the digits in the tens place: 1+2+1=4, to get 4 tens and 3 more, so 43.

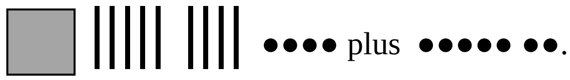
We can also model the addition using base 10 blocks. When we do this, the patterns are closely connected to the traditional algorithm. Represent both numbers, 26 and 17, and combine the collections into one:

|| ●●●●● ● plus | ●●●●● ●● equals ||| ●●●●● ● ●●●●● ●●.

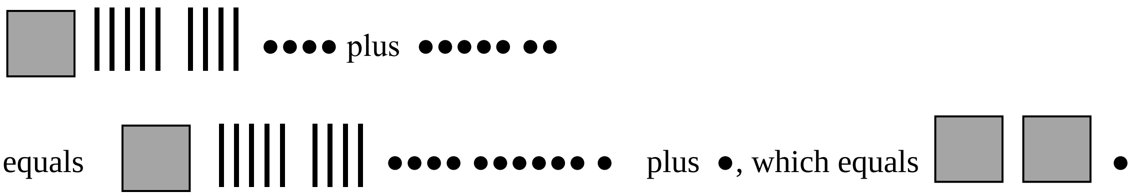
Since we have more than 10 single dots, gather two sets of five of them to make an additional rod. This models the "carrying" idea in the traditional addition algorithm.

||| ●●●●● ● ●●●●● ●● equals |||| ●●●. So the answer is 43.

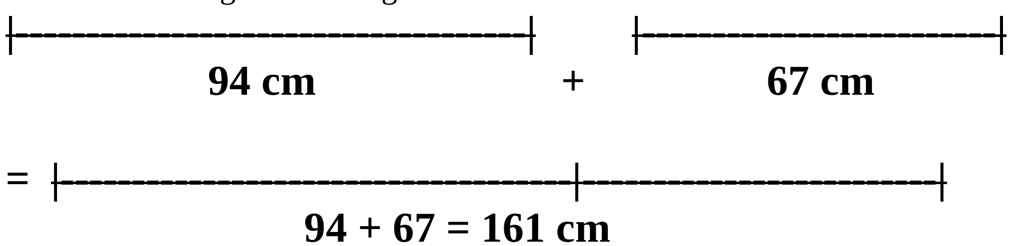
The base 10 block model for addition provides a good framework for addition by a single digit, which is fundamental to any style of addition. For example, if we add 194 + 7, then we would model this with




Take 6 of the extra 7 dots to make the next set of 10, i.e. a 10th rod. The ten rods make a new square, and we still have one dot left. So the whole solution looks like

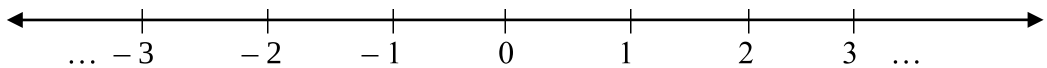



Another way to model addition spatially is by joining lengths on a number line to make one longer segment. For example, to see what it means to add a length of 94 cm to a length of 67 cm, we visualize two line segments of these lengths joined end-to-end to make one line segment of length 94 + 67 cm.




Notice that we did not need to discuss "how" to find the sum 94 + 67 again. In fact, we feel 2-digit addition problems, and even some 3-digit addition problems, are easy enough to be done completely in one's head. Challenge yourself by trying to do the first 8 problems at the end of this section in your head using the left-to-right method. One important advantage of the left-to-right algorithm over the traditional algorithm is the ease with which it blends into estimation. The importance of being able to easily and quickly estimate sums is increasingly important in a fast-paced technological society where calculator applications are readily available whenever we need a precise calculation. For example, consider the calculation 854 + 367. The person whose instincts are to use left-to-right addition immediately knows the answer is more than 1100. The person used to a right-to-left algorithm has to separately learn a new method for estimation. For this reason, we encourage the reader to train themselves to use left-to-right algorithms for all number operations.

When used for counting things or describing measurements, counting numbers are being used to represent **quantities**. Quantities used for measurement do not always have to be whole numbers, and they are naturally ordered on a line we call the **real number line**:




Numbers used to represent quantities include fractions and decimal numbers, and for certain types of scales like temperature, negative numbers as well. The representation of numbers on a line is useful for introducing number comparison concepts such as the size of a number (absolute value) and inequalities.

We conclude this section with some simple calculation problems to illustrate the use of left-to-right algorithms.

Problem 1): 854 + 367?

We will go through the mental steps needed to add these in one's head using the left-to-right method. First add the digits in the hundreds place: 8 + 3 = 11, write this in place of the 8, and remove the 3 in the second summand.

854 + 367 = 1154 + 67

Now add the tens: that is, 115 + 6. We need to add 5 to 115 to get the next multiple of 10, so take this 5 from the 6:

115 + 6 = 115+5+1 = 120 + 1 = 121.

So now we have:

1154 + 67 = 1214 + 7

Now add the ones: we need to add 6 to 1214 to get the next multiple of 10, so

1214 + 7 = 1214 + 6 + 1 = 1220 + 1 = 1221.

So 854 + 367 = 1221.

In looking back at the use of this algorithm, note that it is closely related to what is familiar about counting. The concept of carrying in the traditional algorithm is replaced by gathering what you need to make the next multiple of 10 from the next digit you add.

With the traditional algorithm, addition problems become more difficult when they involve several numbers and the sums of digits involved are often more than 10. Try one of these with the left to right method:

Problem 2): 84 061 + 70 353 + 6 687?

In the traditional method larger problems are always written vertically so that we can clearly see how the place values line up. This also helps with the left-to-right method, though it would not be absolutely required. When you add, it is good practice to take advantage of combinations of numbers that add together to give multiples of 10 wherever you see them:

$$\begin{array}{r} 84\,061 \\ 70\,353 \\ +\ 6\,687 \\ \hline \end{array} \rightarrow \begin{array}{r} 154\,061 \\ 353 \\ +\ 6\,687 \\ \hline \end{array} \rightarrow \begin{array}{r} 160\,961 \\ 53 \\ +\ \ 87 \\ \hline \end{array} \rightarrow \begin{array}{r} 161\,001 \\ 13 \\ +\ \ 87 \\ \hline \end{array} \rightarrow \begin{array}{r} 161\,001 \\ \\ +\ \ 100 \\ \hline 161\,101 \end{array}$$

## Exercises: Section 2.2

Perform the following addition problems. Challenge: Do this completely in your head! (Just think them through using the left-to-right algorithm, and write only the answer.)

| | | | |
|---|---|---|---|
| 1. 43 + 73 | 2. 56 + 24 | 3. 92 + 38 | 4. 104 + 29 |
| 5. 160 + 473 | 6. 776 + 824 | 7. 1 041 + 78 | 8. 5 526 + 3 291 |

Find the indicated sums using the left-to-right method:

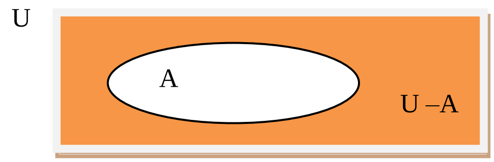
9. $$\begin{array}{r} 6\,732 \\ 566 \\ +\ \ 614 \\ \hline \end{array}$$

10. $$\begin{array}{r} 167\,912 \\ 92\,312 \\ +\ 123\,128 \\ \hline \end{array}$$

11. $$\begin{array}{r} 436 \\ 552 \\ 324 \\ 763 \\ +\ 141 \\ \hline \end{array}$$

# 2.3 Subtraction

Subtraction is the "take away" operation. It models the act of taking away things that have previously been counted. We can illustrate this with sets by using the difference of sets: If the set U has *n* elements and its subset A has *m* elements, then the complementary set U – A has *n* - *m* elements.



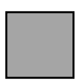

Another way to think of subtraction is to think of counting backwards. For example, to do 103 – 7, we can think of counting backwards 7 from 103. We need to count backwards 3 from 103 to get to 100, and then count back 4 more from 100 to get 96.

103 – 7 = 103 – (3 + 4) = 100 – 4 = 96.

This way of thinking reinforces how a negative sign distributes through a sum, and demonstrates the importance of using parentheses to organize a sequence of operations:

103 – (3 + 4) = 103 – 3 – 4.

In our base 10 block model, to do 103 – 7, the operation corresponds to the "borrowing" idea in the traditional method:

 ●●● minus ●●●●● ●● equals ||||| ||||| ●●● minus ●●●●● ●●,

which is equal to ||||| |||| ●●●●● ●●●●● ●●● minus ●●●●● ●●.

Now we can directly take the 7 dots away from the 13 dots, leaving 6 dots, so the answer is ||||| |||| ●●●●● ●. Again, this provides the framework for subtraction by a single digit that every subtraction algorithm requires.

Once one has mastered subtractions by a single digit, the left-to-right method for subtraction can be seen as a sequence of these operations:

| | | |
|---|---|---|
| 1011 – 827 | = 211 – 27 | (since 10 – 8 = 2) |
| | = 191 – 7 | (since 21 – 2 = (21 – 1) – 1 = 20 – 1 = 19) |
| | = 184. | (since 91 – 7 = (91 – 1) – 6 = 90 – 6 = 84) |

For larger problems, you may want to write the numbers in a column as we did with addition, so that the place values all line up:

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 120 060 | | 50 060 | | 41 060 | | 40 260 | | 40 170 |
| - 79 898 | → | - 9 898 | → | - 898 | → | - 98 | → | - 8 |
| | | | | | | | | 40 162 |

You might also copy the numbers that remain below the previous step:

$$\begin{array}{r} 16\,001 \\ \underline{-\ 8\,763} \\ 8\,001 \\ \underline{-\quad 763} \\ 7\,301 \\ \underline{-\qquad 63} \\ 7\,241 \\ \underline{-\qquad\ \ 3} \\ 7\,238 \end{array}$$

**Estimation shortcuts for addition and subtraction**: People that are good at mental calculation always look for shortcuts to save time or give alternatives to check answers. With addition or subtraction we are able to take advantage of an estimation shortcut when we add or subtract a number that is close to a multiple of a power of 10. Consider these problems:

3 248 + 997: Since 997 = 1 000 – 3,
3 248 + 997 = 3 248 + (1 000 – 3) = (3 248 + 1 000) – 3 = 4 248 – 3 = 4 245.

78 091 – 19 966: Since 19 966 = 20 000 – 34,

$$\begin{aligned} 78\,091 - 19\,966 &= 78\,091 - 19\,996 \\ &= 78\,091 - (20\,000 - 34) \\ &= (78\,091 - 20\,000) + 34 \\ &= 58\,091 + 34 \\ &= 58\,125 \end{aligned}$$

**Negative numbers**

A natural question that arises with subtraction is what happens if a larger number is taken away from a smaller one: If we know how to do 10 – 8, what is 8 – 10? This can be the student's first encounter with negative numbers. For computations that involve negative numbers, we can either use the distribution of negative signs again:

8 – 10 = – ( 10 – 8 ),

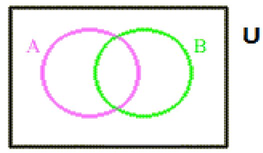
or think of counting backwards to 0 and then keep going backwards into the negative numbers:

8 – 10 = 8 – (8 + 2) = 0 – 2 = – 2.

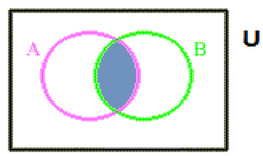
We have included some problems involving negative numbers in the exercises for your practice.

## Systematic Counting

Now that we know two arithmetic operations, we can consider problems that involve both. Counting problems involving sets are easy when all the sets are disjoint, but when the sets overlap, then a Venn diagram is often helpful. When we deal with problems involving two sets, we can in general put them put them both at once inside some larger set which we think of as a **universal** set, which is represented by the outside box in our Venn diagram. The general Venn diagram for two sets A and B with universal set U looks like this:

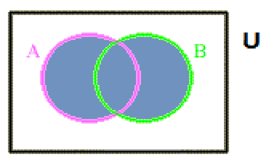



There are several other subsets of U determined by the two sets A and B. In the previous unit we encountered the **intersection** $A \cap B$, the set of things that are common to both A and B:



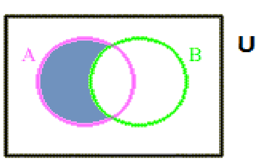

$A \cap B$

We also have their **union**, $A \cup B$, which is the set of things that are either in A or in B.

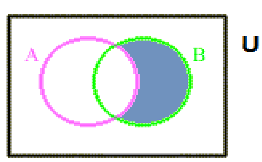



$A \cup B$

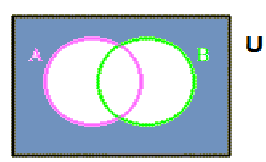
Other non-overlapping subsets determined by A and B are defined by **set differences**. There are three of these: $A – (A \cap B)$, $B – (A \cap B)$, and $U – (A \cup B)$:




$A – (A \cap B)$

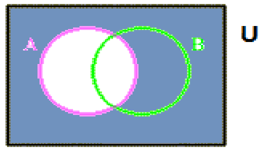



$B – (A \cap B)$

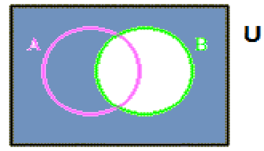



$U – (A \cup B)$

$U – (A \cup B)$ can also be identified by writing it as a complement: $(A \cup B)^c$.
We also have the two complements $A^c = U – A$ and $B^c = U – B$:




$A^c$



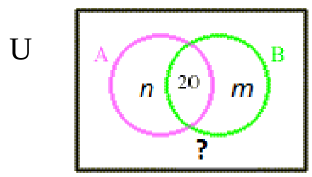

$B^c$

It should not be surprising that there are several ways to represent each subset of U using these operations. To stay organized, keep these basic pictures in mind.

Visualizing the relationship between two sets using the Venn diagram model gives us a means to represent many variations of counting problems. For example, consider the following problem:

**Problem:** A universal set U has 50 elements, of which 40 are in the subset A and 30 are in the subset B. If 20 are in the intersection A ∩ B, how many elements of U lie outside A ∪ B?

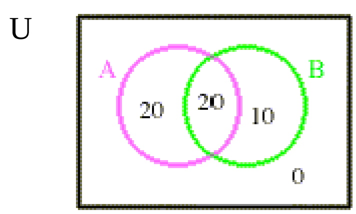
The Venn diagram for this problem looks like this:




We label it this way because we want to know the number of elements in the non-overlapping subsets of U. Write |S| for **the number of elements in the set S**.
We are given that |U| = 50, |A| = 40, |B| = 30, and |A ∩ B| = 20.
We want to find |U – (A ∪ B)|.

Our strategy will be to work our way out from the middle:
A has 40 elements, A ∩ B has 20, so $n$ = |A – (A ∩ B)| = 40 – 20 = 20.
B has 30 elements, A ∩ B has 20, so $m$ = |B – (A ∩ B)| = 30 – 20 = 10.
The number of elements of A ∪ B is the sum of the numbers of elements in each of its three non-overlapping subsets, so |A ∪ B| = 20 + 20 + 10 = 50.
Since |U| = 50, the elements of A ∪ B account for all the elements of U, so |U – (A ∪ B)| = 0.

**Looking back:**
We found that the final distribution of elements of U looks like this:



So we can actually answer any problem about the distribution of elements of U relative to the sets A and B. The idea behind this problem is a fundamental principle of systematic counting called the **inclusion-exclusion principle**:

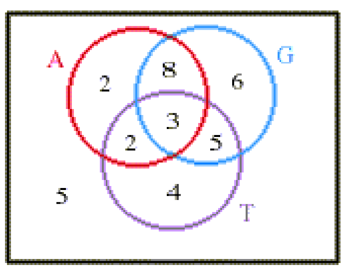
$$|A \cup B| = |A| + |B| - |A \cap B|.$$

To understand it, one can think of trying to count all the elements of $A \cup B$ by first counting all the elements of A, then counting all the elements of B, and then subtracting the number of elements in $A \cap B$, which makes sense because you have counted them *twice*.

Try another example involving three sets:

**Problem:** Among a group of 35 high-school students 15 are studying algebra, 22 are studying geometry, 14 are studying trigonometry, 11 are studying both algebra and geometry, 8 are studying geometry and trigonometry, 5 are studying algebra and trigonometry, and 3 are studying all three of these subjects. How many students are studying only algebra? How many are studying none of the three subjects?

Begin by labelling the three sets A, G, and T inside the universal set U of the 35 high-school students. Then put 3 in the region common to all subjects (namely $A \cap G \cap T$). Next, fill in the regions representing exactly two subjects (for example, the number in A and G but not in T will be 11 - 3 = 8 students, etc.). The procedure is the same no matter how many sets we start with: first fill in the region where all sets intersect, then (by subtraction) fill in the regions where all but one intersect, etc. Just be careful not to count any element twice. You should eventually conclude that 2 study only algebra (since 2 is the size of $A \cap G^c \cap T^c$), while 5 study none of the subjects (because 5 is the number of elements in $A^c \cap G^c \cap T^c$).



## Exercises: Section 2.3

Do the following calculations. Use a shortcut if it will help. Challenge: Try to do them in your head!

1. 73 – 29
2. 56 – 24
3. 92 – 38
4. 104 – 69
5. 473 – 188
6. 824 – 776
7. 1 001 – 578
8. 5 526 – 3 291
9. 54 275 + 9 980
10. 7 989 – 1 098

Find the indicated differences using a left-to-right method.

11. 1 222 222 – 999 987
12. 220 043 – 139 576
13. 100 050 – 7 777
14. 7 777 – 100 050
15. – 423 – 5790
16. – 2 571 + 690

17. Students were asked to name their favourite free Regina newspaper. This is the result of the survey:

    33 like *the Carillon*
    32 like *The Metro*
    28 like *The Prairie Dog*
    11 like *the Carillon* and *The Metro*
    15 like *the Carillon* and *The Prairie Dog*
    14 like *The Metro* and *The Prairie Dog*
    5 like all three
    7 like none of these newspapers

    a) How many students took part in the survey?

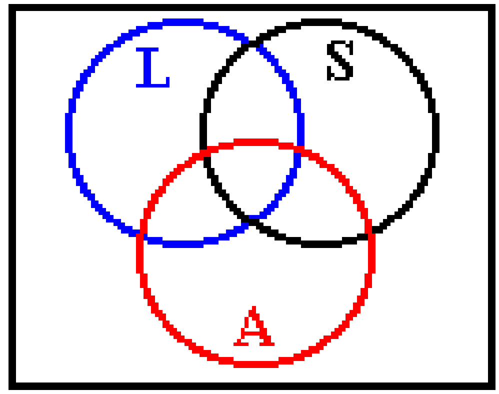
    b) How many students only like *the Carillon*?

    c) How many students like exactly two newspapers?

18. In a survey of 100 University of Regina students, it was found that

    50 receive government loans,
    55 receive private scholarships,
    40 receive financial aid from the university,
    20 receive government loans and private scholarships,
    15 receive government loans and aid from the university,
    30 receive private scholarships and aid from the university, and
    10 receive income from all three sources.

a) Fill in the Venn diagram where L corresponds to the set of students who receive government **L**oans, S corresponds to the set of student with private **S**cholarships, and A corresponds to the set of students receiving **A**id from the university.




b) How many of the 100 surveyed students receive income from none of these three sources?

c) How many of the 100 surveyed students receive private scholarships, but not a government loan?

d) How many of the 100 surveyed students do not have a private scholarship?

# 2.4 Multiplication

Multiplication of counting numbers is just repeated addition. We have already found multiplication necessary for explaining our base-10 numeration system when we talked about larger numbers and expanded notation. The fact that 20 is 2 sets of 10 is another way to say $20 = 2 \times 10$, that 10 sets of 10 is 100 is expressed by saying $10 \times 10 = 100$ or $100 = 10^2$. In any multiplication equation $m \times n = p$, $m$ and $n$ are the **factors** and $p$ is their **product**.

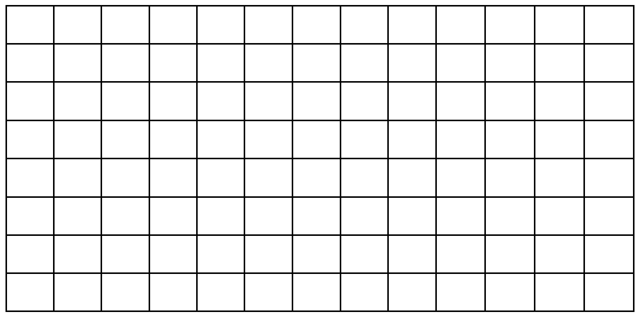
The repeated-addition model for multiplication can be used to expose its properties. Consider the problem $3 \times 7$: This means 3 sets of 7, which is $7 + 7 + 7$. If three people have 7 sticks each and they put them all together, the sticks can be arranged like this:

||||| ||
||||| ||
||||| ||

It is easy to see that if we count these in columns instead of in rows, we would be counting 7 sets of 3, which is $7 \times 3$. So $3 \times 7 = 7 \times 3$. This reasoning can easily be seen to work with any number of sticks instead of 7, and any number of rows instead of 3. This idea is the commutative law for multiplication: $\boldsymbol{n \times m = m \times n}$.

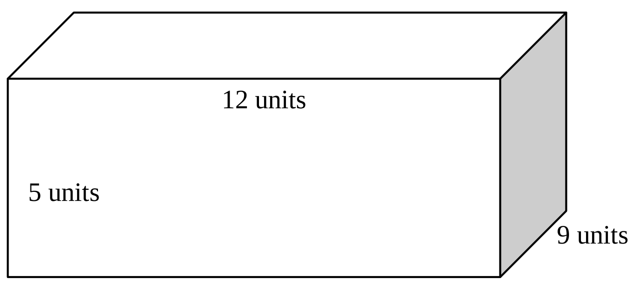
Another law of multiplication we can justify using this model is distributivity. We can view 7 as $5 + 2$, and count the sticks in the first 5 columns separately from those in the last two. This would give the equation $3 \times (5 + 2) = (3 \times 5) + (3 \times 2)$. Again, this can be done whenever the second factor is the sum of two numbers, so it justifies the distributive law: $\boldsymbol{n \times (m + k) = (n \times m) + (n \times k)}$.

Another view of multiplication is the area model. Suppose we want to count the number of squares in a rectangular array that is 13 squares wide and 8 high:



We can count them one-by-one, but it is better to recognize that there are 8 rows of 13 each, so there are $8 \times 13$ squares. If each square has the same unit of area, then $8 \times 13$ is also the number of square units that measures the area of the rectangle. In general, the multiplication $\boldsymbol{m \times n}$ will represent the area of a rectangle that is m units wide and n units tall. This view of multiplication can be used when $\boldsymbol{m}$ and $\boldsymbol{n}$ are arbitrary nonnegative numbers.

This also works for volume: if we want to know the volume of a rectangular solid that is 5 units tall, 9 units wide, and 12 units long, then it is not easy to count all the single cubic units in the solid one-by-one.




However, if we view the solid as having 5 layers with $12 \times 9$ cubic units in each layer, then we can see that the volume is $5 \times (12 \times 9)$. Turning the solid onto its front side, we would see the solid as 9 layers with $5 \times 12$ cubic units in each layer, so its volume is $(5 \times 12) \times 9$. So this is telling us that $5 \times (12 \times 9) = (5 \times 12) \times 9$. Again this will work any solid with counting numbers as its side lengths, so we can generalize this to get the associative law for multiplication: $\boldsymbol{n \times (m \times k) = (n \times m) \times k}$.

Other models for multiplication are also useful. Seeing multiplication of positive numbers as the *scaling* of a length is used in spatial reasoning. For example: we can think of $2 \times 17$ as the length 17 being doubled in size, and $\frac{2}{3} \times 18$ as being two-thirds of 18. Another model for multiplication is the *counting of pairs of independent choices*: if there are 6 choices of notebook and 4 choices of pens, then the number of possible choices of notebook and pen pairs is 6 × 4. This can be seen from the area model, if one fills in the squares in a 6-by-4 rectangular array with all the different pairs of choices, each square would contain a different pair. Multiplication involving negative numbers is best modelled with the number line with a sense of direction. If you multiply by a positive number, you keep going the same direction. If you multiply by a negative, the product goes in the opposite direction. Once one has used the number line with direction model to establish **( – 1) × ( – 1) = 1** and **( – 1) ×** ***n*** **= –** ***n***, the laws of multiplication can be applied to decide whether or not a product is negative or positive.

The three laws for multiplication, along with the commutative and associative laws for addition and our understanding of expanded notation, give the foundation for all calculation algorithms for multiplication. We will look at several shortcut methods that work well in certain cases before introducing a general (left-to-right!) algorithm. We start with the easiest shortcuts:

**Multiplying by 0:** ***n*** **× 0 = 0.**

**Shortcut for multiplying by** $\mathbf{10^{n}}$: Shift the decimal place *n* units to the right. So, for example, 736 × 1000 = 736 000.

This rule includes the shortcut for multiplying by 1. Shift 0 places right. The number doesn't change.

This method also works for negative powers of 10, i.e. decimals like 0.0001, 0.01, 0.1, etc. Instead of shifting to the right, you shift to the left: 70306 × 0.0001 = 7.0306 (The decimal place moved 4 decimal places to the left because 0.0001 = $10^{-4}$.) We need to use this idea whenever we multiply numbers involving decimals together.

**Doubling and Halving:** The Doubling and Halving method can be very useful for mental calculations. Doubling is done by simply adding the number to itself. Halving is done by dividing the number by 2, or by guessing what needs to be doubled to produce the number. The method uses the identity ***n × m = ( n × ½) × (2 × m)***.

**Problem:** Compute 72 × 89 using the doubling and halving method:

Double 89 to get 178, halve 72 to get 36. So 72 × 89 = 36 × 178.
Do it again: Halve 36 to get 18, double 178 to get 356. 36 × 178 = 18 × 356.
And again: Halve 18 to get 9, double 356 to get 712. 18 × 356 = 9 × 712.

At this point we have reduced to a single digit multiplication and we will give a left-to-right algorithm for that later. If you are trying to use doubling and halving and you need to halve an odd number, then think like this:

9 × 712 = [(4 × 2) + 1] × 712 = [(4 × 2) × 712] + [1 × 712] = [(4 × 2) × 712] + 712
= (4 × 1424) + 712

So you should think of halving 9 to get 4, but since 9 is odd you add the number you are doubling at the end.

4 is even so we continue doubling and halving: 4 × 1424 = 2 × 2848 = 1 × 5696 = 5696. Finally, add the penalty of 712 to get the final answer: 72 × 89 = 5696 + 712 = 6408.

Doubling and halving is also a good method to learn for people that never remember multiplication tables. For example: 7 × 6 = 14 × 3 = (28 × 1) + 14 = 28 + 14 = 42.

Usually we double the larger number and halve the smaller, but the opposite also works: For example: 32 × 7 = 16 × 14 = 8 × 28 = 4 × 56 = 2 × 112 = 224.

**Shortcut for multiplying by 5:** Use $n \times 5 = (n \times \frac{1}{2}) \times 10$. So if you can halve it, then you can multiply it by 5. Here are some examples:

348 × 5 = 174 × 10 = 1740. (Note that this shortcut is really a special case of the doubling and halving method.)

37.24 × 5: Half of 372 is 186, so half of 37.24 is going to be 18.62. Therefore, 37.24 × 5 = 186.2.

5 029 × 5: Half of 502 is 251, half of 9 is 4.5. Therefore, half of 5 029 is 2 514.5, so 5 029 × 5 = 25 145.

**The Estimation Shortcut:** When you multiply by a number that is close to a number with only one non-zero digit, it can be useful to use that fact:

**Problem:** 3 058 × 19. Since 19 = 20 – 1, this problem is equivalent to

3 058 × (20 – 1) = (3 058 × 20) – 3 058.

Multiplying by 20 is just doubling and adding a zero, so 3 058 × 20 = 61 160, and 61 160 – 3 058 = 58 102.

This method can even be applied to both factors, one after the other:

**Problem**: 1002 × 9 999. This is (1000 + 2) × 9 999, so that's 9 999 000 + (2 × 9 999).
2 × 9 999 = 2 × (10 000 – 1) = 20000 – 2 = 19 998.
So the answer is 9 999 000 + 19 998 = 10 018 998.

As slick as these shortcuts can be, they are not going to work effectively for every problem. It takes experience to be able to recognize that a there is a good shortcut for a multiplication problem in general. When there are too many digits involved in a multiplication problem, there often is no good alternative to a general algorithm.

The foundation for any multiplication algorithm will be one's proficiency with single-digit multiplication, so knowledge of multiplication tables at an instant recall level is still absolutely essential. The traditional algorithm for multiplication is a right-to-left algorithm, which is great for getting the last digit of your answer right but not the best for estimation purposes. So we will introduce a left-to-right method.

**General left-to-right algorithm:** In its simplest form, the algorithm is very similar to the traditional algorithm except pairs of digits are multiplied in the opposite order A multiplication problem is gradually converted to an addition problem. First let's look at multiplying by a single digit.

**Problem:** 358 × 7. Think of 358 in expanded notation: 358 = 300 + 50 + 8. Multiplying these numbers by 7 we get 358 × 7 = 2100 + 350 + 56 = 1906.

The number of summands in the addition problem arising from a single digit multiplication will be the number of nonzero digits of the other factor. To avoid getting overwhelmed, you should add them up as you go along.

**Problem**: 120 486 × 8. 100 000 × 8 = 800 000. 20 000 × 8 = 160 000, add these to get 960 000. 400 × 8 = 3 200, so now we have 963 200. 80 × 8 = 640, which gets us up to 932 840. Finally, adding 6 × 8 = 48 gives the answer 932 888.

Now let's look at multiplication of two 2-digit numbers. We still think of the numbers in expanded notation. The important thing is to multiply each pair of digits involved exactly once as you convert the problem into an addition problem.

**Problem**: 92 × 67. Written out in full detail according to the laws of multiplication, we are trying to do this:

$$\begin{aligned} 92 \times 67 &= (90 + 2) \times (60 + 7) = [(90 + 2) \times 60] + [(90 + 2) \times 7] \\ &= (90 \times 60) + (2 \times 60) + (90 \times 7) + (2 \times 7) \\ &= 5400 + 120 + 630 + 14 \\ &= 6164 \end{aligned}$$

Of course, that is a lot to write. So again it is better mentally to do one multiplication at a time and keep a running total as you go. That way you only have to remember one number and add it to the next product at each step.

So for 92 × 67: Start with 90 × 60 = 5400. Then add this to 2 × 60 = 120 to get 5520. Add 5520 to 90 × 7 = 630 to get 6020 + 130 = 6150. Finally, add this to 2 × 7 = 14 to get the answer: 6164.

At this point one may want to compare this left-to-right method to the doubling and halving one with this example:

92 × 67 = (184 × 33) + 92 = (368 × 16)+ 184 + 92 = (736 × 8) + 276
= 5888 + 276 (we are experts at single-digit multiplication now!?)
= 6164.

(The shortcut does not save much time for us with this one, in my opinion.)

Now try multiplication of a 3-digit number by a 2-digit number. The method is the same. There are just more pairs of single digits that have to be multiplied together. But it still can be done in your head or with minimal doodling.

**Problem:** 724 × 59: Start with 700 × 50 = 35 000. (Think of 7 times 5 = 35, followed by the number of zeroes you see.) Then add this to 20 × 50 = 1 000 to get 36 000. Add this to 4 × 50 = 200 to get 36 200. Now move to the next digit in the second factor.
Add 36 200 to 700 × 9 = 6 300 to get 40 200 + 2 300 = 42 500.
Add 42 500 to 20 × 9 = 180 to get 42 680.
Add 42 680 to 4 × 9 = 36 to get the answer, 42 716.

Compare this method for this problem to what we would get using the estimation shortcut:
724 × 59 = (724 × 60) – 724 = 43 440 – 724 = 42 716.
(The shortcut is faster with this one.)

When we move to multiplication of numbers involving 3 nonzero digits or more, remembering the running total with consistency becomes too difficult for most, and we require an organized method to write things down. The main advantage of the traditional method was that it was optimal in terms of the space needed to write everything down. Our left-to-right algorithm is really a sequence of better and better estimates, so it's advantage is to build a foundation for good estimation and mental calculation habits. Another desirable feature is to have a method that makes it easier to avoid errors. When you go to the trouble to write down your calculations, you want to have a cumbersome method that leads to errors.

**Problem**: 87.12 × 656: Start with the products made by the first digits of the factors: 8 × 6 = 48. Ignore the decimal place until the end, so we treat this as being 8 000 × 600 = 4 800 000. This is the first estimate.

We added 5 zeroes, so now do the products for which we would need to add 4 zeroes: the 8 × 5, and the 7 × 6. Add them to our first estimate to get the next estimate.

| | | |
|---|---|---|
| 4 800 000 | | 5 220 000 |
| 420 000 | → | + 400 000 |
| + 400 000 | | 5 620 000 |

Now add the products for which we would add 3 zeroes: the 8 × 6, 7 × 5, and 1 × 6:

| | | | |
|---|---|---|---|
| 5 620 000 | 5 688 000 | 5 718 000 | 5 723 000 |
| 48 000 → | 35 000 → | 5 000 → | + 6 000 |
| 35 000 | + 6 000 | + 6 000 | 5 729 000 |
| + 6 000 | | | |

Now do the products for which we would add 2 zeroes: 7 × 6, 1 × 5, and 2 × 6:

| | | |
|---|---|---|
| 5 729 000 | 5 733 200 | 5 733 700 |
| 4 200 → | 500 → | + 600 |
| 500 | + 600 | 5 734 300 |
| + 600 | | |

Now the products for 1 zero: 1 × 6, and 2 × 5:

| | |
|---|---|
| 5 734 300 | 5 734 360 |
| 60 → | + 100 |
| + 100 | 5 734 460 |

Finally, add the last product: 2 × 6, and position the decimal place 2 units left.

5 734 460
+ 12
5 734 472 → The answer is 57 344.72

There are many methods for multiplication. Here is a left-to-right method we call the "zigzag multiplication". It is a slightly modified form of the "lattice method" used in medieval times. It converts the multiplication problem to an addition problem in a very smooth fashion.

**Problem:** 2.154 × 3.068. Again think of 2 124 × 3 068, and worry about the decimal place later.

The idea is to write all the single-digit products down in an organized fashion that converts the whole problem into an addition problem directly. We write each single-digit product as a pair in a good position. Start with the first digits: 3×2, and write it as 06 (use two digits). Then write the next product 3×1=03 below the first and *shifted over one place*. Continue making products with the 3. Write the next product 3×5=15 *beside* the first, and the 3×4=12 beside the second. Since the 3 really means 3 000, add three zeroes to the last product. So the products with the 3 look like this:

0606
 0312000

Now place the products with the 60. You need to move two place values to the right from where you started with the products for 3000.  Add one zero at the end.  It should line up with the products from the previous step.

```
0606
 0312000
   1212
    06240
```

Now place the products for the 8.  Shift over one place from where you started the products for 60.  Then add them up and position the decimal place 6 places left at the end to get the answer.   The complete solution looks like this:

```
    2.154
  × 3.068
---------
0615
 0312000  → 6462000
   1230   → 6585000
    06240 → 6591240
    1640  → 6607640
      0832 → 6608472
-------------------
 6.608472 (answer)
```

## Exercises: Section 2.4

Find the answer.  Challenge:  Do these in your head.

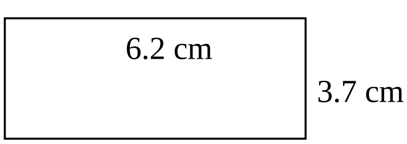
| | | | |
|---|---|---|---|
| 1. 27 × 4 | 2. 83 × 7 | 3. 602 × 11 | 4. 531 × 80 |
| 5. 24 × 79 | 6. 701 × 75 | 7. 88 × 69 | 8. 94 × 29 |

Choose the most efficient method to do each calculation.  Then find the answer.   Show all your work.

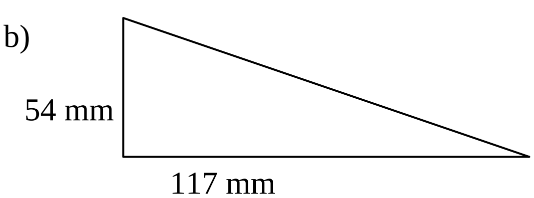
| | | | |
|---|---|---|---|
| 9. 692 × 28 | 10. 2 563 × 64 | 11. 876 × 909 | 12. 9 555 × 222 |
| 13. 4.95 × 0.13 | 14. – 37 × 61 | 15. – 206 × – 27 | 16. 84.461 × 32 |

17. Find each area or volume.

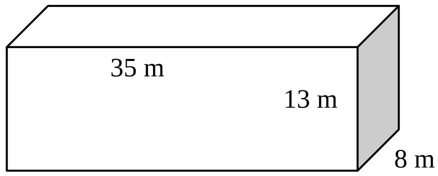
a)



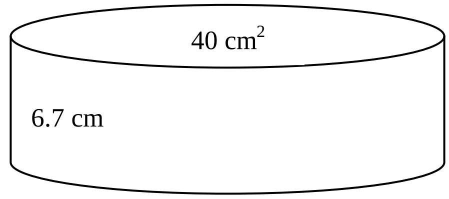

b)



c)




d)




18. There are 6 models of compact car to choose from. Each has 3 upgrade options and comes in 5 available colours. How many different cars could be bought?

19. An integer is a **perfect square** if it is equal to $n^2$ for some integer ***n.*** It is helpful to know the squares of the single-digits: 0, 1, 4, 9, 16, 25, 36, 49, 64, and 81. With experience, most can remember the first 20 perfect squares.

a) Explain why a positive integer whose last digit is 2, 3, 7, or 8 cannot be a perfect square.

b) Explain why a negative integer cannot be a perfect square.

c) Which of the following are perfect squares: 625, 729, 1 024, 6 241, or 10 000?

d) Given that 271 441 is a perfect square, find its *square root*, i.e. the *n* for which it is equal to $n^2$. (Hint: Use a guess and check strategy.)

20. Observe that $15^2 = 225$, $25^2 = 625$, $35^2 = 1225$, and $45^2 = 2025$. Find a pattern for the squares of 2-digit numbers ending in 5, and show why it is correct.

21. An integer is a **perfect cube** if it is equal to $n^3$ for some integer ***n.***

a) Find the smallest 10 perfect cubes, starting with 0.

b) Which powers of 10 are perfect cubes?

c) Given that 456 533 is a perfect cube, find its *cube root*, i.e. the integer *m* for which it is equal to $m^3$. How does your answer in a) help you with your first guess?

# 2.5 Division

The word "divide" means to separate. In arithmetic, to "divide by 2" means to separate into two equal parts. If we start with a line segment of length p, then $p \div 2$ is the length of one of the parts if we divided that segment into two equal parts, $p \div 3$ is the length of one of the parts if it was divided into three equal parts, $\boldsymbol{p} \div 4$ is the length of one of the parts if it was divided into three equal parts, etc. Dividing ***p*** by 1 doesn't change the length at all, the one part still has length ***p***. Thinking of division in this way, we can see that we cannot accommodate "division by 0". If we start with something of length ***p,*** we can't divide it into zero equal parts, because we can't turn something into nothing. So dividing by 0 is simply declared to be ***undefined***. It just cannot be made to make sense.

**N ÷ 0 is undefined. We cannot divide by 0.**

As long as we are dividing by a nonzero number, division can be thought of as the opposite process to multiplication. When $\boldsymbol{p} \div \boldsymbol{d} = \boldsymbol{q}$ and $\boldsymbol{d} \neq 0$, it means mean $\boldsymbol{q} \times \boldsymbol{d} = \boldsymbol{p}.$ In the equation $\boldsymbol{p} \div \boldsymbol{d} = \boldsymbol{q}$, ***p*** is called the dividend, ***d*** the divisor, and ***q*** the quotient.

When doing division calculations, one has to come to grips with the fact that the result of dividing one counting number by another may not be another integer. For example, the answer to 7 ÷ 3 is not an integer, because there is no integer that can be multiplied by 3 to give 4. **Rational numbers** directly represent answers to division problems, in fact fractions are defined by:

$$\boldsymbol{p} \div \boldsymbol{d} = \frac{\boldsymbol{p}}{\boldsymbol{d}}$$

So the real answer to 7 ÷ 3 is the rational number $^7/_3$, which is equal to the mixed numeral $2\frac{1}{3}$. Before rational numbers are introduced, elementary school children are taught to write the answer to 7 ÷ 3 as 2 R 1, and say "2 remainder 1". Uniqueness of this answer is a result of the **division algorithm** property of integers:

**If *d* is any positive divisor and *p* is any integer dividend, then there is a *unique* quotient *q* and nonnegative integer remainder *r* < *d* for which $p = (d \times q) + r$.**

So when we say 7 ÷ 3 = 2 R 1, we mean that these are the unique integers for which 7 = (3 × 2) + 1 and the remainder is a positive integer less than 3.

The goal of division calculation problems is always be to give the unique quotient and remainder guaranteed by the division algorithm. As with multiplication, there are many useful shortcuts for division.

**Shortcut for dividing by $10^n$.** Shift the decimal point ***n*** places left to divide by $10^n$. This is exactly the opposite shift as for multiplication. To divide by 0.1, 0.01, etc., we would shift a negative number of places left, making the actual shift to the right.

This idea also helps when we want to divide by a decimal number. Just shift to get rid of decimal places at the beginning. For example: 2 346 ÷ 3.75 = 234 600 ÷ 375

**Shortcut for dividing by 2:** This is halving. Guess what needs to be added to itself to give the number.

**Shortcut for dividing by 5:** Divide by 10 and then double. For example,

80 ÷ 5 = (80 ÷ 10) × 2 = 8 × 2 = 16.
660 ÷ 5 = (660 ÷ 10) × 2 = 132.

If there is a remainder, it will appear as a decimal: For example,

1023 ÷ 5 = (1023 ÷ 10) × 2 = 102.3 × 2 = 204.6.

Since 0.6 of 5 is 0.6 × 5 = 3, the remainder is 3. 1023 ÷ 5 = 204 R 3.

**Estimation shortcut:** As with multiplication, if the dividend is close to a nice multiple of the divisor, you can use that to simplify the problem. For example, for 3 007 ÷ 15, we can see that 3 000 ÷ 15 = 200 since 15 × 2 = 30. So (3 000 + 7) ÷ 15 = 200 + (7 ÷ 15) = 200 R 7. If you are using rational numbers, you would write the answer as the mixed numeral $200\,^{7}/_{15}$.

Similarly, 9 989 ÷ 25 = (10 000 – 11) ÷ 25 = 400 – (11 ÷ 25)

Since the remainder for the division algorithm cannot be negative, the quotient should be one less, and add the divisor to the remainder to make it positive:

400 – (11 ÷ 25) = 399 + [(25 – 11) ÷ 25] = 399 + (24 ÷ 25). So the answer is 399 R 14.

**Fast division for single-digit divisors**: When we have to divide by a single digit, the traditional algorithm is very effective, and it works left-to-right. The most basic technique is to simply "subtract multiples" of the divisor until you find the remainder, and then add up the factors corresponding to the multiples that were subtracted to get the quotient.

**Problem:** 7193 ÷ 9: Start by finding the largest multiple of 9 smaller than 71. We sometimes say "9 goes into 71 about 7 times" when we do this. Subtract it, keeping all the other digits in position. Then do the same for the first pair of digits in the remainder, and keep going until you can't subtract multiples of 9 anymore:

```
 7193
-63__  → 700 × 9
  893
 - 81_ → 90 × 9
   83
-  81  → 9 × 9
    2
```

This means that 7193 = (700 + 9 + 9) × 9 + 2, so 7193 ÷ 9 = 799 R 2.

Once you get used to the method, you can subtract the multiples in your head and not write so much:

$71^{8}9^{8}3 \div 9 = 799$ R 2

You might compare this method for this problem to what you would do using the estimation shortcut:

7193 ÷ 9 = (7200 – 7) ÷ 9 = 800 – (7 ÷ 9) = 799 + [(9 – 7) ÷ 9] = 799 R 2. Not bad either.

**Common divisor shortcut:** The idea for this shortcut is to divide by a common divisor first to reduce the size of the numbers in the problem.

**Problem:** 10 000 ÷ 140. Both numbers are multiples of 10, so divide both by 10 first. You should record that you divided by 10. We need this at the end.
Now we are doing 1000 ÷ 14. Both are divisible by 2. Divide both by 2 to reduce to 500 ÷ 7. This is a single-digit divisor, so just calculate it: $50^{1}0 \div 7 = 71$ R 3

The shortcut to use this for 10 000 ÷ 140 is that the quotient will be the same, and the remainder will be the product of this remainder times the products of the common divisors that have been factored out. So the shortcut written out looks like this:

10 000 ÷ 140 = 1000 ÷ 14 = 500 ÷ 7 = 71 R (3 ×10 × 2) = 71 R 60.
(÷10) (÷2)

Try another one:

**Problem:** 213 312 ÷ 132. In the next unit we will see how to recognize that both of these are divisible by 11. Since 11 times tables are very easy, we can divide by 11 the same way we divide by a single digit:

$21\ {}^{10}3\ {}^{4}3\ {}^{10}1\ {}^{2}2 \div 11 = 19\ 392$, and $13\ {}^{2}2 \div 11 = 12$.

Both 19 392 and 12 are divisible by 2, so now we can reduce to 9 696 ÷ 6.

$9\ {}^{3}6\ 9\ {}^{3}6 \div 6 = 1616$ R 0. So the answer to 213 312 ÷ 132 = 1616.

**General method for division:** The most straightforward general method for division is the "subtract multiplies" method, often traditionally known as "long division". To save on decision-making, subtracting doubles or multiplying by 5 can fast guesses. It can help to write general division problems with the traditional "surd" symbol $\overline{)\quad}$, especially when dividend or divisor is a decimal or when the desired quotient is a decimal number.

**Problem:** 69 704 ÷ 79.

```
      ______
 79 )69 704
    - 63 2__   → 800 × 79  A good first guess comes from 64 ÷ 8.
        6 504
     -   6 32_  →80 × 79
          184
      -    158 → 2 × 79
            26
```

Therefore, 69 704 ÷ 79 = 882 R 26.

**Problem:** 21 332.22 ÷ 27.3. Express your answer as a decimal number.

Start by shifting two decimal places to the right to turn this into an integer division problem.

```
         ______781.4__  ←  (This is where we write the answer, which develops as we go.)
2730 ) 213 3222
       - 136 50__      → 500 × 2730
           76 8222
         - 54 60__     → 200 × 2730
           22 2222
         - 13 650_____→ 50 × 2730     (Note: we can write the number of hundreds, 7
             8 5722                   in the answer on the line above now.)
         -  8 190_____→ 30 × 2730
               3822
           -   2730___→ 1 × 2730   (We now know the answer is 78?...
                1092
             -     819  → 0.3 × 2730  (We now know the answer is 781.?...)
                  273
              -    273 → 0.1 × 273
                   0
```

Therefore, 213 31.2 ÷ 27.3 = 781.4.

## Exercises: Section 2.5

Solve the following single-digit division problems. Only write what is absolutely necessary.

1. 846 ÷ 6  2. 60 120 ÷ 8  3. 13 896 ÷ 9  4. 62 178 ÷ 4  5. 21 846 ÷ 7

Use the common divisor shortcut to do the following division problems:

6. 728 ÷ 39 (Hint: 13)   7. 9 936 ÷ 81   8. 1 199 ÷ 88   9. 76 752 ÷ 864

Solve the following division problems. If a shortcut is available that will help, use it. List the shortcuts you use.

10. 9 935 ÷ 45   11. 1 484 ÷ 42.4

12. 13 296 ÷ 240   13. 813 944 ÷ 2 877

14. The algebraic properties of division are very limited, which is another reason to be careful with division.

a) Give an example to show that $\boldsymbol{p \div q \neq q \div p}$ in general for nonzero integers $\boldsymbol{p}$ and $\boldsymbol{q}$.

b) Which of the following are true? Give an example to justify your answers.
- i) $(\boldsymbol{n} + \boldsymbol{m}) \div \boldsymbol{d} = (\boldsymbol{n} \div \boldsymbol{d}) + (\boldsymbol{m} \div \boldsymbol{d})$ when $\boldsymbol{d} \neq 0$.
- ii) $\boldsymbol{n} \div (\boldsymbol{m} + \boldsymbol{d}) = (\boldsymbol{n} \div \boldsymbol{m}) + (\boldsymbol{n} \div \boldsymbol{d})$ when $\boldsymbol{d}+\boldsymbol{m}, \boldsymbol{d}, \boldsymbol{m}$ are all $\neq 0$.
- iii) $\boldsymbol{n} \div (\boldsymbol{m} \div \boldsymbol{d}) = (\boldsymbol{n} \div \boldsymbol{m}) \times \boldsymbol{d}$ when $\boldsymbol{d}$ and $\boldsymbol{m}$ are both $\neq 0$.

# 2.6 Arithmetic in Other Bases

To better understand our system, it is helpful to investigate other number systems. Many societies have used bases other than ten. Number systems were based on primary groupings of 20 in the Inuit areas of North America, in most of Mexico, and in Central America. The Celtic people of northwestern Europe, the Ainu of northeastern Asia, the Yoruba, Igbo, Banda of Africa, and the native Australians of Victoria also used base-20 systems. The Bushman of Africa used a base-2 system, as digital computers do today. The Babylonians used a base-60 system, as reflected in our use of seconds, minutes and degrees.

It is clear that number systems are grounded in our surroundings. Many Native American languages have numeral words that illustrate their origins.

An Inuit person describes how they count to 100: “We went by hands and then by feet. Two hands are 10 and [two hands with] one foot is 15. The other foot makes 20. When you have 20, that’s one person. One person plus five fingers is 25 and so on. Five people make 100 and 100 means a bundle. Often the foxes and sealskins were bundled into 100.” (From Oshaweetok as quoted by Michael P. Closs, <u>Native American Mathematics</u>, pp 5.)

In the Mayan empire, which existed more than 4000 years ago, a number system was developed which used base 20. Some historians credit the Mayans as being the first to create a symbol for zero and use it as a placeholder. We will examine the Mayan number system in more detail later in this unit.

There are several real-world situations where the traditional base for counting is not 10. For example:

- counting hours - if it is 10:00 a.m., what time will it be 30 hours from now? (base 24)
- counting days of the week - if today is Wednesday, what day will it be 452 days from now? (base 7)
- counting degrees - if you are facing north and spin clockwise 810 degrees, which direction will you be facing? (base 360)
- counting minutes and seconds – How many minutes is 240 seconds? (base 60)
- counting eggs – if you go to the store to buy 60 eggs, how many dozens to you buy? (base 12)

**Counting in base b**

The key to numbers in other bases is counting in multiples of powers of the base, as we do in base 10. We can think of numbers in base *b* in a similar fashion to the base 10 block representation, except now we need to have *b* dots to make a rod, *b* rods to make a square, *b* squares to make a large block, etc.

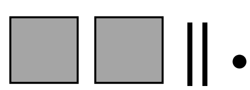
For example, suppose we want to represent 25 dots in base 3:

●●●●● ●●●●● ●●●●● ●●●●● ●●●●●

We first gather them into as many sets of 3 as we can:

**(●●●)(●● ●)(●●●)(● ●●)(●●●) (●●●)(●● ●)(●●●)●**

Make each set of 3 into a rod, and keep the one left over: 

Each set of 3 rods becomes a square: 

There are no collections of 3 left. When we count 25 dots in base 3, we get $(2 \times 3^2) + (2 \times 3^1) + (1 \times 3^0)$. (Here, use the convention that $3^0 = 1$.) So the base 3 name of this number is "two-two-one base 3" and written as $221_3$.

In general, if we wish to work in a base *b* we think of the number n in powers of *b* as follows:

$n = (a_m \times b^m) + (a_{m-1} \times b^{m-1}) + (a_{m-2} \times b^{m-2}) + \ldots + (a_3 \times b^3) + (a_2 \times b^2) + (a_1 \times b) + a_0,$

where each $a_m$ lies in the range from 0 to $b$-1. Thus, in base 10, we have $b = 10$ and the $a_m$'s are the digits with values 0, 1, 2, ..., 9. Of course, when b is larger than 10, we must use extra symbols for the digits. In this text we use A as a digit representing the number ten, B for eleven, C for twelve, etc.

**Converting between bases**

Sometimes we may wish to convert from one base to another, for example change a number in base 10 to its equivalent number in base 3. Let's consider what is going on in such a conversion. If we want to convert 204 from base 10 to base 3, we will be expressing 204 as a sum of multiples of powers of 3 instead of powers of 10. Recalling how we counted 25 in base 3, we first find out how many multiples of 3 there are and see what is left over:

204 ÷ 3 = 68 R 0, so 204 is 68 sets of 3.

Then we need to find out how many sets of $3^2$ there are in the 68 sets of 3, so divide 68 by 3:

68 ÷ 3 = 22 R 2, so 68 sets of 3 is the same as 22 sets of $3^2$ and 2 sets of 3.

Now find out how many sets of $3^3$ there are in the 22 sets of $3^2$.

22 ÷ 3 = 7 R 1, so 7 sets of $3^3$ and 1 set of $3^2$.

Finally, since 7 ÷ 3 = 2 R 2, we have 2 sets of $3^4$ and 1 set of $3^3$. Therefore, all together when we count 204 things in base 3 we get $(2 \times 3^4) + (2 \times 3^3) + (1 \times 3^2) + (2 \times 3) + 0$, which is base 3 expanded notation for the base 3 number $22120_3$.

This process works for any base. If we convert 204 into base 7, we would do this:

204 ÷ 7 = 29 R 1, and 29 ÷ 7 = 4 R 1, so 204 = $411_7$.

To convert 67 352 from base 10 to base 8:

67 352 ÷ 8 = 8 419 R 0,
8 419 ÷ 8 = 1 052 R 3,
1 052 ÷ 8 = 131 R 4,
131 ÷ 8 = 16 R 3,
16 ÷ 8 = 2 R 0. → so 67 352 = $203430_8$.

What about converting 6 $556_7$ to base 12? When we convert between bases where both are different from 10, it is often easier for us to convert to base 10 first because we have more familiarity with arithmetic in base 10. To convert from base b to base 10, just expand the number in base b and calculate that in base 10:

$$6\,544_7 = (6 \times 7^3) + (5 \times 7^2) + (4 \times 7) + 4$$
$$= (6 \times 343) + (5 \times 49) + (4 \times 7) + 4$$
$$= 2\,058 + 245 + 28 + 4$$
$$= 2\,335$$

Then convert this base 10 number to base 12:

$$2\,335 \div 12 = (2400 - 65) \div 12 = 200 - (65 \div 12) = 200 - (5 + (2 \div 12))$$
$$= 195 - (2 \div 12) = 194 + ((12 - 2) \div 12) = 194 \text{ R } 10,$$
$194 \div 12 = 16$ R 2,
$16 \div 12 = 1$ R 4.
We use the letter A to represent 10 as a digit in base 12. So $6544_7 = 142A_{12}$.

In some special situations it is easy to convert from one base to another. Computers use a binary (base 2) system. Computer code is usually written in hexadecimal (base 16), because it is very easy to convert between base 2 and base 16.

For example, let's consider the base 2 number $10\ 1110\ 1010\ 0111_2$. and see how to write it directly as a base 16 number. The point is that each 4 digits of a base 2 number represent a single digit of a base $2^4$ number. So $0111_2 = 4+2+1 = 7$ can be written as a single digit in base 16. Convert each 4 digit part of the base 2 number the same way:

$0010 \mid 1110 \mid 1010 \mid 0111 = (2) \mid (8+4+2) \mid (8+2) \mid (4+2+1) = 2 \mid 14 \mid 10 \mid 7 = 2EB7_{16}$.

Recall that in base 16 we use A for the digit ten, B for the digit eleven,..., and F for 15. One nice thing about this process is that it is completely reversible (by similar reasoning), so that given $65C3_{16}$ we can quickly rewrite it as $6|5|12|3 = 0110\ 0101\ 1100\ 0011_2$.

**Arithmetic in other bases**

Arithmetic can be done in other bases using the same ideas developed in the previous sections, with care taken to write the calculations in the other base. Rather than looking for shortcuts when working in a base we are not used to, use expanded notation to think carefully about what the numbers in the new base mean.

**Problem**: $4052_6 \times 31_6$. We will stick to an elementary approach. First, use expanded notation in base 6 to break this into easier problems:

$$4052_6 \times 31_6 = (4000_6 + 50_6 + 2) \times (30_6 + 1)$$
$$= [(4000_6 + 50_6 + 2) \times 30_6] + [4052_6 \times 1]$$
$$= (4000_6 \times 30_6) + (50_6 \times 30_6) + (2 \times 30_6) + (4052_6 \times 1)$$

Now we calculate these products in base 6. The important thing to remember is that the product of single digits has to be written in base 6. For example, the first multiplication $4 \times 3$ in base 6 is $20_6$, because $4 \times 3$ things is the same number of things as two sets of six. For $5 \times 3$, we think of this as being 2 sets of 6 plus 3, so that's $23_6$. The rules for adding zeroes to multiply by powers of $10_b$ work the same in any base, as does the rule for multiplying by 1.

$$\begin{aligned} 4052_6 \times 31_6 &= (4\,000_6 \times 30_6) + (50_6 \times 30_6) + (2 \times 30_6) + (4\,052_6 \times 1) \\ &= (200\,000_6 + 2\,300_6 + 100_6) + 4\,052_6 \end{aligned}$$

Now, to finish, add these numbers in base 6:

$$\begin{aligned} 4052_6 \times 31_6 &= (200\,000_6 + 2\,300_6 + 100_6) + 4\,052_6 \\ &= 202\,400_6 + 4\,052_6 \\ &= 210\,400_6 + 52_6 \\ &= 210\,452_6 \end{aligned}$$

Try another one:

**Problem:** $2102_3 \times 121_3$. Do the same steps as above working in base 3 this time:

$$\begin{aligned} 2102_3 \times 121_3 &= (2\,102_3 \times 100_3) + [(2\,000_3 + 100_3 + 2) \times 20_3] + (2\,102_3 \times 1) \\ &= 210\,200_3 + (2\,000_3 \times 20_3) + (100_3 \times 20_3) + (2 \times 20_3) + 2\,102_3 \\ &= 210\,200_3 + 110\,000_3 + 2\,000_3 + 110_3 + 2\,102_3 \\ &= 1\,020\,200_3 + 2\,110_3 + 2\,102_3 \\ &= 1\,020\,200_3 + 11\,212_3 \\ &= 1\,101\,200_3 + 212_3 \\ &= 1\,102\,112_3 \end{aligned}$$

**Problem**: $2B\,79A_{12} \div 99_{12}$. This time we use the general repeated-subtraction method of long division, and do every calculation in base 12. Recall that the digit B means 11 and A means 10 when we work in base 12.

$$\begin{array}{rlll} 99_{12}) & \overline{2B\,79A_{12}} & & \\ & \underline{-\,25\,30\,0}_{\ 12} & \rightarrow 300_{12} \times 99_{12} & (\text{since } 3 \times 99_{12} = 230_{12} + 23_{12} = 253_{12}) \\ & 6\,49A_{12} & & \\ & \underline{-\ \ 5\,830}_{\ 12} & \rightarrow 70_{12} \times 99_{12} & (\text{since } 7 \times 99_{12} = 530_{12} + 53_{12} = 583_{12}) \\ & 86A_{12} & & \\ & \underline{-\,816}_{\ 12} & \rightarrow A_{12} \times 99_{12} & (\text{since } A \times 99_{12} = 760_{12} + 76_{12} = 816_{12}) \\ & 54_{12} & & \end{array}$$

Therefore, $2B\,79A_{12} \div 99_{12} = 37A_{12}$ R $54_{12}$.

## Exercises: Section 2.6

Convert the following. In base 16 we use A for the digit ten, B for the digit eleven,..., F for 15.

1. $782_{10}$ to base 7
2. $36_8$ to base 6
3. $712_9$ to base 10
4. $262_9$ to base 3
5. $111_8$ to base 2
6. $101\ 010_2$ to base 16
7. $102_3$ to base 4
8. $442_5$ to base 3
9. $368_9$ to base 6
10. $11\ 021_3$ to base 5
11. You are given three distinct base 8 digits: *x*, *y*, *z*. When you form all of the 6 possible two digit numbers from these and add them up, you find that their sum is $504_8$. What are *x*, *y*, *z*?
12. You are given three distinct base 10 digits: *x*, *y*, *z*. When you form all of the 6 possible two digit numbers from these and add them up, you find that their sum is 484. What are *x*, *y*, *z*?
13. You are given three distinct base 7 digits: *x*, *y*, *z*. When you form all of the 6 possible two digit numbers from these and add them up, you find that their sum is $514_7$. Can you find *x*, *y*, *z*? Why does it not work?
14. Why do mathematicians tend to mix up Halloween and Christmas? (Hint: First figure out why this riddle would be placed in this section.)

Carry out the following calculations in the indicated base (Remember in bases higher than 10 we will use A for the digit ten, B for the digit eleven, C for the digit twelve, etc.):

15. $$\begin{array}{r} 1\ 432_5 \\ 233_5 \\ +\ \underline{341_5} \end{array}$$

16. $$\begin{array}{r} 732_8 \\ 566_8 \\ +\ \underline{6\ 614_8} \end{array}$$

17. $$\begin{array}{r} 6\ 7B9_{12} \\ 9\ 23A_{12} \\ +\ \underline{1\ 238_{12}} \end{array}$$

18. $$\begin{array}{r} 6\ 105_8 \\ -\ \underline{627_8} \end{array}$$

19. $$\begin{array}{r} 11\ 0101_2 \\ -\ \underline{1\ 1010_2} \end{array}$$

20. $$\begin{array}{r} 9\ 7B9_{16} \\ -\ \underline{6\ A3A_{16}} \end{array}$$

21. $1\,011_2 \times 111_2$

22. $3\,212_4 \times 223_4$

23. $436_{12} \times 78_{12}$

24. $2A3B_{12} \div 6A_{12}$

25. $1101_2 \div 10_2$

26. $24\,043_5 \div 32_5$

# 2.6 Appendix -- An Algebraic Adventure

You might recall that multiplying polynomials in algebra was a mystifying experience, but now that we have gained experience working in other bases it probably deserves another look. It's actually very easy, just like multiplying in base *x* but we do not even have to worry about the carries! For example, consider the following multiplication. All we really need is a units column, an *x* column, an $x^2$ column, etc.:

$$\begin{array}{r} 3x + 2 \\ \times \quad 5x + 7 \\ \hline 21x + 14 \\ 15x^2 + 10x \quad\;\; \\ \hline 15x^2 + 31x + 14 \end{array}$$

Really very simple! Notice how we have added in "base *x*", nothing to it. Similarly,

$$\begin{array}{r} 4t^2 + 5t - 3 \\ \times \qquad 4t - 6 \\ \hline -24t^2 - 30t + 18 \\ 16t^3 + 20t^2 - 12t \qquad\;\; \\ \hline 16t^3 - 4t^2 - 42t + 18 \end{array}$$

Division of polynomials is also the same idea as our division algorithms in base 10, base 5, or whatever, but again, it's really easier because we do not need to carry. Here is an example of long division of polynomials arising from the above example:

$$\begin{array}{rl} 4t - 6 \qquad\qquad & \\ 4t^2 + 5t - 3 \;\big)\; 16t^3 - 4t^2 - 42t + 18 & \leftarrow \text{(Think of solving } 4t^2 \times \_\_\_\_ = 16t^3\text{)} \\ -\;(16t^3 + 20t^2 - 12t) \qquad & \leftarrow \text{(Multiply your answer above by the divisor} \\ -24t^2 - 30t + 18 & \text{and subtract that polynomial from the dividend)} \\ -\;(-24t^2 - 30t + 18) & \leftarrow \text{Repeat the process} \\ 0 & \end{array}$$

Therefore, $(16t^3 - 4t^2 - 42t + 18) \div (4t^2 + 5t - 3) = 4t - 6$.

Multiplication and division of powers of an arbitrary base are what is behind the **power rules** we use in algebra:

$$b^n \times b^m = b^{n+m}, \qquad b^n \div b^m = b^{n-m}, \text{ and } (b^n)^m = b^{n\times m}.$$

## Exercises: Section 2.6 Appendix

Perform the following multiplications and divisions:

1. $\begin{array}{r} 2y + 3 \\ \underline{\times\ 6y + 1} \end{array}$

2. $\begin{array}{r} 6n^2 - 2n + 12 \\ \underline{\times \qquad 4n\ -\ 7} \end{array}$

3. $\begin{array}{r} 5z^2 - 6 \\ \underline{\times\ 3z + 11} \end{array}$

4. $(2y^2 + y - 3) \div (y - 1)$

5. $(4x^3 - 11x^2 + 31x - 42) \div (x^2 - x + 6)$

6. $(5z^4 - 80) \div (5z^3 + 10z^2 + 20z + 40)$

# 2.7 Roman Numerals

Our base-10 system is known as the Hindu-Arabic numeration system. It originated in India, then it spread to China and the Arab world. Around 800 years ago, in 1202, Fibonacci, an Italian merchant and scholar, brought Arabic numerals to Europe with his influential book *Liber Abaci*. Much of the notation and methods of arithmetic he introduced are what we still use today. Of course, it took some centuries for this new number system to replace the well-established Roman Numerals in Europe – and in fact we still see Roman numerals used for decorative purposes in our society today.

The Romans used a base-10 number system, but their symbols are different from our Arabic numerals and they are combined using a different strategy. Their symbols are:

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| **Roman Symbol** | **I** | **V** | **X** | **L** | **C** | **D** | **M** |
| **Arabic Equivalent** | **1** | **5** | **10** | **50** | **100** | **500** | **1 000** |

It is easy to imagine the origin of most of them: I (one finger), V (one hand), X (two hands, one up one down), C (from Latin centum = 100 as found in our word cent and century), and M (from the Latin mille = 1000, as found in our words mil, mile, and millennium). The symbols L and D on the other hand seem to have no such helpful origin.

The basic rule for writing a number using Roman numerals is to present it as economically as possible -- use at most four I's, X's, or C's, and at most one V, L, or D. So a group of five I's is replaced by a V, two V's by an X, and so on. We arrange symbols in a number in decreasing order of magnitude (M before D, D before C, …). For example, 1787 = MDCCLXXXVII.

Roman numerals were in use for more than a thousand years, so the rules for writing them evolved over the centuries – there is no universally accepted set of rules.

Addition and subtraction with Roman numerals come down to combining like symbols. We group the symbols, working our way from the smallest magnitude to the largest, this works like our traditional algorithms do in base 10. For example: Suppose we want to add 1787 + 824 + 287 in Roman numerals. The three numbers are written as MDCCLXXXVII, DCCCXXIIII, and CCLXXXVII:

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | M | D | CC | L | XXX | V | II | 1787 |
| | | D | CCC | | XX | | IIII | 824 |
| + | | | CC | L | XXX | V | II | 287 |
| | M | (DD) ↙ | (CCCCC) ↙ | (LL) ↙ | (XXXXX) ↙ | (VV) ↙ | (IIIII) ↙ | |
| | M | D | CC C | L | XXX X | V | III | |
| | MM | D | CCC | L | XXXX | V | III | 2 898 |

- Starting from the right with the "I" column, there are eight I's. Five I's make a V leaving, three I's.
- In the "V" column, there are 2 V's plus the one "carried" from the I column. Two V's make an X, leaving one V.
- In the "X" column, there are eight X's plus the one carried. Five X's make an L, leaving four X's with an L carried, and so on.

For subtraction, one borrows from the next largest magnitude. It is instructive to compare subtraction in our number system with subtraction using Roman numerals. Try 31 – 18.

| **Arabic numerals** | | | | **Roman numerals** | | | | |
|---|---|---|---|---|---|---|---|---|
| 31 | | 2 (11) | | XXX I | | XX (VV) I | | XX V IIIII |
| - 18 | ⇒ | - 1 8 | | - XVIII | ⇒ | - X V III | ⇒ | - X V III |
| | | 1 3 | | | | | | X III |

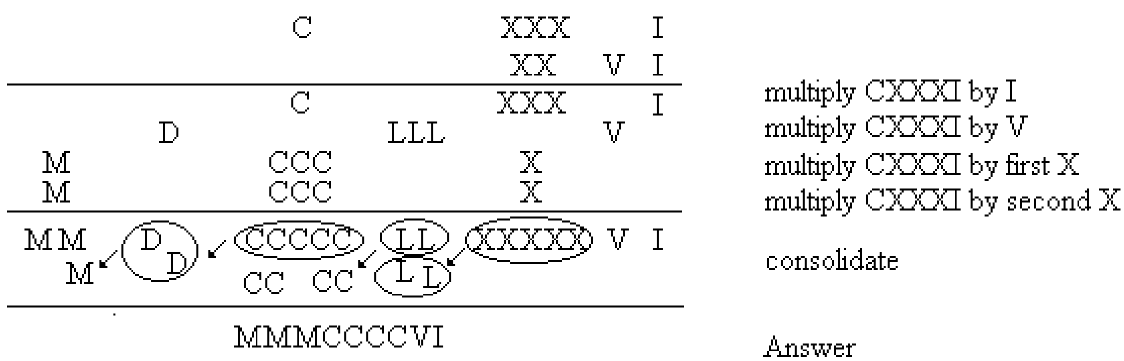
In the Roman system, one physically borrows to turn XXXI into XXVIIIIII. Then one physically takes away X, V and III from XX, V, and IIIII to get XIII.

Multiplication also comes down to a matter of bookkeeping. Unlike our system where we must memorize extensive multiplication tables, Roman children had to learn a few simple rules such as

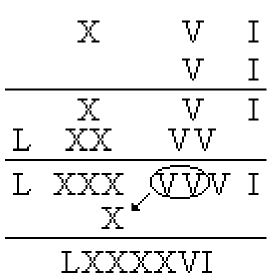
- Multiplication by I copies the number you are multiplying.
- Multiplication by X slides each symbol over two places; it increases each I to X, each V to L, each X to C, each L to D, and each C to M.

The only place one has to think a bit is for a product such as V by V (which yields XXV), V by L (giving CCL), and L by L (giving MMD).

Let us first multiply CXXXI by XXVI (which should give us 131 × 26 = 3406). Note that for each partial product, we shift the multiple and product to the left two columns for each X, one for each V, and none for each I.




Here is an example that shows how V times V can be accommodated: multiply XVI by VI (which should give us 16 × 6 = 96).




Multiply XVI by I
V times I equals V, V times V equals XXV, V times X equals L

## Exercises: Section 2.7

Write the following Roman numerals in decimal notation (base 10).

1. DCXII 2. MDCCCCLXXVI 3. MDCCLXXVI

Write the following as Roman numerals.

4. 189 5. 948 6. 2 120

Add the following (without converting to decimal form).

7. MMDCXXXXII
+ CCCLXXXVI

8. MMMMMDLXXVI
+ CCLXXXI

Subtract the following (without converting to decimal form).

9. DCCCCLXXXXVIII
\- CCCLXXVI

10. MMLXXXXIIII
\- MDCCCXXXIIII

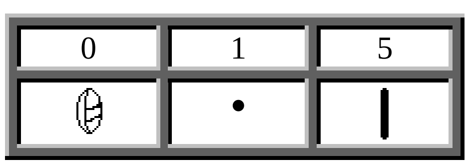
Multiply the following (without converting to decimal form).

11. CCXIII
by XXXVII

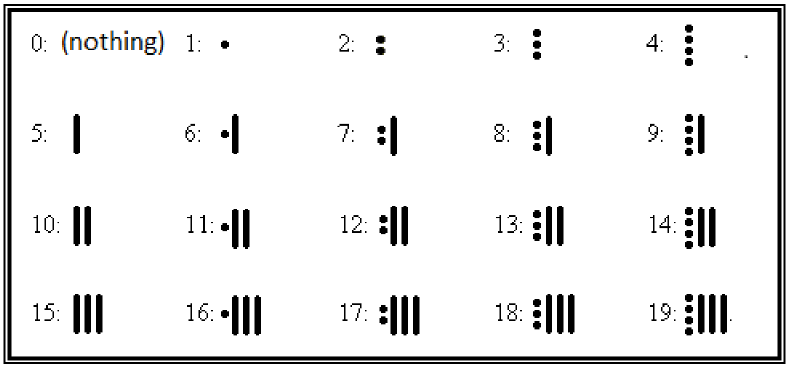
12. LXVII
by XXVI

# 2.8 Mayan Numerals

The Mayan civilization flourished in Mexico and Central America during the six hundred period AD 300-900. Among their remarkable achievements was the development of a place-value number system. The Mayan system is base 20 (vigesimal) rather than base 10 (decimal). Their digits are based on three symbols:




Some refer to these symbols as shells, pebbles, and sticks, which may have been the original counting items. These symbols can be combined to construct the twenty digits from 0 to 19 as follows:




19 is represented by 4 pebbles and three sticks, , which is the maximum number of symbols that can occupy one place. If we add one pebble to 19, we will have 5 pebbles, which becomes 1 stick, giving us four sticks. These four sticks are represented in the second place as one pebble, and then the first place is left with a shell, zero, . Symbols in the second place are multiples of 20. Hence, five pebbles make one stick (5), and four sticks make one pebble and one shell (20).

For calendrical purposes, the Mayans modified the system slightly, but the details do not concern us here. You can find further information about Mayan arithmetic in reference books or at http://www.mayacalendar.com/mayacalendar/Home.html.

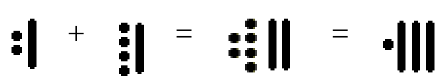
Now that we know the basics, let us look at how we convert from the Mayan system to our decimal system. Consider the Mayan numeral:

In decimal form this would represent: $2(20^3) + 14(20^2) + 6(20) + 18$

$= 2 \times 8\,000 + 14 \times 400 + 6 \times 20 + 18$

$= 16\,000 + 5\,600 + 120 + 18 = \underline{21\,738}$

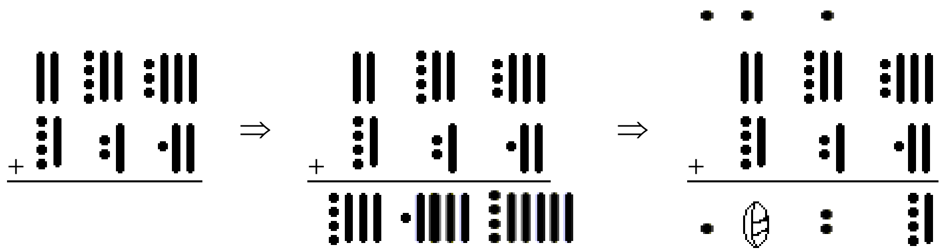
Addition of Mayan numerals is very similar to addition in our base-10 system.




Here we can say:

- 2 pebbles + 4 pebbles = 6 pebbles, and 1 stick + 1 stick = 2 sticks, so we have 6 pebbles and 2 sticks.
- 5 pebbles makes one stick, so 6 pebbles makes 1 stick and 1 pebble, so all together we get 1 pebble and three sticks, which makes 16.

When working with larger numbers we find that addition can get more complicated. Here is a problem that works through the possible complications:



This is how it works:

- Start by simply adding the pebbles and sticks down each column.
- Now we must put this number into the Mayan base 20 format. In the one's column there are 5 sticks and 4 pebbles, 4 sticks combine to make 20 which is carried over as one pebble in the 20's column, leaving 1 stick and 4 pebbles.
- In the 20's column there is 1 pebble plus the one that was carried over giving 2 pebbles and 4 sticks. We carry 20 (the four sticks) to make one pebble in the 400's column and leaving 2 pebbles in the 20's column.
- Now in the 400's column we have 3 sticks and 5 pebbles (1 carried); 5 pebbles makes one stick, giving us 4 sticks which carry over to the 8 000's column as a pebble. There is nothing left in the 400's column, which is represented by a shell.

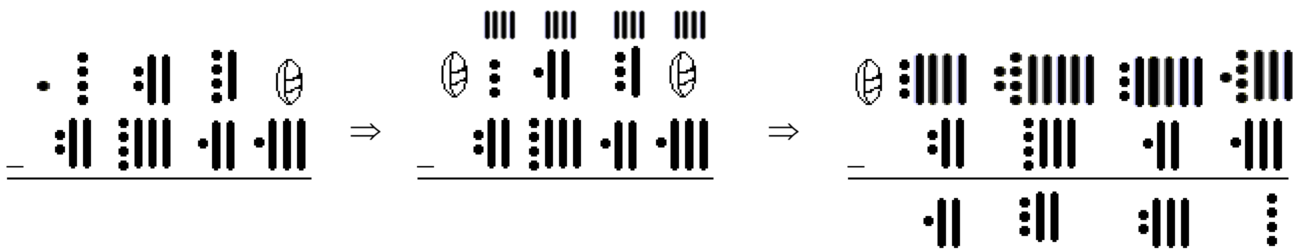
If you feel confident you may be able to add and carry out the conversion between pebbles and stones at the same time, but be cautious about keeping everything in the right position. You must always remember that everything stays in base 20.

When subtracting we must also look at how many pebbles and sticks we have in each position. But in order to subtract we may need to make some sticks into pebbles. For example:

This is what we did:

- There are three sticks subtract 2 pebbles and 1 stick, so we must make one of the sticks in the group of three into 5 pebbles so we can subtract the two pebbles.
- Then we have 5 pebbles subtract 2 pebbles, and 2 sticks subtract 1 stick, leaving us with 3 pebbles and 1 stick.

Borrowing is also a necessary technique for subtracting Mayan numbers. Let's look at the following example:



This may look really confusing, but it is not that difficult. Here is what happened:

- Starting in the one's column we needed to borrow one pebble from the 20's column, giving us four sticks in the one's column, and leaving 3 pebbles and 1 stick in the 20's column.
- In the 20's column we must borrow from the 400's column, this column is made up of 20's so when we take one pebble we are taking a 20 giving us 3 pebbles and five sticks.
- The 400's column is left with 1 pebble and 2 sticks, so we will borrow again from the 8000's column, to get 4 more sticks, giving 6 in total.
- Moving to the 8 000's column, we borrow the one pebble from the 160 000's column, leaving it with a shell, zero.
- Now it looks like we should be able to subtract, but we must take pebbles from pebbles, so we must change some sticks back into pebbles. We did this in the one's and 400's columns.
- Finally we can subtract. Working down the columns, subtract pebbles from pebbles and sticks from sticks.

We can also easily convert from decimal to Mayan numerals. Lets look at 19 578 using our two methods of Section 2.3 (with the method on the left streamlined to read from bottom to top instead of left to right):

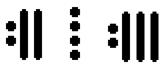
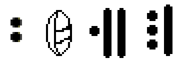
19 578 ÷ 20 = 978, remainder 18 = 
978 ÷ 20 = 48, remainder 18 = 
48 ÷ 20 = 2, remainder 8 = 
2 ÷ 20 = 0, remainder 2 = 

| | | | | |
|---|---|---|---|---|
| | | | | 19578 |
| **2** () | × | 8000 | = | 16000 |
| | | | | 3578 |
| **8** () | × | 400 | = | 3200 |
| | | | | 378 |
| **18** () | × | 20 | = | 360 |
| | | | | 18 |
| **18** () | × | 1 | = | 18 |
| | | | | 0 |

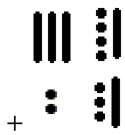
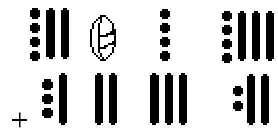
We get 19 578 = .

## Exercises: Section 2.8

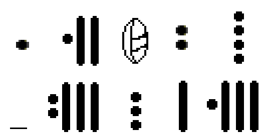
Write the following Mayan numerals in decimal notation (base 10).

1. 
2. 
3. 

Add the following (without converting to decimal form)

4. 
5. 
6. 

Subtract the following (without converting to decimal form).

7. 
8. 
9. 

Write the following in Mayan numerals.

10. 238
11. 1280
12. 239 742

# 2.9 Unit Review Exercises

Convert the following to decimal form (base 10).

1. $43_5$
2. $3C4_{16}$
3. $2\,212_7$
4. $3\,435_6$
5. $11\,010_2$
6. $13\,131_4$

Convert the following from decimal form to the indicated base.

7. 847 to base 3
8. 2 630 to base 16
9. 583 to base 2
10. 3 987 to base 6

Convert the following to the indicated base.

11. $344_5$ to base 7

12. $162_9$ to base 2

13. $2\ 312\ 131_4$ to base 11

14. $3\ 724_8$ to base 6

15. Using the shortcut method, convert $110\ 110\ 100\ 101_2$ to base 16.

16. Using the shortcut method, convert $8\ 271_9$ to base 3.

Perform the following operations.

17. $9A6_{11} + 812_{11}$

18. $F\ B3A_{16} + 62D_{16}$

19. $421_7 - 226_7$

20. $432_5 - 123_5$

21. $101\ 101_2 - 1\ 011_2$

22. $B2A_{12} - 8A9_{12}$

23. $253_6 \times 41_6$

24. $2D3_{16} \times 8E_{16}$

25. $14\ 153_7 \div 26_7$

26. $3\ 102_4 \div 13_4$

27. $17\ B48_{12} \div A4_{12}$

28. $11\ 200_5 \div 41_5$

Do the following calculations using the best available method. State the method(s) you used.

29. 112 021 + 10 212 101

30. 14 275 - 7 213

31. 1 434 ÷24

32. $(75)^3$

33. 3 255 × 14

34. 2 073 + 4 702

35. 894 × 36

36. 90 270 ÷ 27

37. 2 660 × 51

38. 88 007 – 4 993

39. – 144.2 ÷ 3.5

40. 509 990 ÷ 17

41. Joan collects eggs from her chickens each day, and puts them into egg cartons of one dozen eggs each. During one 90-day period she collected 3456 eggs.

a) Express this number in base 12 notation.

b) Determine the number of egg cartons Joan used when she collected 3456 eggs.

42. A spaceship brought back from an asteroid a fragment on which was inscribed:

((((/////
<u>*(((///</u>
**((//

Interpreting this as a mathematical problem, can you determine in what the base the author was working, what the process was, and can you translate it into modern terminology, base 10?

43. Convert 1 999 to Roman numerals.

44. Convert MMDCCCLXVII to decimal form.

Perform the following calculations without converting to decimal form.

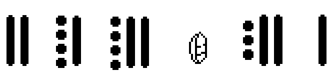
45. MCCCLXXXVII
MDCCXXXVIII
<u>+ MDCCLXXII</u>

46. MMMMMMMXII
<u>- MMCCCXXXVI</u>

47. Multiply CLXXXVII by XXVI

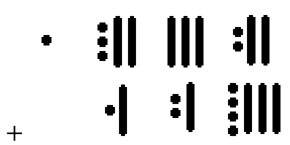
48. Convert 975 837 to Mayan numerals.

49. Convert the following Mayan numeral to decimal form:

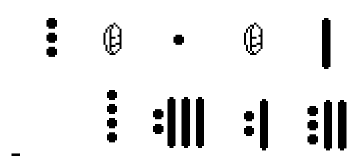


Perform the following calculations without converting to decimal form.

50.



51.



52. To what base $b$ is 6 times 6 equal to $28_b$?

53. $abc_9$ is a 3-digit number when written in base 9 notation; the *same* number in base 6 notation is $cba_6$ Find $a$, $b$, and $c$.

54. a) You have a balance and the following standard weights: 1 kg, 2 kg, 4 kg, 8 kg, 16 kg, 32 kg. What combination of these weights in one pan will balance a 46 kg package in the other?

b) What connection is there between the previous exercise and binary numbers?

55. A *gross* is a dozen dozen; and a *great gross* is a dozen gross. A man wishes to add two numbers: the first number is 3 great gross and 5 gross and A dozen and 3; the second number is 7 great gross and 5 gross and B dozen (where A stands for ten and B for eleven). Perform the job for him in the spirit of addition, base 12.

56. You are in Saudi Arabia where the currency is in riyals, qurush, and halals. Your hotel bill (and you may assume it is correctly added) reads:

| | **Riyals** | **Qurush** | **Halas** |
|---|---|---|---|
| Room | 14 | 8 | 3 |
| Meals | 21 | 9 | 1 |
| Extras | 6 | 7 | 2 |
| Total | 42 | 5 | 1 |

From this information, how many halals are there in one qurush, and how many qurush are there in one riyal?

57. In 1971 Britain completed the changeover to decimal currency. Before then there were *twelve pence* (d) *in a shilling* (s), and *twenty shillings in a pound* (£). In the former system, how much change would you get from a £10 note if the three items you bought cost £3. 16s. 8d, £2. 17s. 9d, and £2. 5s. 4d?

58. What is the sum of all digits in the number $10^{95} - 95$?

59. Octavius Fingers surveyed thirty-five students to find out which languages they could read. The unfortunate Mr. Fingers was born with no thumbs so he naturally did his counting in base 8. He found that $17_8$ can read French; 5 can read both French and German; $26_8$ can read German; $10_8$ can read both German and Russian; $21_8$ can read Russian; 3 can read all 3; $13_8$ can read both French and Russian.

a) How many can read only German?

b) How many cannot read any of the 3 languages?

# Unit 3
# Modular and Calendar Arithmetic

## 3.1 Modular Arithmetic and Congruences

There once was a student named Ben,
Who could only count modulo ten,
He said, "When I go
Past my little toe,
I shall have to start over again."

We have seen that there are many real-world situations where we need to count in a base other than 10:

- counting in hours - if it is 10:00 a.m., what time will it be 30 hours from now?
- counting in days - if today is Wednesday, what day will it be 452 days from now?
- counting degrees - if you are facing north and spin clockwise 810 degrees, which way will you be facing?

In the first case, the hour repeats after every multiple of 12 is reached, so we are only interested in the fact that 30 is 6 more than 24 and that 10 (a.m.) plus 6 hours equals 16:00, which leaves a remainder of 4 when we subtract 12. Hence if it is 10:00am then 30 hours later it will be 4:00pm. In the second case, the day of the week repeats every 7 days, and 452 leaves a remainder of 4 when we divide it by 7. In detail, $452 = 7 \times 64 + 4$, so 452 days equals 64 weeks plus 4 days. The number of weeks is not relevant here -- just the days that are left over, which we add to our present day. Hence if today is Wednesday, then it will be Sunday in 452 days. In the third case, 810 leaves a remainder of 90 when we subtract multiples of 360. Thus if you are facing north and spin clockwise 810 degrees you will be facing east. So in each situation, we only need to keep track of our remainders when we do our arithmetic.

"Arithmetic with remainders" is called **modular arithmetic**. In a more formal manner, if ***m*** is a positive integer, and ***a*** and ***b*** are any integers, we say that

***a* is congruent to *b* modulo *m*** if $\boldsymbol{a \div m}$ and $\boldsymbol{b \div m}$ have the same remainder.

We denote this relationship by

$$\boldsymbol{a \equiv b \text{ mod } m},$$

and refer to ***m*** as the **modulus** of this congruence relationship.

The mathematician Carl Friedrich Gauss first developed this idea of congruence in the late 18th century. In our examples above, we have 30 ≡ 6 (mod 12); 452 ≡ 4 (mod 7); 810 ≡ 90 (mod 360).

Equivalent ways of stating the congruence relationship ***a*** ≡ ***b*** (mod ***m***):

- ***a* – *b*** ≡ 0 mod ***m***; i.e. ***a* – *b*** is a multiple of ***m*.**
  For example: 10 ≡ – 2 mod 6, because 10 – (– 2) = 12, which is a multiple of 6, and multiples of 6 have zero remainder when divided by 6.
  Also, – 7 ≡ 2 mod 3, since (– 7) – 2 = – 9, which is a multiple of 3.
- ***a*** = *t**m*** + ***b*** for some integer *t*.
  For example: 39 ≡ 4 mod 7, because 39 = (5×7) + 4.
- ***a*** and ***b*** have the same last digit in base ***m***
  For example: 39 = $54_7$, so 35 ≡ 4 mod 7.

NOTE: We will be restricting ourselves to the integers when using congruences. Congruence is not defined for non-integer numbers.

More examples:

| | | |
|---|---|---|
| 10 ≡ 1 mod 3; | 5 ≡ – 3 mod 4; | 1 023 ≡ 1 mod 2; |
| 1 023 ≡ – 1 mod 2; | 27 ≡ 5 mod 11; | – 18 ≡ 3 mod 7 |

## Exercises: Section 3.1

Fill in the blank with the whole number that is less than the modulus and satisfies the congruence.

1. 83 ≡ ___ mod 2
2. $2^3$ ≡ ___ mod 5
3. 26 ≡ ___ mod 12
4. 78 ≡ ___mod 7
5. 95 ≡ ___ mod 11
6. 16 ≡ ___ mod 2
7. 15 ≡ ___ mod 6
8. 81 ≡ ___mod 9
9. 30 ≡ ___ mod 8
10. – 5 ≡ ___ mod 8
11. – 3 ≡ ___ mod 9
12. 60 ≡ ___ mod 12

Determine which of the following statements are True and which are False.

13. $738 \equiv 3 \pmod{11}$

14. $24 \equiv 3 \pmod{8}$

15. $75\,651 \equiv 0 \pmod{3}$

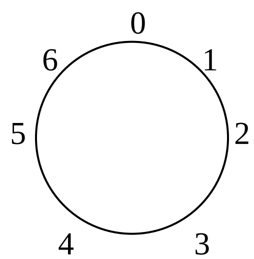
16. $194 \equiv 3 \pmod{6}$

17. $4\,678 \equiv 2 \pmod{4}$

18. $45\,627 \equiv 1 \pmod{3}$

# 3.2 Arithmetic with Congruences

Modular arithmetic is arithmetic involving congruence with of a fixed modulus. A good model to think of for addition and subtraction modulo ***n*** is to think of the numbers 0, 1, 2, ... , ***n*** – 1 written clockwise around a clock, with addition counting clockwise and subtraction counted counter-clockwise. So if we want to add 5 and 3 modulo 7, we would count 3 forward around the clock, starting at 5:




If we count 3 clockwise from 5, we stop at 1.
So $(5 + 3) \equiv 1 \bmod 7$.

When we do modular arithmetic we treat multiples of the modulus as being 0. For example, if we add 23 modulo 5 to 14 modulo 5 naively, we end up with 37 modulo 5, which we can then reduce to 2 modulo 5; briefly, $23 + 14 = 37 \equiv 2 \pmod{5}$. But if we start with the idea that 5 behaves like 0 when we compute with congruences modulo 5, we can eliminate larger numbers in the problem early on:

$$\begin{aligned}(23 + 14) &= ((4 \times 5) + 3 + (2 \times 5) + 4) \\ &\equiv (0 + 3 + 0 + 4) \bmod 5 \\ &\equiv (5 + 2) \bmod 5 \\ &\equiv (0 + 2) \bmod 5 \\ &\equiv 2 \bmod 5\end{aligned}$$

So when we do arithmetic with integer congruences, we have a choice whether to do arithmetic with the integers themselves first and then reduce modulo the modulus, or to reduce first and then do the integer arithmetic with the remainders. Often doing the integer arithmetic with the remainders is faster because you can make the integers you have to work with smaller, and thus do a lot more of the work in your head.

Here is a multiplication example: 16 modulo 3 times 29 modulo 3 is 464 modulo 3, which is congruent to 2 modulo 3. If we reduce first, the calculation looks like this:

$$\begin{aligned}(16 \times 29) &= (((5\times3)+1) \times ((9\times3)+2)) \\ &\equiv (((5\times0)+1) \times ((9\times0)+2))) \bmod 3 \\ &\equiv (\ 1 \times 2\ ) \bmod 3 \\ &\equiv 2 \bmod 3\end{aligned}$$

Sometimes adding multiples of the modulus simplifies matters. For example, to calculate 3 – 19 modulo 5, one option is to first add 4 × 5:

$$3 - 19 \equiv 20 + 3 - 19 \equiv 4 \bmod 5.$$

As another example, to calculate $2^{28}$ modulo 10 first observe that $2^5 = 32 \equiv 2 \bmod 10$. Thus for calculations in modulo 10, we can replace $2^5$ with 2. Hence

$$\begin{aligned}2^{28} &= 2^{25} \times 2^3 = (2^5)^5 \times 2^3 \\ &\equiv (2)^5 \times 2^3 \bmod 10 \\ &\equiv 2 \times 2^3 \bmod 10 \\ &\equiv 16 \bmod 10 \\ &\equiv 6 \bmod 10.\end{aligned}$$

Arithmetic with a fixed modulus is similar to ordinary arithmetic in that we can construct addition and multiplication tables. For example, suppose we want to construct tables for addition and multiplication modulo 6:

| **+** | **0** | **1** | **2** | **3** | **4** | **5** |
|---|---|---|---|---|---|---|
| **0** | 0 | 1 | 2 | 3 | 4 | 5 |
| **1** | 1 | 2 | 3 | 4 | 5 | 0 |
| **2** | 2 | 3 | 4 | 5 | 0 | 1 |
| **3** | 3 | 4 | 5 | 0 | 1 | 2 |
| **4** | 4 | 5 | 0 | 1 | 2 | 3 |
| **5** | 5 | 0 | 1 | 2 | 3 | 4 |

Addition mod 6

| **×** | **0** | **1** | **2** | **3** | **4** | **5** |
|---|---|---|---|---|---|---|
| **0** | 0 | 0 | 0 | 0 | 0 | 0 |
| **1** | 0 | 1 | 2 | 3 | 4 | 5 |
| **2** | 0 | 2 | 4 | 0 | 2 | 4 |
| **3** | 0 | 3 | 0 | 3 | 0 | 3 |
| **4** | 0 | 4 | 2 | 0 | 4 | 2 |
| **5** | 0 | 5 | 4 | 3 | 2 | 1 |

Multiplication mod 6

Note that in modular arithmetic we can have ***a*** times ***b*** is congruent to 0 even when ***a*** and ***b*** are both non-zero, for example 3 × 2 ≡ 0 mod 6. This outcome is quite different from our usual arithmetic and suggests that *division in modular arithmetic should be approached with caution*.

Keeping in mind this warning about division, we can deal with equations in modular arithmetic in a manner very similar to that which we are used to in ordinary arithmetic.

For example, if we are asked to solve $3z + 2 \equiv 4 \bmod 7$, then we are asked to find all the integers that satisfy the congruence. Observe that the equation $3z + 2 = 4$ has no integer solution. However, $4 \equiv 11 \bmod 7$, and the equation $3z + 2 = 11$ has the integer solution $z = 3$, so 3 is a solution to the original congruence $3z + 2 \equiv 4 \bmod 7$. So a congruence represents several equations at once, only some of which may have solutions.

To solve for an integer missing in a congruence, we first reduce algebraically using modular arithmetic. For $3z + 2 \equiv 4 \bmod 7$, we can reduce to the linear congruence $3z \equiv 2 \bmod 7$ just by subtracting 2 from both sides. Up to congruence, we then only have to look for solutions to $3z \equiv 2 \bmod 7$ for $z$ in the set $\{0, 1, 2, 3, 4, 5, 6\}$, because every integer is congruent modulo 7 to exactly one of these. The row in the multiplication table mod 7 corresponding to multiplying by 3 reads: 0, 3, 6, 2, 5, 1, and 4. So the only solution to $3z \equiv 2 \bmod 7$ up to congruence is $z \equiv 3 \bmod 7$. This is also the only solution up to congruence to the congruence $3z + 2 \equiv 4 \bmod 7$ that we started with.

Note that $z = 3$ is not the unique integer solution to $3z + 2 \equiv 4 \bmod 7$! (For instance, $z = 10$ works as well.) What it really means for $z \equiv 3 \bmod 7$ to be the unique solution up to congruence is that the set of integer solutions to $3z + 2 \equiv 4 \bmod 7$ is precisely equal to the set of all integers that are congruent to 3 modulo 7. So the solution set is

$$\{3, 3 \pm 7, 3 \pm (2 \times 7), 3 \pm (3 \times 7), 3 \pm (4 \times 7), ... \} = \{..., -18, -11, -4 , 3, 10, 17, 24, ...\}.$$

This means there are really infinitely many solutions to this congruence. This is typical of solutions to congruence equations that have solutions. Here are some more examples:

**Examples:**

Solve $4z + 1 \equiv 2 \bmod 9$.

- Let's rewrite this as: Solve $4z \equiv 1 \bmod 9$.
- Now try $z = 0, 1, 2, ..., 8$. For these values of $z$, we have that $4z = 0, 4, 8, 3, 7, 2, 6, 1$, and 5. So the only solution up to congruence is $z = 7$, where $4z = 28 \equiv 1 \bmod 9$.
- Thus, $4z + 1 \equiv 2 \bmod 9$ has solution set
  $\{7, 7 \pm 9, 7 \pm 18, 7 \pm 27,...\} = \{..., -20, -11, -2, 7, 16, 25, 34,...\}$.

Solve $5z - 1 \equiv 3 \bmod 8$.

- Let's rewrite this as: Solve $6z \equiv 4 \bmod 8$.
- Now try $z = 0, 1, 2, ..., 7$. This time we find two solutions up to congruence:
  $z \equiv 2 \bmod 8$ and $z \equiv 6 \bmod 8$
- Thus, $5z - 1 \equiv 3 \bmod 8$ has solution set
  $\{2, 2 \pm 8, 2 \pm 16,...\} \cup \{6, 6 \pm 8, 6 \pm 16,...\} = \{..., -14, -10, -6, -2, 2, 6, 10, 14, ...\}$.

Solve $2z \equiv 1 \bmod 4$. Equivalently, $2z \equiv 1 \bmod 4$.

- We try $z = 0, 1, 2, 3$ and find that $2z = 0, 2, 0$, and 2. None of these are congruent to 1 mod 4, so $2z \equiv 1$ mod 4 has <u>no</u> solutions. The solution set is the empty set.
- In other words, $2z - 1$ cannot be congruent to 0 mod 4 for any $z$.

What happens with a quadratic? Consider $z^2 + 3 \equiv 0$ mod 7.

- When working in mod 7, we need only check if 0, 1, 2, …, 6 are solutions. Checking for $z^2 + 3 \equiv 0$ mod 7 we find that $z = 2$ and $z = 5$ are solutions.
- Once we have the solutions from 0, 1, 2, …, 6 we can generate the rest of the solutions by adding and subtracting multiples of 7.

Lets try a cubic problem: Solve $3z^3 \equiv 1$ mod 5.

- By substitution we can see that $z = 3$ is the only solution in 0, 1, 2, 3, 4. Thus $3z^3 \equiv 1$ mod 5 has solution set { 3, 3 ± 5, 3 ± 10, …}.

## Exercises: Section 3.2

Do the following modular arithmetic exercises.

1. (16 + 27) (mod 3)
2. (16 + 27) (mod 4)
3. (134 – 17) (mod 17)
4. (239 - 122) (mod 3)
5. (8 × 5) (mod 6)
6. (32 × 19) (mod 4)
7. Construct the addition and multiplication tables for mod 7 arithmetic.

With the help of the tables from exercise #7, solve the following exercises in mod 7 arithmetic by listing all solutions from 0 to 6.

8. $4x \equiv 2$ (mod 7)
9. $2x \equiv 4$ (mod 7)
10. $3x \equiv 6$ (mod 7)
11. $6x \equiv 3$ (mod 7)

Find the smallest positive solution to the following:

12. $3\,456 \equiv x$ mod 11
13. $823 \times 234 \equiv z$ mod 8
14. List all numbers between – 20 and 20 that are congruent to 4 modulo 7.

15. List all number between – 20 and 20 that are congruent to 4 modulo 12.

For exercises #16 to #19, find all possible positive solutions, if any, which are less than the given modulus.

16. $7x \equiv 3 \bmod 1$

17. $4z \equiv 1 \bmod 6$

18. $2x^2 \equiv 5 \bmod 9$

19. $z^3 \equiv 3 \bmod 5$

# 3.3 Calendar Arithmetic

When we ask what day of the week it will be 20 days from now, we are actually asking a question about congruences modulo 7. Observe that $20 \equiv -1 \bmod 7$. So if today is a Tuesday, 20 days from now is one day short of three weeks from now, which is a Monday.

If we know, for example, that January 1, 2010 was a Friday, can we use congruences to tell which day of the week September 8, 2010 was?

This can be done using congruences modulo 7 and knowing how many days are in each month:

After 31 days of January it will be February 1. After the 28 days in February of 2010 – it isn't a leap year – it will be March 1. After the 31 days of March, 30 days of April, 31 days of May, 30 days of June, 31 days of July, and 31 days of August, it will be September 1. September 8 is 7 days after September 1. Therefore, September 8, 2010 happens 31+28+31+30+31+30+31+31+7 days after January 1, 2010.

Since January 1, 2010, was a Friday and

$$\begin{aligned} 31+28+31+30+31+30+31+31+7 &\equiv 3 + 0 + 3 + 2 + 3 + 2 + 3 + 3 + 0 \bmod 7 \\ &\equiv 5 \bmod 7 \\ &\equiv -2 \bmod 7, \end{aligned}$$

September 8, 2010 must have been 2 days before a Friday, which was a Wednesday.

In order to be able to handle dates that are several years apart, one needs to know how many days there are in a year:

There are 365 days in a common year and 1 extra day in a leap year. The extra day in a leap year has been an extra day in February (the $29^{th}$ ). Leap years occur in every year whose number is divisible by 4, except for years that are multiples of 100, which are only leap years when they are multiples of 400. So the years 1800 and 1900 were not leap years, but 2000 was a leap year.

The current calendar that we use is called the *Gregorian* Calendar. It was introduced by Pope Gregory in Rome on March 21, 1582. It took a few centuries before it replaced the Julian Calendar in other parts of Europe. England began using it on Thursday, September 14, 1752. The last European country to switch was Greece, on March 1, 1923. Although it is the universal standard for the business world, there are still some small parts of the world that do not use the Gregorian calendar.

Knowing the number of days in each year, when leap years occur, and the number of days in each month is all that is needed to be able to solve "day of the week" questions using congruence modulo 7.

**Problem:** Find the day of the week on the day Canada became an independent country: July 1, 1867.

From above we know that January 1, 2010 was a Friday. January 1, 1868 was 2010 – 1868 = 142 years earlier. The leap years between these dates are 1868, 1872, ... , 2008 with the 1900 omitted and the year 2000 included. We can use the idea of the Fence Post Problem (see Section 2.3.7) to count the number of terms in this sequence. If every $4^{th}$ year was included, there would be $((2008 – 1868) \div 4) + 1 = 35 + 1 = 36$ years listed. Since 1900 was not a leap year, there are actually only 35. Each leap year has 1 extra day. So January 1, 1868 was $(142 \times 365) + 35$ days before Friday, January 1, 2010.

Note that $365 \equiv 1 \pmod 7$. Thus

$$(142 \times 365) + 35 \equiv (2 \times 1) + 0 \pmod 7$$
$$\equiv 2 \pmod 7.$$

Therefore, January 1, 1868 occurred 2 days before a Friday. So it was a Wednesday.

To finish the problem, we have to count the days back from January 1, 1868 to July 1, 1867. There are 31 days in July, 31 days in August, 30 days in September, 31 days in October, 30 days in November, and 31 days in December. So Wednesday, January 1, 1868 was 31+31+30+31+30+31 days after July 1, 1867. Since

$$31+31+30+31+30+31 \equiv 3 + 3 + 2 + 3 + 2 + 3 \pmod 7$$
$$\equiv 2 \pmod 7,$$

July 1, 1867 was 2 days before a Wednesday, which means it was a Monday.

## Exercises: Section 3.3

1. Show that January 1, 2000 was a Saturday.

2. a) Show that June 30, 1912 was a Sunday (the date of the Regina “cyclone”).

   b) June 30, 2002, the $90^{th}$ anniversary for the Regina cyclone, was celebrated in Regina as it was a Sunday. Prove that June 30, 2002 fell on a Sunday.

3. a) Find the day that Saskatchewan joined Canada as a province, September 1, 1905.

   b) What day of the week will Saskatchewan celebrate its $150^{th}$ birthday in 2055?

4. What day of the week did the United States celebrate its bicentennial, July 4, 1976?

5. On April 26, 1986, technicians at the Chernobyl Power Plant in the Ukraine (former Soviet Union) allowed the power in the fourth reactor to fall to low levels as part of a controlled experiment that went wrong. The reactor overheated causing a meltdown of the core. Two explosions blew the top off the reactor building releasing clouds of deadly radioactive material in the atmosphere for 10 days after the explosion. On what day of the week did the explosion occur? The clouds were no longer emerging from the plant on what day of the week?

6. The Maple Leaf flew over Canada, as it’s official flag, for the first time on February 15, 1965. On what day of the week did this historic event happen?

7. The first day of the new millennium was January 1, 2001, which was also Julian Day 2 451 911. The Julian Day system was invented by $16^{th}$ century scholar Joseph Scalinger to honour his father, Julius. It is the precise number of days that have passed since Julian Day 0, January 1, 4713 BC. The date changes at precisely noon at the 0 meridian. Since Regina stays on Central Standard Time, this always happens at 6 A.M. in Regina. Astronomers use this system to record the official times of celestial events, because it is not dependent on local time zones.

8.
   a) Given that Julian Day 0 was a Monday, what day of the week was January 1, 2001?

   b) What day of the week will Julian Day 2 455 555 be?

9. The date chosen for Easter Sunday depends on both the sun and the moon. It is always *the first Sunday after the first full moon occurring on or following the vernal equinox*. The date for Easter in the $21^{st}$ century can be calculated using the following algorithm:

Let the number of the year be $Y$.

Step 1: Calculate $B \equiv Y \pmod{19}$, with $0 \leq B < 19$.
Step 2: Calculate $C \equiv Y \pmod{4}$, with $0 \leq C < 4$.
Step 3: Calculate $D \equiv (19 \times B) + 24 \pmod{30}$, $0 \leq D < 30$.
Step 4: Calculate $E \equiv (2 \times C) + (4 \times Y) + (6 \times D) + 5 \pmod{7}$, $0 \leq E < 7$.
Step 5: Easter Sunday is $D+E$ days after March 22, with 2 exceptions:

- if $D = 29$ and $E = 6$, then Easter Sunday is April 19; and
- if $D = 28$ and $E = 6$, then Easter Sunday is April 18.

For example, when $Y = 2010$, we calculate $B = 15$, $C = 2$, $D = 9$, and $E = 4$. So Easter Sunday in 2010 was 13 days after March 22, which is April 4.
Use this to calculate the date Easter will be this year and next year.

# 3.4 The "Casting Out 9's" trick

An arithmagician says

- "Pick a number; do not let me see it!
- Now scramble the digits in any way you wish to form a new number.
- Subtract the smaller of these two numbers from the larger.
- Now circle any non-zero digit of the resulting number and add up the non-circled digits.
- Tell me the answer!"

"14", comes the response.

- "Then you circled a 4!"

It turns out that this is correct! How do they do such tricks?

The idea behind this trick is that

$$m - n \equiv ((\text{sum of digits of } m) - (\text{sum of digits of } n)) \bmod 9,$$

for any nonnegative integers $m$ and $n$ (see below). However, since the digits of $n$ constitute a rearrangement of the digits of $m$, the difference [(sum of digits of $m$) - (sum of digits of $n$)] is 0. Thus, $m - n \equiv 0 \bmod 9$. So, to get the answer, we need only to subtract the response given the arithmagician from the next closest multiple of 9 that is a larger multiple of 9.

Consider the example:

$$\begin{array}{rl} 7\,214 & (m) \\ -\ \underline{4\,172} & (n) \\ 3\,042 & \end{array}$$

circle the 3. Add 0 + 4 + 2 = 6, so the response is 6.
Note that 9 – 6 = 3, the circled number.

How do we see that $[m\text{-}n] \equiv$ [(sum of digits of $m$) - (sum of digits of $n$)] mod 9? It comes from the fact that for any integer $m$, $m \equiv$ sum of the digits of $m$ (mod 9). For example, we claim that $7\,843 \equiv 7 + 8 + 4 + 3 \pmod 9$. This is correct because

$$\begin{aligned} 7\,843 &= 7 \times 1\,000 + 8 \times 100 + 4 \times 10 + 3 \\ &= 7 \times (999 + 1) + 8 \times (99 + 1) + 4 \times (9 + 1) + 3 \\ &\equiv 7 \times 1 + 8 \times 1 + 4 \times 1 + 3 \\ &\equiv 7 + 8 + 4 + 3 \bmod 9. \end{aligned}$$

Excursion: Note that adding the digits of a number gives us an easy way to check if it is divisible by 9 or not. Since the number and the sum of its digits are both the same mod 9, the number is divisible by 9 (i.e. congruent to 0 mod 9) precisely when the sum of its digits is equal to 9. When adding up the digits, you can discard multiples of 9 as they arise, hence the name "casting out nines." The same trick works when checking for divisibility by 3. Why? You will see this idea in more detail in Section 3.3.

The arithmagician continues. "Well, you did so well on that one, try another.

- Pick a number.
- Double it!
- Add 7.
- Multiply by 5.
- Now, subtract your original number.
- Throw away any one non-zero digit and tell me the rest of the digits in any order you wish."

"6 and 8", came the reply.

- "Aha! You tossed away a 3!"

Example: $m = 53$; $2m = 106$; $106 + 7 = 113$; $113 \times 5 = 565$; $565 - 53 = 512$; toss the 5 and answer 2 and 1. He'll say 5.

Note here that $5 \times (2m+7) - 9m + 35 \equiv 8 \bmod 9$. Thus the number you end up with has a remainder of 8 when divided by 9. Equivalently, the sum of the digits of the number that you end up with must have a remainder of 8 when divided by 9. Thus, if given 2 and 1 you need 5 more to make the remainder 8.

"O.K. You are catching on. One last trick.

- Pick a number, any number.
- Multiply it by 3.
- If the result is odd, add 25, otherwise add 34.
- Now divide by 2.
- O.K! Now, add 11.
- Multiply all this by 6 and finally divide by 9.
- Find the remainder! Got it?
- It's 6!"

Example: $m = 73$; $3m = 219$; $219 + 25 = 244$; $244 \div 2 = 122$; $122 + 11 = 133$;
$6 \times 133 = 798$; $798 = 9 \times 88 + 6$.

In general, $6((^1/_2)(3m + 25) + 11) = 9m + 141 \equiv 6 \bmod 9$ or $6((^1/_2)(3m + 34) + 11) = 9m + 168$ which leaves a remainder of 6 when divided by 9.

## Exercises: Section 3.4

1. Check 126 (86 – 29) = 7 249 and 127 (86 – 29) = 7 239 by casting out nines. Which equation is more likely to be correct?

2. Use the method of casting out nines to determine the value of 123 456 789 (mod 9).

3. Calculate 7 267 (mod 3). Calculate 7 267 (mod 9).

# 3.5 Multiple Congruences

In preparation for a parade a bandleader wants to arrange his charges for the march. When he arranges them in rows of 3, one is left over; in rows of 4, two are left over; and in rows of 5, three remain. What is the smallest number of members that could be in his band?

In this problem we have to solve a number of congruences at the same time to find the number of band members, N. We deduce from the information given that $N \equiv 1 \bmod 3$;

$N \equiv 2 \bmod 4$; and $N \equiv 3 \bmod 5$. If we look at these congruences separately and find their positive solutions we have:
$N \equiv 1 \bmod 3$: {1, 4, 7, 10, 13, 16, 19, ... }
$N \equiv 2 \bmod 4$: {2, 6, 10, 14, 18, 22, ...}
$N \equiv 3 \bmod 5$: {3, 8, 13, 18, 23, 28, 33, ...}

What we now have to do is find the smallest integer that is common to all three solution sets. This could be difficult in general, but here we do not have to look too far to find that 58 is in all three sets; i.e. N = 58.

What's a good way to locate the solution N = 58?

- Notice the place where the first two sets initially agree -- at 10.
- Since the first set goes up in 3's and the second in 4's they will agree again 12 later, and another 12 later, ...; namely they agree at 10, 22, 34, 46, ...
- Similarly, we see the second and third sets agree at 18 and as they grow by 4's and 5's they agree at 18, 38, 58, ... . By this observation we can reduce our search time considerably.

## Exercises: Section 3.5

1. a) The students in a Math 101 class decide to divide themselves into groups of equal size to study multiple congruences. When they try groups of 4, 3 are left over; groups of 5, 2 are left; finally they try groups of 7 and discover that if the professor joins in then they can use 7. The professor, reasoning that he never really understood modular arithmetic anyway, thinks it will probably do him some good and agrees to join in. What is the number of students in the class if you are told it is between 150 and 250?

   b) Solve for $s$ where $150 \leq s \leq 250$: $s \equiv 3 \pmod 4$, $s \equiv 2 \pmod 5$, and $s \equiv -1 \pmod 7$

   c) Do you notice any similarities between part a) and part b)?

2. Solve for $s$: $s \equiv 3 \pmod 4$ and $s \equiv 5 \pmod 7$

3. Solve for $s$ where $-90 \leq s \leq 90$. $s \equiv -2 \pmod 9$, $s \equiv 0 \pmod 5$, and $s \equiv 3 \pmod 4$

4. Solve for $s$ where $1 \leq s \leq 120$. $s \equiv 1 \pmod 3$, $s \equiv 2 \pmod 4$, and $s \equiv 3 \pmod{10}$

5. John and Alfred work in the library. When they put the returned books in piles of 5 there were 2 books left over. While John went to get more, Alfred made stacks of 7 books and 4 were left over. When John returned with 3 more books, they made stacks of 14 books and none were left over. How many books could they have started with? (Just give the smallest possible correct answer that is positive.)

# 3.6 Unit Review Exercises

Solve the following.

1. $43 \equiv$ ___ (mod 2)
2. $89 \equiv$ ___ (mod 12)
3. $16 \equiv$ ___ (mod 3)
4. $2\,000 \equiv$ ___ (mod 7)
5. $23 + 47 \equiv$ ___ (mod 4)
6. $83 + 46 \equiv$ ___ (mod 11)
7. $8 \times 14 \equiv$ ___ (mod 5)
8. $13 \times 48 \equiv$ ___ (mod 9)
9. $6\,(52+14) \equiv$ ___ (mod 8)
10. $8\,(43 + 80) \equiv$ ___ (mod 6)

Find all the solutions, if any, to:

11. $8x \equiv 3 \pmod 8$
12. $3z \equiv 5 \pmod{11}$

Fill in the blanks with the smallest positive integer possible.

13. $241 \equiv$ ___ (mod 16)
14. (7 × ___) $\equiv 2$ (mod 6)
15. 4(__) $\equiv 3$ (mod 7)
16. $(\_\_)^3 \equiv 2 \pmod 5$
17. $(3 \times 366 + 14 \times 365) \equiv$ ___ (mod 7)

Solve for $x$ in each of the following.

18. $x \equiv 2001^{2000} \pmod 3$

19. $x \equiv 17(430 + 2600) \pmod 7$

20. $4x \equiv 5 \pmod 9$

21. $17 + x \equiv (11 \times 21) \pmod{13}$

22. Susan and Mike came home and found their marbles all together in a can. They do not know how many marbles they have, but Susan remembers that when she puts their marbles into groups of 10, 2 are left over, and she has more than 10 marbles. Mike remembers that when he puts their marbles into groups of 12, 4 are left over and he has more than 12 marbles. Help Mike and Susan find out how many marbles they have.

23. For the common year 2002, January 1 was Tuesday, and December 31 was also Tuesday. Is it always true that the first and last days of a year fall on the same day of the week? What about 2000?

24. In the Gregorian calendar can Christmas (December 25) fall on the same day of the week in two consecutive years? Why or why not?

25. Thanksgiving Day in Canada is always the second Monday in October. What is the least possible number of days between Thanksgiving and Christmas?

26. Today is Thursday, October 3, 2002. On what day of the week does October 3 fall 100 years from today, October 3, 2102? (Assume that no nuclear holocaust will affect the calendar; the day can be found without any need for the day-of-the-week formula!)

27. King Hussein ascended the throne of Jordan on August 11, 1952. His 46-year reign ended with his death on February 7, 1999. On what day of the week was Hussein proclaimed king and on what day did he die?

28. a) Susan is a nurse with a work schedule of three days on duty followed by two days off. On August 1, 2000, she was back to work after having had the previous two days off. Was she scheduled to work on the following Christmas day?

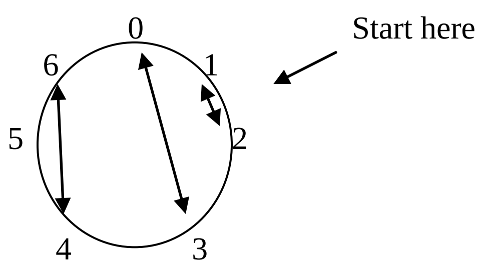
    b) Using the fact that Christmas fell on Monday in 2000, what day of the week was August 1, 2000? (Do not use any calendar or days of the week formula here, except perhaps to check your answer.)

29. Explain how Christmas can fall on Saturday one year, then on Thursday ten years later. Explain why Wednesday and Friday are the only other possibilities.

30. Determine the value of 10 691 033 (mod 9) by casting out nines.

31. Explain how to determine the value of 983 215 – 589 123 (mod 9) with no computations at all (and no calculations!).

32. The clock model for modular arithmetic can be used to set up *round-robin tournaments* for any number of players. For example, if there are 6 players playing in the tournament, and every player is supposed to play every other player exactly once, then we number the players 1 to 6. Since there is an even number of players, we add an imaginary player numbered 0, and write the numbers 0 to 6 around a clock. (We only have to do this when there are an even number of players, so if there were 7 players we would just place 1 to 7 around the clock.)
We then draw arrows on the inside of the clock connecting pairs of numbers that give all possible positive differences. This can always be done starting on one side of the clock and moving to the other side. Connect the first pair you encounter, then the next pair, then the next, until there is just one number left on the other side.




List the games indicated by the arrows. So here we would have 1vs2, 0vs3, and 4vs6. Player 5 is not paired with anyone, so they get a bye in the first round. Anyone paired with 0 also gets a bye in that round.

By spinning the numbers on the clock forward one step at a time, not moving the inside arrows, and playing the games between pairs of players indicated by the arrows at each step, explain why a round-robin tournament will be created after a full spin of the clock– every player will play every other player exactly once.

Use this idea to list the sequence of games in a round robin tournament with
a) 5 players. b) 8 players.

\

# Unit 4
# Number Theory

> "Do you know what the foundation of mathematics is? The foundation of mathematics is numbers. If anyone asked me what makes me truly happy, I would say: numbers. And do you know why?
>
> Because the number system is like human life. First you have the natural numbers. The ones that are whole and positive. The numbers of the small child. But human consciousness expands. The child discovers longing, and do you know what the mathematical expression is for longing?
>
> The negative numbers. The formalization of the feeling that you are missing something. And human consciousness expands and grows even more, and the child discovers the in-between spaces. Between stones, between pieces of moss on the stones, between people. And between numbers. And do you know what that leads to? It leads to fractions. Whole numbers plus fractions produce the rational numbers. And human consciousness does not stop there. It wants to go beyond reason. It adds an operation as absurd as the extraction of roots. And produces irrational numbers.
>
> It's a form of madness. Because the irrational numbers are infinite. They cannot be written down. They force human consciousness out beyond the limits. And by adding irrational numbers to rational numbers, you get real numbers.
>
> It does not stop. It never stops. Because now, on the spot, we expand the real numbers with the imaginary ones, square roots of negative numbers. These are numbers we cannot picture, numbers that normal human consciousness cannot comprehend. And when we add the imaginary numbers to the real numbers, we have the complex number system. The first numbers system in which it's possible to explain satisfactorily the crystal formation of ice. It's like a vast, open landscape. The horizons. You head towards them and they keep receding."
>
> Smilla Jasperson in Peter Høeg's Miss Smilla's Feeling for Snow

Number theory begins with the integers. Especially interesting are the multiplicative properties of the integers, which come from the so-called prime numbers – those integers greater than one that have no divisors other than themselves and one. The systematic study of the integers goes back to the Pythagoreans (circa 570 B.C.), for whom prime numbers had a mystical significance. The primes were of particular interest to Euclid (circa 300 B.C.) -- he gave us the first proofs of the infinitude of primes.

Properties of integers enable us to solve problems in which we think that there is not enough information available to obtain a solution. For example:

**The Van Problem:** Suppose you are a teacher and you are taking your class on a field trip. A number of parents offer their vans as a means of transportation. When you try to put 6 children into each van, you notice that one child is left over. However, you can put an equal number of children into each van, with none left over, if there is one less van. How many children are there?

We will come back to solve this later -- give it a try now!

# 4.1 The Prime Numbers

Recall that whenever a positive integer is written as a product $n = d \times q$ of positive integers $d$ and $q$, then we say that $d$ (or $q$) is a **divisor** (or **factor**) of $n$, and $n$ is divisible by (or a **multiple** of) $d$. For convenience, when we want to say "$d$ is a divisor of $n$", we will write

$$d \mid n$$

and say "$d$ divides $n$". If $d$ is not a divisor of $n$ then we write $d \nmid n$. For example, $3 \mid 39$ but $3 \nmid 40$.

The set of **prime numbers** is the set of positive integers greater than 1 with no proper divisors (i.e. divisors other than themselves or 1). Integers greater than 1 that are not prime numbers are called **composite numbers**.

Composite numbers will always be a product of primes. This is because the smallest positive integer greater than 1 that divides $n$ has to be prime. If $d$ was the smallest positive divisor of $n$ other than 1, with $n = d \times q$ and $d$ not prime, then we would have $d = a \times b$ with $1 < a < d$. But then we could write $n = a \times (b \times q)$, so $a$ would be a positive divisor of $n$ other than 1 that is smaller than $d$. So this indirect argument proves $d$ has to be prime.

This means that the prime numbers are the building blocks of the set of positive integers greater than 1 with regards to multiplication, so it is of fundamental importance to know which numbers are prime. The list of smallest prime numbers is

$$2, 3, 5, 7, 11, 13, 17, 19, 23, 29, \ldots$$

The number 2 is the only even prime since $2 \mid n$ whenever $n$ is even. But how many primes are there? How can we find them?

Euclid proved that there are infinitely many primes. The proof is based on the idea that for any finite set of primes, we can always find a prime number that is not in that set.

To understand the idea, it might help to consider a small example. Start with the three prime numbers 3, 5, and 7. From them, we form the number $N = 3 \times 5 \times 7 + 1 = 106 = 2 \times 53$. Note that the prime factors of $N$ -- 2 and 53 -- are different from the numbers we started with.

We now look at an arbitrary finite collection of primes and see that the same trick produces new prime numbers.

- Start with a finite set of the prime numbers $\{p_1, p_2, p_3, \ldots, p_n\}$.
- Form the number $N = (p_1 \times p_2 \times p_3 \times \ldots \times p_n) + 1$
- Note that $N \equiv 1$ modulo each prime in the set, since $p_1 \times p_2 \times p_3 \times \ldots \times p_n \equiv 0 \bmod p_1$, $p_1 \times p_2 \times p_3 \times \ldots \times p_n \equiv 0 \bmod p_2$, and so on. This means that none of the primes in this list can divide $N$ because there is always a remainder of one.
- Because $N$ is a positive integer greater than 1, its smallest divisor greater than 1 is a prime number, and this prime number cannot be one of the ones in our finite set.

This shows that for any finite list of prime numbers, we can always find another prime number not in that list. Therefore, the list of prime numbers must go on forever, and so *the set of prime numbers is infinite.*

About 200 B.C. Eratosthenes gave us a method to find primes. For illustrative purposes, consider the following chart of the numbers 2, 3, ... , 100.

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 |
| 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 | 30 |
| 31 | 32 | 33 | 34 | 35 | 36 | 37 | 38 | 39 | 40 |
| 41 | 42 | 43 | 44 | 45 | 46 | 47 | 48 | 49 | 50 |
| 51 | 52 | 53 | 54 | 55 | 56 | 57 | 58 | 59 | 60 |
| 61 | 62 | 63 | 64 | 65 | 66 | 67 | 68 | 69 | 70 |
| 71 | 72 | 73 | 74 | 75 | 76 | 77 | 78 | 79 | 80 |
| 81 | 82 | 83 | 84 | 85 | 86 | 87 | 88 | 89 | 90 |
| 91 | 92 | 93 | 94 | 95 | 96 | 97 | 98 | 99 | 100 |

- We know 2 is a prime. Circle it and cross out all the remaining multiples of 2;

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | (2) | 3 | ~~4~~ | 5 | ~~6~~ | 7 | ~~8~~ | 9 | ~~10~~ |
| 11 | ~~12~~ | 13 | ~~14~~ | 15 | ~~16~~ | 17 | ~~18~~ | 19 | ~~20~~ |
| 21 | ~~22~~ | 23 | ~~24~~ | 25 | ~~26~~ | 27 | ~~28~~ | 29 | ~~30~~ |
| 31 | ~~32~~ | 33 | ~~34~~ | 35 | ~~36~~ | 37 | ~~38~~ | 39 | ~~40~~ |
| 41 | ~~42~~ | 43 | ~~44~~ | 45 | ~~46~~ | 47 | ~~48~~ | 49 | ~~50~~ |
| 51 | ~~52~~ | 53 | ~~54~~ | 55 | ~~56~~ | 57 | ~~58~~ | 59 | ~~60~~ |
| 61 | ~~62~~ | 63 | ~~64~~ | 65 | ~~66~~ | 67 | ~~68~~ | 69 | ~~70~~ |
| 71 | ~~72~~ | 73 | ~~74~~ | 75 | ~~76~~ | 77 | ~~78~~ | 79 | ~~80~~ |
| 81 | ~~82~~ | 83 | ~~84~~ | 85 | ~~86~~ | 87 | ~~88~~ | 89 | ~~90~~ |
| 91 | ~~92~~ | 93 | ~~94~~ | 95 | ~~96~~ | 97 | ~~98~~ | 99 | ~~100~~ |

- the least number remaining, 3, is then prime. Circle it and cross out all the remaining multiples of 3;

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | **2** | **3** | ~~4~~ | 5 | ~~6~~ | 7 | ~~8~~ | ~~9~~ | ~~10~~ |
| 11 | ~~12~~ | 13 | ~~14~~ | ~~15~~ | ~~16~~ | 17 | ~~18~~ | 19 | ~~20~~ |
| ~~21~~ | ~~22~~ | 23 | ~~24~~ | 25 | ~~26~~ | ~~27~~ | ~~28~~ | 29 | ~~30~~ |
| 31 | ~~32~~ | ~~33~~ | ~~34~~ | 35 | ~~36~~ | 37 | ~~38~~ | ~~39~~ | ~~40~~ |
| 41 | ~~42~~ | 43 | ~~44~~ | ~~45~~ | ~~46~~ | 47 | ~~48~~ | 49 | ~~50~~ |
| ~~51~~ | ~~52~~ | 53 | ~~54~~ | 55 | ~~56~~ | ~~57~~ | ~~58~~ | 59 | ~~60~~ |
| 61 | ~~62~~ | ~~63~~ | ~~64~~ | 65 | ~~66~~ | 67 | ~~68~~ | ~~69~~ | ~~70~~ |
| 71 | ~~72~~ | 73 | ~~74~~ | ~~75~~ | ~~76~~ | 77 | ~~78~~ | 79 | ~~80~~ |
| ~~81~~ | ~~82~~ | 83 | ~~84~~ | 85 | ~~86~~ | ~~87~~ | ~~88~~ | 89 | ~~90~~ |
| 91 | ~~92~~ | ~~93~~ | ~~94~~ | 95 | ~~96~~ | 97 | ~~98~~ | ~~99~~ | ~~100~~ |

- the least number remaining, 5, is then prime. Circle it and cross out all the remaining multiples of 5

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | **2** | **3** | ~~4~~ | **5** | ~~6~~ | 7 | ~~8~~ | ~~9~~ | ~~10~~ |
| 11 | ~~12~~ | 13 | ~~14~~ | ~~15~~ | ~~16~~ | 17 | ~~18~~ | 19 | ~~20~~ |
| ~~21~~ | ~~22~~ | 23 | ~~24~~ | ~~25~~ | ~~26~~ | ~~27~~ | ~~28~~ | 29 | ~~30~~ |
| 31 | ~~32~~ | ~~33~~ | ~~34~~ | ~~35~~ | ~~36~~ | 37 | ~~38~~ | ~~39~~ | ~~40~~ |
| 41 | ~~42~~ | 43 | ~~44~~ | ~~45~~ | ~~46~~ | 47 | ~~48~~ | 49 | ~~50~~ |
| ~~51~~ | ~~52~~ | 53 | ~~54~~ | ~~55~~ | ~~56~~ | ~~57~~ | ~~58~~ | 59 | ~~60~~ |
| 61 | ~~62~~ | ~~63~~ | ~~64~~ | ~~65~~ | ~~66~~ | 67 | ~~68~~ | ~~69~~ | ~~70~~ |
| 71 | ~~72~~ | 73 | ~~74~~ | ~~75~~ | ~~76~~ | 77 | ~~78~~ | 79 | ~~80~~ |
| ~~81~~ | ~~82~~ | 83 | ~~84~~ | ~~85~~ | ~~86~~ | ~~87~~ | ~~88~~ | 89 | ~~90~~ |
| 91 | ~~92~~ | ~~93~~ | ~~94~~ | ~~95~~ | ~~96~~ | 97 | ~~98~~ | ~~99~~ | ~~100~~ |

- and now 7

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | **2** | **3** | ~~4~~ | **5** | ~~6~~ | **7** | ~~8~~ | ~~9~~ | ~~10~~ |
| **11** | ~~12~~ | **13** | ~~14~~ | ~~15~~ | ~~16~~ | **17** | ~~18~~ | **19** | ~~20~~ |
| ~~21~~ | ~~22~~ | **23** | ~~24~~ | ~~25~~ | ~~26~~ | ~~27~~ | ~~28~~ | **29** | ~~30~~ |
| **31** | ~~32~~ | ~~33~~ | ~~34~~ | ~~35~~ | ~~36~~ | **37** | ~~38~~ | ~~39~~ | ~~40~~ |
| **41** | ~~42~~ | **43** | ~~44~~ | ~~45~~ | ~~46~~ | **47** | ~~48~~ | ~~49~~ | ~~50~~ |
| ~~51~~ | ~~52~~ | **53** | ~~54~~ | ~~55~~ | ~~56~~ | ~~57~~ | ~~58~~ | **59** | ~~60~~ |
| **61** | ~~62~~ | ~~63~~ | ~~64~~ | ~~65~~ | ~~66~~ | **67** | ~~68~~ | ~~69~~ | ~~70~~ |
| **71** | ~~72~~ | **73** | ~~74~~ | ~~75~~ | ~~76~~ | ~~77~~ | ~~78~~ | **79** | ~~80~~ |
| ~~81~~ | ~~82~~ | **83** | ~~84~~ | ~~85~~ | ~~86~~ | ~~87~~ | ~~88~~ | **89** | ~~90~~ |
| ~~91~~ | ~~92~~ | ~~93~~ | ~~94~~ | ~~95~~ | ~~96~~ | **97** | ~~98~~ | ~~99~~ | ~~100~~ |

The numbers that are left, {2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97}, are the primes less than or equal to 100.

This is a tedious method to determine primes; many sophisticated algorithms have been developed to determine whether or not a given number is prime. Such information is not important solely for academic reasons; it also has very important applications in encryption codes used by banks and credit card companies.

A simple algorithm to check whether or not a number $n$ is prime is to successively try to divide the number by all the small primes up to its square root. If none of them divide $n$, then $n$ is prime. Why? If $\boldsymbol{n}$ is not prime, then we could write it as the product $\boldsymbol{n = d \times q}$**,** where $\boldsymbol{d}$ is its smallest divisor greater than 1 and $\boldsymbol{d \leq q < n}$**.** But then $\boldsymbol{d^2 \leq n}$, so $\boldsymbol{n}$ has a proper divisor that is less than its square root.

For example, if $n = 113$, then we try to divide 113 by 2, 3, 5 and 7; since none of them are divisors of 113 we conclude that 113 is prime. We need not try to divide by primes greater than 7 as the next prime is 11, which is greater than the square root of 113.

Primes tend not to follow any identifiable patterns. You might notice in the small chart above that there are primes close together (in fact just 2 apart), the so-called *twin primes*:

3 & 5, 5 & 7, 11 & 13, 17 & 19, 29 & 31, 41 & 43, 59, & 61, 71 & 73.

It is not known if there are infinitely many such twin primes. This *Twin Primes problem* is just one of the many unanswered questions that remain concerning prime numbers.

## Exercises: Section 4.1

1. In our notes, we saw how Eratosthenes used a table to find prime numbers from 1 to 100. Use this method to find the primes between 101 and 200.

Determine whether the numbers given in exercises #2 to #7 are prime or composite.

| | | | | | |
|---|---|---|---|---|---|
| 2. | 637 | 3. | 383 | 4. | 522 604 |
| 5. | 317 | 6. | 457 | 7. | 725 |

8. Suppose we want to see why 713 is prime.

   a) Why is it unnecessary to prove directly that 15 <u>does not divide</u> 713?

   b) Why is it unnecessary to prove directly that 37 <u>does not divide</u> 713?

   c) Give the minimum list of numbers that you would have to try dividing into 713 to show that it is prime.

   d) Is 713 a prime? Why or why not?

   e) Exactly one of 703, 731, 715, and 733 is prime. Which one?

9. It is easy to see that there is only one prime triple (3, 5, 7) in which three successive primes are only 2 apart. Why?

# 4.2 Prime Factorization, Factor Trees, and Divisibility Rules

A **prime factorization** of a positive integer greater than 1 is the number written as a product of primes. An elementary way to find the prime factorization of a number is to repeatedly divide by smallest prime divisors. For example, the number 4620 is even, hence divisible by 2, and so we can write 4620 = 2 × 2310. 2310 is even, and 2310 = 2 × 1155, so 4620 = $2^2$ × 1155. 1155 is odd, so not divisible by 2. The next smallest prime is 3. 1155 ÷ 3 = 385, so 4620 = $2^2$ × 3 × 385. 385 ÷ 3 = 128 R 1, so 385 is not divisible by 3. The next smallest prime is 5, and 385 ÷ 5 = 77. Since 77 = 7 × 11, and both 7 and 11 are prime, we can conclude that

$$4620 = 2^2 \times 3 \times 5 \times 7 \times 11$$

is a prime factorization of 4620.

The prime factorization will always be unique as long as the prime factors are written in increasing order. This is what is known as the **Fundamental Theorem of Arithmetic**. It can be seen intuitively from the way we found the prime factorization of 4620: the smallest divisor greater than 1 that we find at each step will always be unique.

Repeatedly finding the smallest prime divisor is not always an efficient way to find the prime factorization of a number. For example, suppose we want to find the prime factorization of 13 013. 13 is a prime factor of 13 013 that is easy to find. We have
13 013 = 13 × 1 001. To check whether or not $13^2|13\,013$, consider 1 001 ÷ 13:

```
 1 001
– 650  = (50 × 13)
   351
– 260  = (20 × 13)
    91
  – 91 = (7 × 13)
     0
```

Since the remainder is 0, 1 001 is divisible by 13, and we have

$$\begin{aligned} 13\,013 &= 13 \times 1\,001 = 13^2 \times 77 \\ &= 7 \times 11 \times 13^2. \end{aligned}$$

So this is the prime factorization of 13 013.

**Factor Trees**

A **factor tree** is a visual representation that helps us to find the prime factorization of a number. For example, if we want to find prime factorization of 5 040 using a factor tree, we could start with the easiest factorization of 504 we can think of: 5040 = 10 × 504. We draw the first branches of the factor tree like this:

```
   5040
   /  \
 10    504
```

Since the prime factorization of 10 is 2 × 5, we can finish that branch:

```
        5040
        /  \
      10    504
     /  \
    2    5
```

Note that once we reach a prime factor, then that will be the end of the branch. If the factor is composite, then the branch can be continued further. Now work on the 504 factor. 504 is divisible by 4 since 504 ÷ 4 = 126. Since the prime factorization of 4 is 2 × 2, we can grow our factor tree this far:

```
       5040
       /  \
     10    504
    / \    /  \
   2   5  4    126
         / \
        2   2
```

Now consider 126. 126 = 2 × 63, 63 = 7 × 9, and 9 = 3 × 3. So our complete factor tree for 5040 looks like this:

```
       5040
       /  \
     10    504
    / \    /  \
   2   5  4    126
         / \   /  \
        2   2 2    63
                  /  \
                 7    9
                     / \
                    3   3
```

Collecting the numbers at all the ends of branches of our factor tree gives us the final prime factorization of 5040: $5040 = 2^4 \times 3^2 \times 5 \times 7$.

**Divisibility Rules**

We can see from the previous prime factorization examples that it would be valuable to have an easy rule that says when a number is divisible by a given prime or other smaller number. Such rules are known as **divisibility rules**. We have already been using a few of the easier divisibility rules without stating them:

***Divisibility rule for* 2**: An integer is divisible by 2 if and only if it is even.

***Divisibility rule for* 10**: An integer is divisible by 10 if and only if its last digit is a 0. There are similar rules for divisibility by 100, 1000, etc.

***Divisibility rule for* 5**: An integer is divisible by 5 if and only if its last digit is a 0 or a 5. There is a similar rule for divisibility by 25: the 2 last digits should be 00, 25, 50, or 75.

We have seen one more rule earlier in this text, that you may perhaps realize can be interpreted as a divisibility rule:

***Divisibility rule for* 9**: An integer is divisible by 9 if and only if the sum of its digits is divisible by 9.

This fact is what is behind the "casting out 9s" trick. It follows from the fact that for any digit $\boldsymbol{d}$, and any power $\boldsymbol{n}$, $( \boldsymbol{d} \times 10^{\boldsymbol{n}} ) \equiv ( \boldsymbol{d} \times 1^{\boldsymbol{n}} ) \bmod 9$. In fact, since 10 is also congruent to 1 modulo 3, the same kind of rule can be used for the prime number 3.

***Divisibility rule for* 3**: An integer is divisible by 3 if and only if the sum of its digits is divisible by 3.

Let's see how this rule works. Suppose we want to test if 834 567 is divisible by 3.

$$\begin{aligned} 834\,567 &= ( 8 \times 10^5 ) + ( 3 \times 10^4 ) + ( 4 \times 10^3 ) + ( 5 \times 10^2 ) + ( 6 \times 10^1 ) + 7 \\ &\equiv [( 8 \times 1^5 ) + ( 3 \times 1^4 ) + ( 4 \times 1^3 ) + ( 5 \times 1^2 ) + ( 6 \times 1^1 ) + 7] \bmod 3 \\ &\equiv ( 8 + 3 + 4 + 5 + 6 + 7) \bmod 3 \\ &\equiv 0 \bmod 3 \end{aligned}$$

Therefore, 834 567 is divisible by 3. In practice, we go directly to the step where we consider the sum of the digits mod 3.

The **divisibility rule for 6** is quite simple: In order to be divisible by 6, a number has to be even, and it has to be divisible by 3. You just combine the rules for 2 and 3 together.

Congruence concepts are very useful for establishing divisibility rules. Another example of this is the divisibility rule for 4. In order to be divisible by 4, an integer must first be even. But not every even integer is divisible by 4, so we need something more specific.

Since 4 divides 100, an integer will be congruent to 0 modulo 4 if and only if the number formed by its last two digits is congruent to 0 modulo 4. This gives us the

***Divisibility rule for* 4:** An integer is divisible by 4 if and only if the number formed by its last two digits is divisible by 4.

For example, 834 567 is not divisible by 4 because it is odd. To check whether or not the even integer 834 566 is divisible by 4, the rule says that we need to only check if 66 is divisible by 4. Subtract convenient multiples of 4 to reduce it: 66 – 40 = 26, and 26 ÷ 4 = 8 R 2. Therefore, 66 is not divisible by 4, so neither is 834 566.

The ***divisibility rule for 12*** works like the one for 6. The number has to be divisible by both 4 and 3.

There is a divisibility rule for 7, but it is so complicated and hard to remember that the simple technique of "subtracting multiples of 7" to reduce the number modulo 7 is actually easier to use!. Here is the (complicated) **divisibility rule for 7*:*** To find if a number is divisible by 7, take the last digit and multiply it by 2, then subtract it from the remaining digits. If the new number is divisible by 7, so is the original. (You might have to repeat this process several times before you get a small enough number.)

For example, suppose we want to know if 6930 is divisible by 7 using its divisibility rule:

693 – (2 × 0) = 693. Do it again:
69 - 3(2) = 69 - 6 = 63. Since 63 is divisible by 7, so is 6930.

On the other hand, if we just subtract multiples of 7 (which is exactly the same as dividing without keeping track of the quotient), we can do better:

$$\begin{array}{r} 6930 \\ -\ 6300 \\ \hline 630 \end{array}$$
← this is 90 × 7, so 6930 is divisible by 7.

This "subtracting multiples" idea is very useful as a general approach to divisibility by any positive integer where an easy rule is not available.

The idea for the **divisibility rule for 8** is almost the same as the one for 4, except we need to use 200 instead of 100, because 200 ≡ 0 mod 8 and 100 is not. So if we want to check if the number 2 453 544 is divisible by 8, we first reduce it modulo 200 to get 144, then subtract multiples of 8. Since 144 – 80 = 64 is divisible by 8, so is 2 453 544.

What about divisibility by 11? A number is divisible by 11 when it is congruent to 0 (mod 11). Because 10 ≡ – 1 (mod 11),

odd powers of 10 are congruent to –1 (mod 11);
even powers of 10 are congruent to 1 (mod 11).

This leads to the idea of an **alternating sum** of the number's digits. An example should make it clear what this means. For the value of 2 468 753 (mod 11),

$$\begin{aligned}
2468753 &= (2 \times 10^6)+(4 \times 10^5)+ (6 \times 10^4)+(8 \times 10^3)+ (7 \times 10^2)+ (5 \times 10) + 3 \\
&\equiv [(2 \times (-1)^6)+(4 \times (-1)^5)+(6 \times (-1)^4)+(8 \times (-1)^3)+(7 \times (-1)^2)+(5 \times (-1))+3] \bmod 11 \\
&\equiv [2 - 4 + 6 - 8 + 7 - 5 + 3] \bmod 11 \\
&\equiv (18 - 17) \bmod 11 \\
&\equiv 1 \bmod 11
\end{aligned}$$

Since 2 468 753 $\equiv$ 1 mod 11, we conclude that 2 468 753 is not divisible by 11.

In general, add every second digit starting at the right, then subtract the sum of the remaining digits from that total. The resulting alternating sum is congruent modulo 11 to the original number.

***Divisibility rule for 11:*** An integer is divisible by 11 if and only if the alternating sum of its digits is divisible by 11.

There is another complicated **divisibility rule for 13**: To find if a number is divisible by 13, take the last digit and multiply it by 9, then subtract it from the remaining digits. If this new number is divisible by 13, so is the original. (You might have to do this process several times until you get a small enough number.) We remark that the rules stated here for 7 and 13 are not unique, you will find various versions of them in the literature.

For example, to see if 586 is divisible by 13 using its rule:
58 - 6(9) = 58 - 54 = 4, which is not divisible by 13, so neither is 586.
Again the procedure of subtracting multiples of 13 is easier to remember and usually just as efficient:
584 – 650 = – 66, and – 66 + 65 = – 1 is not divisible by 13, so neither is 584.

For questions about divisibility by any odd prime other than 3 and 11, we usually default to the **"subtracting multiples" method**. In fact, this technique allows us to determine whether numbers in the millions are divisible by a 2-digit number in less than 10 steps.

For example, suppose you want to know whether or not 2 448 988 is divisible by 17. You just subtract the easy-to-calculate multiples of 17 (that you create by doubling or multiplying by 5) from the front of the number and repeat this until you reach 0 or a number smaller than your divisor.

```
  2 448 986
- 1 7
    748 986
  - 68          ←(68 = 34×2)
      68 986
   - 68
          986
        - 85
          136
        -   85
            51
       -   34
           17
         - 17
            0
```

Therefore, 2 448 986 is divisible by 17.

Here is another example of this general procedure. To check whether or not 76 906 249 is divisible by the prime 31:

```
  76 906 249
 - 62
  14 906 249
 - 12 4         ←(62 × 2)
    2 506 249
  - 2 48        ←(124 × 2)
       26 249
    -  24 8
        1 449
      - 1 24
          209
        - 124
           85
         - 62
           23
```

This shows that the remainder of 76 906 249 ÷ 31 will be 23, so this number is not divisible by 31.

## Exercises: Section 4.2

Find the prime factorization of each of the following.

| | | | | | |
|---|---|---|---|---|---|
| 1. | 120 | 2. | 630 | 3. | 1 800 |
| 4. | 168 | 5. | 88 | 6. | 409 |

Determine all possible values of the digit *x* that satisfies each divisibility statement for exercises #9 to #15.

| | | | | | |
|---|---|---|---|---|---|
| 7. | $2 \mid 463\ 97x$ | 8. | $3 \mid 463\ 97x$ | 9. | $11 \mid 463\ 97x$ |
| 10. | $2 \mid 123\ x24$ | 11. | $8 \mid 123\ x24$ | 12. | $3 \mid 123\ x24$ |
| 13. | $3 \mid 4x2\ 379$ | 14. | $9 \mid 4x2\ 379$ | 15. | $2 \mid 4x2\ 379$ |

Determine the smallest number that is divisible by all of the numbers in each group listed.

| | | | |
|---|---|---|---|
| 16. | 2, 3, 5, 6, 7, 8 | 17. | 2, 4, 6, 8, 10 |
| 18. | 3, 4, 5, 6 | 19. | 2, 3, 4, 8, 9, 10 |

Using the methods discussed in this section (not your calculator), determine whether each number in exercises #20 to #27 is divisible by:

a) 2 b) 3 c) 4 d) 5 e) 6 f) 8 g) 9 h) 11 i) 12

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 20. | 3 156 | 21. | 572 | 22. | 6 240 | 23. | 123 789 |
| 24. | 50 193 | 25. | 4 704 | 26. | 1 001 | 27. | 360 360 |

28. a) Devise a divisibility test for 15. (Hint: Compare the divisibility test for 6)
b) What is the smallest possible multiple of 15 that is made up of only the digits 0, 4, 7, each appearing the same number of times. (Hint: Use your knowledge here; do **not** use trial and error.)

29. How many 3-digit positive integers have digits whose product is 24?

Using the methods discussed in this section (not your calculator), determine whether each number in exercises #30 to #37 is divisible by:

a) 7 b) 13

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 30. | 3 156 | 31. | 572 | 32. | 6 240 | 33. | 123 789 |
| 34. | 50 193 | 35. | 4 704 | 36. | 1 001 | 37. | 360 360 |

38. Determine whether 1 831 019 is divisible by

a) 17 b) 19 c) 23 d) 29 e) 79

39. Use the method of subtracting multiples to check whether or not 360 360 is divisible by 7 or is divisible by 13.

# 4.3 How many divisors?

**Problem:** How many divisors does a number have?

For example,

- $36 = 2^2 3^2$ has 9 divisors: 1, 2, 3, 4, 6, 9, 12, 18, and 36;
- $100 = 2^2 5^2$ has 9 divisors: 1, 2, 4, 5, 10, 20, 25, 50 and 100.

On the other hand, $210 = 2^1 3^1 5^1 7^1$ has 16 divisors: 1, 2, 3, 5, 6, 7, 10, 14, 15, 21, 30, 35, 42, 70, 105 and 210. It is not too easy to list them all; is there a shortcut to finding out how many divisors there are?

Let's do something that seems a bit off topic. Suppose that we have 4 hats, 5 shirts, 3 pairs of pants, and 5 pairs of sandals and we wish to select one article from each category to make an outfit. How many different outfits can we create?

- Clearly we have 4 choices for a hat;
- these together with the 5 possible choices for a shirt make $4 \times 5 = 20$ choices for hat and shirt;
- these 20 choices along with the 3 possible pairs of pants increase our options to $20 \times 3 = 60$;
- these 60 combinations along with the 5 pairs of sandals bring us up to 300 combinations (why?).
- That is, all told there are $4 \times 5 \times 3 \times 5 = 300$ choices.

Let us suppose that we do not HAVE to choose one article of clothing from every category. (As we go through this, note that this does make it possible that you will be going naked!) This gives us $5 \times 6 \times 4 \times 6 = 720$ choices.

Consider the number $\boldsymbol{n} = 2^3 3^4 5^2 7^4$. How many distinct divisors does $\boldsymbol{n}$ have? Let's call this number **d(*n*)**. (Note how this relates to the example about getting dressed.) When we build a divisor, $\boldsymbol{m}$ of $\boldsymbol{n}$, it must include some of the primes 2, 3, 5, and 7 (and only these primes since they are the only ones that make this number) and it cannot have a power of any of these primes larger than we see appearing in $\boldsymbol{n}$.

If we are building a divisor $\boldsymbol{m}$ using symbols, it must look like $2^a 3^b 5^c 7^d$ where $a$, $b$, $c$ and $d$ are integers such that $0 \le a \le 3$, $0 \le b \le 4$, $0 \le c \le 2$, and $0 \le d \le 4$. That is, $\boldsymbol{m}$ cannot have a factor of $2^4$ in it because then $\boldsymbol{m}$ could not divide $\boldsymbol{n}$; $\boldsymbol{m}$ cannot have a factor of $3^{10}$ in it because then $\boldsymbol{m}$ could not divide $\boldsymbol{n}$; and so on for the other prime divisors of $\boldsymbol{n}$. So, how many $\boldsymbol{m}$'s can we build?

- How many choices are there for $a$ (that is for the $2^a$ part of $\boldsymbol{m}$)?
  $a$ can be 0, 1, 2, or 3; **4** choices in all.
- How many choices are there for $b$ (that is for the $3^b$ part of $\boldsymbol{m}$)?
  $b$ can be 0, 1, 2, 3 or 4; **5** choices in all.
- How many choices are there for $c$ (that is for the $5^c$ part of $\boldsymbol{m}$)?
  $c$ can be 0, 1, or 2; **3** choices in all.
- How many choices are there for $d$ (that is for the $7^d$ part of $\boldsymbol{m}$)?
  $d$ can be 0, 1, 2, 3 or 4; **5** choices in all.

Thus, the number of ways to build a number $\boldsymbol{m}$ is exactly the same as the number of ways we could put an outfit together from the clothes that we described earlier, namely **d(*n*)** $= 4 \times 5 \times 3 \times 5 = 300$ distinct divisors. Some of them are:

- $2^0 3^0 5^0 7^0 = 1$ take 0 of each prime
- $2^0 3^2 5^1 7^0 = 45$ take two 3's and one 5, none of the rest
- $2^1 3^1 5^1 7^1 = 210$ take one of each prime
- $2^1 3^0 5^2 7^2 = 2\,450$ take one 2, two 5's, two 7's, no 3's

etc.

Notice what we are saying here. The number $\boldsymbol{n}$ had different prime divisors (in this case 2, 3, 5 and 7). When we decide how many of each prime should or could appear in the divisor $\boldsymbol{m}$, what happens is that we have one more choice than the power to which the prime appeared in $\boldsymbol{n}$ (this is because we are allowed to take 0 of any particular prime). Thus, in our example we see that $n$ the number had the primes 2, 3, 5, and 7 appearing 3, 4, 2, and 4 times respectively and the number of divisors turns out to be

$$(3 + 1) \times (4 + 1) \times (2 + 1) \times (4 + 1) = 4 \times 5 \times 3 \times 5 = 300.$$

Similarly, $7^8 13^4$ has $(8 + 1) \times (4 + 1) = 9 \times 5 = 45$ divisors.

In general if the prime factorization of $\boldsymbol{n}$ is

$$\boldsymbol{n} = p_1{}^{a_1} p_2{}^{a_2} \ldots p_m{}^{a_m}$$

then

$$\mathbf{d}(\boldsymbol{n}) = (a_1 + 1) \times (a_2 + 1) \times (a_3 + 1) \times \ldots \times (a_k + 1).$$

Here are some more examples:

- $2^4 3^2 5^1 7^8$ has $(4 + 1) \times (2 + 1) \times (1 + 1) \times (8 + 1) = 5 \times 3 \times 2 \times 9 = 270$ divisors.
- $2^3 5^1 13^2$ has $(3 + 1) \times (1 + 1) \times (2 + 1) = 4 \times 2 \times 3 = 24$ divisors.
- $7^2 23^3$ has $(2 + 1) \times (3 + 1) = 3 \times 4 = 12$ divisors.

## Exercises: Section 4.3

Find the number of divisors for each of the following.

| | | | | | |
|---|---|---|---|---|---|
| 1. | $2^3 3^3 5^3$ | 2. | $3^5 13^2 23^1$ | 3. | 80 |
| 4. | 385 | 5. | 49 | 6. | $2^1 3^1 5^1 7^1 11^1$ |
| 7. | 782 | 8. | 64 | 9. | 112 |
| 10. | 121 | 11. | 211 | 12. | 279 |

13. The hallway in College West has 100 lockers all in a row. Each is unlocked and has the key in the lock. During the day a succession of students walk by. The first turns every key (thus locking them all); the second passerby turns every second key (thus unlocking half of them); the third passerby turns every third key;..., the $100^{th}$ passerby turns only the $100^{th}$ key. Which lockers are locked after the 100 students have passed by?

    **Hint**: A locker is locked or unlocked according to the number of divisors it has; in fact, we do not need to know the precise number of divisors but whether or not the number of divisors is ___ or ___. Try using 25 lockers instead of 100 at first.

# 4.4 The Greatest Common Divisor and Least Common Multiple

**Greatest Common Divisor**

Given the positive integers $m$ and $n$, a **common divisor** is an integer $d$ such that $d \mid m$ and $d \mid n$. The largest such integer $d$ is called the **greatest common divisor**, the **gcd**, of $m$ and $n$.

In this section we will look at methods of finding the gcd of two or more positive integers. We look at the gcd problem in two ways.

Suppose that we know $m = 2^3 7^2 13^4 17^3$ and that $n = 2^4 5^2 7^5 17^2$. A common divisor must be made up of prime powers that are common to both $m$ and $n$. The possible primes here are 2, 7 and 17 (we exclude the primes 5 and 13 since they only appear in one of the numbers). Since we are looking for the greatest common divisor we must take the largest power of each of these primes that is common to both $m$ and $n$ -- $2^3$ divides both but $2^4$ does not!

Thus we construct the gcd of these numbers as $d = 2^3 7^2 17^2$.

In this manner, once we know the prime factorization of the numbers $m$ and $n$ it is easy to read off their gcd. The method is not restricted to just two numbers of course.

Suppose that we want the gcd of 60, 210 and 450.

- Rewrite these numbers as $2^2\mathbf{3^1 5^1}$, $\mathbf{2^1 3^1 5^1} 7^1$ and $\mathbf{2^1} 3^2 5^2$.
- The gcd is then $d = 2^1 3^1 5^1 = 30$.

We can also find the gcd if we are given $m$ and $n$ but not their prime factorization. Sometimes it is difficult to factor numbers, and this makes the above method impractical. In this case we can turn to what is known as the **Euclidean Algorithm**. The idea is based on the fact that when we have two integers $m$ and $n$, with $m > n$, then

$$\gcd(m,n) = \gcd(m - n, n).$$

This is true because if $d \mid m$ and $d \mid n$ then it must be that $d \mid (m - n)$. For example, if we want to find gcd(243,189), just keep subtracting the smaller number from the larger one, until one number in the pair divides the other, and that will be the gcd:

gcd(243,189) = gcd(54,189) = gcd(54,135) = gcd(54,81) = gcd(54,27) = 27.

A variation of this method can be used to find the gcd of three or more numbers. Just keep subtracting the smallest number in the collection from <u>all</u> the others until one of the numbers is a divisor of all the rest. For example, if we want to find gcd(60, 210, 450) this way, we would do this:

gcd(60, 210, 450) = gcd(60,150,390) = gcd(60,90,330) = gcd(60,30,270), which is 30 because all three of these numbers are multiples of 30.

Try another one: gcd(213, 249, 276)?

gcd(216,249,276) = gcd(216,36,63) = gcd(180,36,27) = gcd(153,9,27), which must be 9 because 153 and 27 are both divisible by 9.

The Euclidean Algorithm is a faster variation of this that works for two numbers. At each stage, you *divide* the larger number by the smaller one, and replace it by the remainder. Let us illustrate by trying to find gcd(12 378, 3 054).

Since 12 378 = 4 × **3 054** + **162,** we have that gcd(12 378, 3 054) = gcd(162, 3 054).

This reduces the problem to one with smaller numbers. We thus can repeat the process and write

$$3\,054 = 18 \times \mathbf{162} + \mathbf{138}$$

and conclude that we want the gcd of 162 and 138. Repeating we write

$$162 = 1 \times \mathbf{138} + \mathbf{24}$$
$$138 = 5 \times \mathbf{24} + \mathbf{18}$$
$$24 = 1 \times \mathbf{18} + \mathbf{6}$$
$$18 = 3 \times \mathbf{6} + 0.$$

The last non-zero remainder, 6 in this case, is the required gcd. (This is so since it is clear that 6 is the gcd of 6 and 18, and thus is the gcd of 18 and 24, and thus is ... the gcd of 3 054 and 12 378.)

Another example: find the gcd of 252 and 198.

$$252 = 1 \times \mathbf{198} + \mathbf{54}$$
$$198 = 3 \times \mathbf{54} + \mathbf{36}$$
$$54 = 1 \times \mathbf{36} + \mathbf{18}$$
$$36 = 2 \times \mathbf{18} + 0$$

So the gcd of 252 and 198 is 18.

Pairs of numbers for which the gcd is 1 are called **relatively prime numbers** (1 is their only common divisor). For example, any two different prime numbers are relatively prime, but so are 15 and 77 (neither of which are prime). If you apply the Euclidean algorithm to two consecutive integers, you see right away that they are relatively prime. In general, two positive integers are relatively prime exactly when the last nonzero remainder in the Euclidean algorithm when it is applied to these numbers is a 1. For example, if we are asked whether or not 954 and 187 are relatively prime, we can look at what happens in the Euclidean algorithm:

$954 = 5\times187 + 19$
$187 = 9\times19 + 16$
$19 = 16 + 3$
$16 = 5\times3 + 1$ ← The last remainder is 1, so 954 and 187 are relatively prime.

**Least Common Multiple**

Given the positive integers ***m*** and ***n***, a common multiple is an integer ***q*** such that $\boldsymbol{m} \mid \boldsymbol{q}$ and $\boldsymbol{n} \mid \boldsymbol{q}$. The least such integer ***q*** is called the **least common multiple**, the **lcm**, of ***m*** and ***n***. In this section we will look at two methods of finding the lcm of two or more positive integers.

Suppose that we know $\boldsymbol{m} = 2^3 7^2 13^4 17^3$ and that $\boldsymbol{n} = 2^4 5^2 7^5 17^2$. A common multiple must contain all the prime powers that appear in <u>at least one</u> of ***m*** and ***n***. The possible primes here are 2, 5, 7, 13 and 17. We must take the <u>largest</u> power of each of these primes that <u>appears in either of</u> ***m*** and ***n***. Since we are looking for the <u>least</u> of the common multiples, we take the larger power between $2^3$ (from ***m***) and $2^4$ (from ***n***), but not from both. Similarly, we take the larger power between $5^0$ and $5^2$, the larger power between $7^2$ and $7^5$, and so on.

Thus, we construct the lcm of these numbers as $q = 2^4 5^2 7^5 13^4 17^3$.

In this manner, it is easy to read off the lcm of the numbers ***m*** and ***n*** once we know their prime factorization. The method is not restricted to just two numbers.

Suppose that we want the lcm of 60, 210 and 450.

* Rewrite these numbers in their prime factor form: $2^2 3^1 5^1$, $2^1 3^1 5^1 \mathbf{7^1}$ and $2^1 3^2 5^2$.
* The lcm is then $\boldsymbol{q} = 2^2 3^2 5^2 7 = 6\,300$.

A particular relationship between ***m***, ***n***, their gcd, and their lcm reveals another way to find the lcm of **two** numbers. Let us look at the pair of numbers we worked with above, $m = 2^3 7^2 13^4 17^3$ and $n = 2^4 5^2 7^5 17^2$.

Their gcd is $\boldsymbol{d} = 2^3 7^2 17^2$ and their lcm is $\boldsymbol{q} = 2^4 5^2 7^5 13^4 17^3$.
Look at $\boldsymbol{m} \times \boldsymbol{n}$ and compare it to the product of their gcd and lcm:

$$\begin{aligned} \boldsymbol{m} \times \boldsymbol{n} &= 2^3 7^2 13^4 17^3\, 2^4 5^2 7^5 17^2 \\ &= 2^3 2^4 5^2 7^2 7^5 13^4 17^3 17^2 \\ &= 2^7 5^2 7^7 13^4 17^5 \end{aligned}$$

$$\begin{aligned}\text{gcd} \times \text{lcm} = \boldsymbol{d} \times \boldsymbol{q} &= 2^3 7^2 17^2\ 2^4 5^2 7^5 13^4 17^3 \\ &= 2^3 2^4 5^2 7^2 7^5 13^4 17^2 17^3 \\ &= 2^7 5^2 7^7 13^4 17^5\end{aligned}$$

The same thing! That is, $\boldsymbol{m \times n = d \times q}$. The important idea is that we pick the smaller prime power common to both numbers to find the gcd, while we pick the largest prime power that appears to find the lcm. So when there are just two numbers, we pick both of the prime power factors that appear, and so the product of these prime powers will be the same.

So, let us find the lcm of 252 and 198. We saw earlier that the gcd is 18; thus,

$$252 \times 198 = 18 \times \text{lcm},$$

and so the lcm is

$$q = (252 \times 198) / 18 = 2\,772.$$

Of course you should note that if, for example, we know ***m, n*** and their lcm we could just as easily use this method to find the gcd; i.e.

$$252 \times 198 = \text{gcd} \times 2\,772$$

and thus the gcd is

$$(252 \times 198) \div 2\,772 = 18.$$

It is important to remember that this gcd method for finding the lcm, and vice versa, only works for two numbers. To find the lcm of three or more numbers, one can find the lcm of the first two, then use that lcm with the third number and so on. So if we want to find lcm of 60, 210, and 450, we can do this in two steps:

The lcm of 60 and 210 is (60 × 210) ÷ 30 = 420 (30 is the gcd of 60 and 210). The lcm of 420 and 450 is (420 × 450) ÷ 30 = 6 300 (30 is also the gcd of 420 and 450). Therefore, the lcm of 60, 210 and 450 is 6 300.

We have to be careful to use the gcd to find the lcd when more than two numbers are involved. Use this idea: lcm(***k,m,n***)= ***d*** ×lcm(***k/d,m/d,n/d***) when d=gcd(k,m,n). Applying this to lcm(60,210,450), we have gcd=30, so

lcm(60,210,450) = 30 × lcm(2,7,15)

Since lcm(2,7,15) can be seen to be 2×7×15 using prime factorization, we have lcm(60,210,450) = 30 × 210 = 6 300.

## Exercises: Section 4.4

In exercises #1 to #6, use prime factorization to find the greatest common divisor of the given pairs of numbers.

| | | | | | |
|---|---|---|---|---|---|
| 1. | 60 and 140 | 2. | 72 and 412 | 3. | 27 and 243 |
| 4. | 128 and 418 | 5. | 60 and 84 | 6. | 130 and 455 |

In exercises #7 to #12, use the Euclidean Algorithm to find the greatest common divisor of the given pairs of numbers.

| | | | | | |
|---|---|---|---|---|---|
| 7. | 84 and 132 | 8. | 30 and 385 | 9. | 310 and 460 |
| 10. | 234 and 470 | 11. | 60 and 204 | 12. | 455 and 4 235 |

For exercises #19 to #21, use prime factorization to find the greatest common divisor of the given three numbers.

| | | | | | |
|---|---|---|---|---|---|
| 13. | 54, 90, and 252 | 14. | 36, 174, and 378 | 15. | 198, 336, and 945 |

16. Use the Euclidean Algorithm twice (applying it to only two numbers at a time) to find the greatest common divisor of 92, 322, and 391.

17. If *x*, *y*, and *z* are primes and *a*, *b*, *c* are natural numbers such that $a > b > c$, what is the greatest common divisor of $x^a y^c z^b$ and $x^b y^a z^c$?

18. Rachel has 240 nickels and 288 pennies and she wants to place them in stacks so that each stack has the same number of coins, and so that each stack contains only one type of coin and there are none left over. What is the largest number of coins she can have in a pile?

19. Scott has a stack of two-by-four pieces of lumber. Some are 60 inches long, and some are 72 inches long. He wants to cut them all so all the pieces are the same length. What is the longest piece he can have without having any lumber left over?

20. a) Show that $n$ and $(n + 1)$ are relatively prime for any positive integer $n$.

b) Why is it true that whenever we pick 101 numbers from the set {1, 2, 3,…, 200}, we know we can find two numbers that are relatively prime? (Hint: Look at part a)

For exercises #21 to #30, use prime factorization to find the least common multiple for each set of numbers.

21. 18 and 24

22. 28 and 42

23. 63 and 90

24. 24 and 36

25. 2, 3, 7, and 9

26. 6, 7, and 9

27. 12, 18, and 30

28. 18, 36, and 54

29. 24, 60, and 70

30. 2, 6, 23, and 46

For exercises #31 and #32, use the given gcd to find the lcm of each set of numbers without factoring.

| | **Numbers** | **gcd** | **Lcm** |
|---|---|---|---|
| 31. | 12 and 18 | 6 | |
| 32. | 72 and 270 | 18 | |

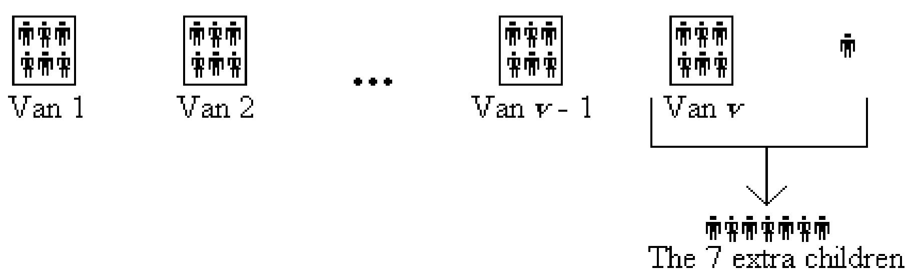
33. Erin and Elyse work on an assembly line, inspecting toy trucks. Erin inspects the wheels of every twelfth truck, and Elyse inspects the frame of every twenty-seventh truck. If they both start work at the same time, which truck will be the first that they will both inspect?

34. Tanner and Brodie are racing on their motorbikes around a circular track. Tanner completes one lap every 45 seconds and Brodie takes 40 seconds to complete one lap. If they both start at the same time, how long will it take before they both reach the starting point simultaneously?

# 4.4 Appendix – The Van Problem

We have developed enough ideas in this unit to let us solve the problem we mentioned at the beginning: Suppose you are a teacher and you are taking your class on a field trip. A number of parents offer their vans as a means of transportation. When you try to put 6 children into each van, you notice that one child is left over. However, you can put an equal number of children into each van, with none left over, if there is one less van. How many children are there?

Although this problem looks unanswerable, we will see that it is easier than it first appears. Let the number of children be $c$ and the original number of vans be $v$. We are then told that $c = 6v + 1$ (one child is left over when we have 6 children in each of $v$ vans):




Taking away the last van leaves 6 + 1 = 7 children to divide evenly among the remaining $v$ - 1 vans. But what numbers divide evenly into 7? Since 7 is a prime, we know there are only two numbers that to divide it evenly -- 1 and 7.

a) We can put each of the 7 remaining children into different vans, implying that there are seven vans remaining, so $v - 1 = 7$. Thus $v$ must equal 8, so there were 8 vans to begin with. We know that the number of children is equal to $6v + 1$ and $v = 8$, so there must be $c = 6(8) + 1 =$ **49 children**.

b) The other option is to put all of the 7 remaining children into a single remaining van. Since only one van remains and one van was taken away, we determine that there originally were 2 vans. By substituting 2 into our formula, we see that there are $c = 6(2) + 1 =$ **13 children**.

Note that the number of vanless children after one van has been removed is equal to the number of children added to each remaining van times the number of remaining vans.

Since 1 and 7 are the only factors of 7, we can conclude that the only possible solutions to this problem are 13 and 49 children.

## Exercises: Section 4.4 Appendix

1. Suppose some people organize car-pools for work, knowing that more than one person will have drive each day. They try to put four people in each vehicle, but one person is left over. However, the group can split themselves up evenly if there is one less vehicle. How many people are car-pooling for work? (Find the smallest possible answer.)

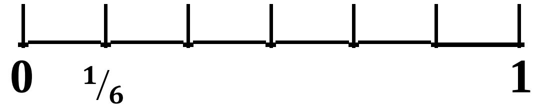
2. A travel company organized a bus trip across the country. They decided to put the same number of people on each bus. When they had 12 people on each bus, one person was left over. The organizers noticed that the number of travellers would be divided evenly if two busses were left behind. How many travellers are going on this bus trip, if the number of travellers is more than 100 but less than 1000?

# 4.5 Arithmetic with Fractions

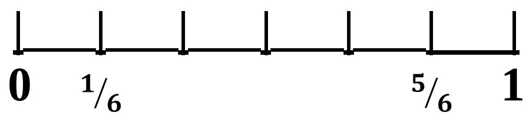
**The Numbers represented by Fractions**

A number in the form $^{\boldsymbol{p}}/_{\boldsymbol{q}}$, where $\boldsymbol{p}$ and $\boldsymbol{q}$ are integers and $\boldsymbol{q} \neq 0$, is called a **fraction**. The integer on the top of the fraction is called its **numerator**, and the integer on the bottom is called its **denominator**. Any number that can be represented as a fraction is defined to be a **rational number**.

Recall that we introduced fractions $^{\boldsymbol{p}}/_{\boldsymbol{q}}$ in Unit 2 to represent the number that is the solution to the integer division problem $\boldsymbol{p} \div \boldsymbol{q}$. Viewing fractions as the solutions to division problems gives us two important visual representations of the quantities represented by fractions. For example, if we want to visualize the fraction $^1/_6$, we can think of a line segment of length 1, divide it into 6 equal parts, and view $^1/_6$ as the length of one of these parts:



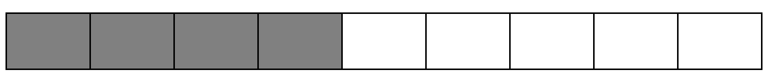
One can then generalize this to understand the fractions $^{1}/_{q}$ with arbitrary positive integer denominators as the length of one part when a line segment of length 1 unit is divided into $q$ equal parts. The fraction $^{p}/_{q}$ will then be the length of the segment formed when p of these line segments of length $^{1}/_{q}$ are joined together. That is, $^{p}/_{q} = p \times {}^{1}/_{q}$. (This also makes sense because $^{p}/_{q} = p \div q = (p \times 1) \div q = p \times (1 \div q) = p \times {}^{1}/_{q}$.) For example, we can understand $^{5}/_{6}$ as as the length of this line segment:




This view of fractions includes the possibility that the denominator $q$ is 1, so it is another way to see that every integer $n$ can be represented as the fraction $^{n}/_{1}$, which makes sense because $n \div 1 = n$.

If the numerator of a fraction $^{p}/_{q}$ of positive integers is larger than the denominator, then the division problem $p \div q$ has a nonzero quotient, and a remainder that is smaller than $q$. If $p = (d \times q) + r$, then we can represent $^{p}/_{q}$ as the **mixed numeral** $d\,{}^{p}/_{q}$ , which is a short way of writing $d + {}^{p}/_{q}$ . So for example, $^{23}/_{7} = 3\,{}^{2}/_{7}$ .

Another reprentation of the quantity represented by a fraction of positive integers is as a part of a whole region. Again, we think of $^{1}/_{q}$ first, then put $p$ of those portions together to get $^{p}/_{q}$. For example, the fraction $^{4}/_{9}$ is represented by the shaded region:



To understand fractions involving negative integers, we use some simple sign patterns to determine if the overall sign of the number is negative or positive:

$$^{(-p)}/_{q} = -\,{}^{p}/_{q} = {}^{p}/_{(-q)}$$

These can be seen from the properties of division and multiplication:

$^{(-p)}/_{q} = (-p) \div q = [(-1) \times p] \div q = (-1) \times (p \div q)$, and
$^{p}/_{(-q)} = p \div (-q) = p \div [(-1) \times q] = [p \div (-1)] \div q = (-p) \div q.$

A negative fraction is positioned on the real number line to the left of 0, just as we do with negative integers.

**Equality of Fractions**

One problem with fractions is that a number can be represented by a fraction in an unlimited number of ways. From the common divisor shortcut for division, we know that for any nonzero integer ***n***, $(\boldsymbol{p} \times \boldsymbol{n}) \div (\boldsymbol{q} \times \boldsymbol{n}) = \boldsymbol{p} \div \boldsymbol{q}$. Therefore,

$$\boldsymbol{p}/\boldsymbol{q} = (\boldsymbol{p} \times \boldsymbol{n})/(\boldsymbol{q} \times \boldsymbol{n}) \text{, for any nonzero integer } \boldsymbol{n},$$

and the equality here means equality as the numbers the two (different) fractions represent. For example, the following fractions all represent the same number:

$$2/3 = 4/6 = 6/9 = 10/15 = -2/-3 = 14/21 = 62/93 = 100/150 .$$

Given a fraction of two positive integers $\boldsymbol{p}/\boldsymbol{q}$, if the gcd of ***p*** and ***q*** is ***d***, then we have $\boldsymbol{p} = \boldsymbol{a} \times \boldsymbol{d}$ and $\boldsymbol{q} = \boldsymbol{b} \times \boldsymbol{d}$, so $\boldsymbol{p}/\boldsymbol{q} = (\boldsymbol{a} \times \boldsymbol{d})/(\boldsymbol{b} \times \boldsymbol{d}) = \boldsymbol{a}/\boldsymbol{b}$.

This is a better representation, because it is a fraction of smaller relatively prime integers. This is the **lowest terms** representation of the rational number. For example, one can see in the previous example that $2/3$ is the lowest terms fraction that represents the number.

**Multiplication and Division of Fractions**

The procedures for the multiplication and division of integer fractions are easier to understand than the procedures for adding and subtracting fractions, so we will look at them first.

The first thing to understand about the multiplication of two fractions is how to multiply two fractions of the form $\boldsymbol{1}/\boldsymbol{q}$ and $\boldsymbol{1}/\boldsymbol{b}$. The meaning of the product $\boldsymbol{1}/\boldsymbol{q} \times \boldsymbol{1}/\boldsymbol{b}$ is that it is the length of one part of a line segment of length $\boldsymbol{1}/\boldsymbol{b}$ that has been divided into ***q*** equal parts. Since it takes ***q*** of these parts to make a line segment of length $\boldsymbol{1}/\boldsymbol{b}$, and b of those to make a line segment of length 1, it will take ***qb*** line segments of length $\boldsymbol{1}/\boldsymbol{q} \times \boldsymbol{1}/\boldsymbol{b}$ to make a line segment of length 1. Therefore,

$$\boldsymbol{1}/\boldsymbol{q} \times \boldsymbol{1}/\boldsymbol{b} = \boldsymbol{1}/(\boldsymbol{q} \times \boldsymbol{b}).$$

Once we understand this, the general pattern for multiplication of fractions comes from the properties of multiplication:

$$\begin{aligned}\boldsymbol{p}/\boldsymbol{q} \times \boldsymbol{a}/\boldsymbol{b} &= (\mathbf{p} \times \boldsymbol{1}/\boldsymbol{q}) \times (\mathbf{a} \times \boldsymbol{1}/\boldsymbol{b}) = (\mathbf{p} \times \mathbf{a}) \times (\boldsymbol{1}/\boldsymbol{q} \times \boldsymbol{1}/\boldsymbol{b}) = (\mathbf{p} \times \mathbf{a}) \times \boldsymbol{1}/(\boldsymbol{q} \times \boldsymbol{b}) \\ &= (\boldsymbol{p} \times \boldsymbol{a})/(\boldsymbol{q} \times \boldsymbol{b}).\end{aligned}$$

So when we multiply fractions, we can think of the procedure as just multiplying straight across, that is, multiply the numerators and multiply the denominators. In practice, we usually combine multiplication of fractions with reduction to lowest terms in the same step. For example: If we want to multiply the two lowest terms fractions $^{5}/_{12}$ and $^{8}/_{15}$, then we can think of the numerators and denominators in factored form first, and cancel common factors in the collective numerator and denominator before multiplying:

$$^{5}/_{12} \times {^{8}/_{15}} = {^{5}/_{(2^2 \times 3)}} \times {^{2^3}/_{(3 \times 5)}} = {^{\cancel{5}}/_{(\cancel{2^2} \times 3)}} \times {^{2^{\cancel{3}}}/_{(3 \times \cancel{5})}} = {^{2}/_{9}}.$$

Division of fractions can be converted to multiplication of fractions using properties of division:

$$^{p}/_{q} \div {^{a}/_{b}} = (p \div q) \div (a \div b) = [(p \div q) \div a] \times b = [p \div (q \times a)] \times b = (p \times b) \div (q \times a)$$
$$= {^{p}/_{q}} \times {^{b}/_{a}}$$

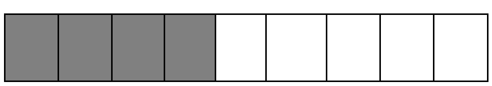
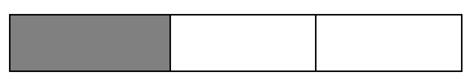
So the procedure for division of fractions is simply to "invert and multiply".

**Addition and Subtraction of Fractions**

The idea for addition of fractions comes from the fact that we can already see how to add fractions that have the same denominator. Since $^{p}/_{q} = p \times {^{1}/_{q}}$ , we can add two fractions with the same denominator $q$ just by adding their numerators:

$$^{p}/_{q} + {^{a}/_{q}} = (p \times {^{1}/_{q}}) + (a \times {^{1}/_{q}}) = (p + a) \times {^{1}/_{q}} = {^{(p+a)}/_{q}}.$$

Next, consider the case where one of the denominators of the two fractions is a multiple of the other. For example, suppose we want to add the fractions $^{4}/_{9}$ and $^{1}/_{3}$. These are the fractions represented by these regions:

So we want the fraction represented by this region:

Fortunately, $^{1}/_{3} = {^{3}/_{9}}$, so we can divide the region representing $^{1}/_{3}$into 3 parts, and the region representing $^{4}/_{9} + {^{1}/_{3}}$ is the same as the one representing $^{4}/_{9} + {^{3}/_{9}}$, which we know from above will be equal to $^{7}/_{9}$ .

This is also the basic idea for adding and subtracting arbitrary integer fractions. We convert both fractions to a convenient denominator that is a multiple of both the original denominators. The best denominator will be the smallest positive integer that is a multiple of both denominators, so this is exactly the lcm of the denominators. Once the two numbers are written as fractions over the same denominator, we can add or subtract numerators to get the result.

For example, suppose we want to find ${}^{19}/_{30} - {}^{11}/_{18}$. The lcm of 30 and 18 is 90. So

$${}^{19}/_{30} - {}^{11}/_{18} = {}^{(19\ \times 3)}/_{(30 \times 3)} - {}^{(11 \times 5)}/_{(18 \times 5)} = {}^{57}/_{90} - {}^{55}/_{90}$$

$$= {}^{2}/_{90} = {}^{1}/_{45}.$$

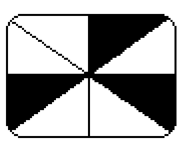
As we can see from the above example, there may be an extra step required after the adding or subtracting is finished in order to express the answer as a lowest terms fraction.

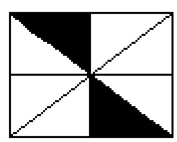
## Exercises: Section 4.5

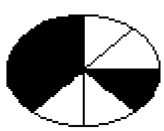
For exercises #1 to #6, reduce the fraction to its lowest terms.

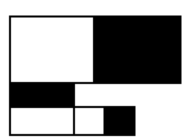
1. $\frac{72}{412}$
2. $\frac{27}{243}$
3. $\frac{128}{418}$
4. $\frac{310}{460}$
5. $\frac{234}{470}$
6. $\frac{455}{4235}$

Write the fraction in lowest terms that represents the shaded portion of each figure.

7. 

8. 

9. 

10. 

11. What number is one half of one quarter of one tenth of 800?

12. Unlike the integers, the rationals have a remarkable density property: between any two rational numbers, we can always find another. One familiar way is to take the average of the two numbers.

    The **average** of a set of ***n*** numbers is (the sum of the ***n*** numbers) ÷ ***n.***

So midway between $^1/_3$ and $^1/_5$ is their average:

$$\frac{1}{2}\left(\frac{1}{3}+\frac{1}{5}\right) = \frac{1}{2}\left(\frac{5}{15}+\frac{3}{15}\right) = \frac{1}{2}\left(\frac{8}{15}\right) = \frac{8}{30} = \frac{4}{15}.$$

Find the average of the following collections of rational numbers:

a) $^2/_3$ and $^2/_9$ b) $^7/_6$, $^5/_{21}$ and $^9/_{14}$ c) $^1/_{60}$, $^1/_{30}$, $^1/_{20}$, and $^1/_{18}$.

# 4.6 Fractions and Decimals

We commonly write rationals in decimal form in our base-10 system; for example, $^1/_4$ represents the same number as 0.25 as follows:

$$0.25 \text{ means } 2 \times \frac{1}{10} + 5\,\frac{1}{10^2}$$

$$\text{adding } \frac{2}{10} + \frac{5}{10^2} \text{ we see that their sum is } \frac{(20+5)}{10^2} = \frac{25}{100} = \frac{1}{4}.$$

In the other direction, if we start with $^1/_4$, we can show that $^1/_4$ = 0.25 by long division.

```
     .25
  4)1.00
     8
     20
     20
      0
```

$$\text{Similarly, } 0.375 = \frac{3}{8} \text{ as } 0.375 = \frac{3}{10} + \frac{7}{10^2} + \frac{5}{10^3} = \frac{(300+70+5)}{1000} = \frac{3(125)}{8(125)} = \frac{3}{8}.$$

On the other hand, starting with $^3/_8$ we can show $^3/_8$ = 0.375 by long division:

```
     .375
  8)3.000
    24
     60
     56
      40
      40
       0
```

A repeating decimal is a number written in decimal notation whose digits to the right of the decimal point eventually consist of endless repetitions of a block of numbers. For example, $^1/_3$ = 0.3333… and $^{2779}/_{110}$ = 25.2636363…. We use a bar over the repeating part of the decimal as shorthand; thus

$^1/_3$ is written as $0.\overline{3}$, while $^{2779}/_{110}$ becomes $25.2\overline{63}$.

$^1/_4$ = 0.250000000… is also a repeating decimal, which we usually write as 0.25.

This leads to the questions: What is the relationship between fractions and decimals? How can we easily go back and forth between fractions and decimals?

In fact, any fraction can be written as a repeating decimal and, conversely, any repeating decimal can be written as a fraction.

We can write any fraction as a repeating decimal by using long division: if we divide a counting number $q$ into 1 by long division, the only possible remainders at each step are the integers 0, 1, 2, ..., $q$-1. This means that within the first $q$ steps of the division process a remainder must repeat, and thus the decimal will repeat. For example, consider $^1/_7$:

```
     .142857
  7)1.000000
     7
     30
     28
      20
      14
       60
       56
        40
        35
         50
         49
          1
```

With a remainder of 1, we are dividing 7 into 1 again and the cycle repeats; thus
$^1/_7 = 0.\overline{142\,857}$.

Another example is ${}^{691}/_{222}$ where long division produces

$$
\begin{array}{r}
3.1126 \\
222\overline{)691.0000} \\
\underline{666}\phantom{.0000} \\
250\phantom{000} \\
\underline{222}\phantom{000} \\
\mathbf{28}0\phantom{00} \\
\underline{222}\phantom{00} \\
\mathbf{58}0\phantom{0} \\
\underline{444}\phantom{0} \\
\mathbf{136}0 \\
\underline{1332} \\
\mathbf{28}
\end{array}
$$

The 28 has repeated (after the decimal point), so continuing the division will result in the 126 being repeated, and hence $691/222 = 3.1\overline{126}$.

In the other direction, suppose that we wish to write the repeating decimal N = 1.282828... in the form ${}^{p}/_{q}$ where $p$ and $q$ are integers. The **period** (the length of the repeating part) is 2 (the 28 is being repeated). If we move the decimal point to the right two digits by multiplying the number by $10^2$ and then subtract N, we find

$$
\begin{array}{rl}
10^2\,N = & 128.282\,828\ldots \\
\underline{N =} & \underline{\phantom{12}1.282\,828\ldots} \\
\mathbf{99\;\; N =} & \mathbf{127.000\ldots} \\
\mathbf{=} & \mathbf{127}
\end{array}
$$

Hence 99N = 127 and hence N = ${}^{127}/_{99}$ which is the same as 1.282 828.... Similarly if M = 0.142 857 142 857 142 857... then the period is 6 and we have

$$
\begin{array}{rl}
10^6\,M = & 142\,857.142\,587\,142\,587\ldots \\
\underline{M =} & \underline{\phantom{142\,857}.142\,587\,142\,587\ldots} \\
999\,999M = & 142\,857
\end{array}
$$

and M = ${}^{142\,857}/_{999\,999}$. Sometimes we want to write the fractions in lowest terms. If we use our knowledge of divisibility by 9, 3 and 11 repeatedly we see that

M = ${}^{15\,873}/_{111\,111}$ = ${}^{5\,291}/_{37\,037}$ = ${}^{481}/_{3\,367}$ which we finally can reduce to M = ${}^{1}/_{7}$.

Let's try N = 34.912 312 312 3... The period in this case is three thus we look at $10^3$N and find

$$
\begin{array}{rl}
10^3\,N = & 34\,912.312\,312\,3\ldots \\
\underline{N =} & \underline{\phantom{34\,8}34.912\,312\,3\ldots} \\
999\;\; N = & 34\,877.4
\end{array}
$$

equivalently (moving the decimal by multiplying both sides by 10),

$$\mathbf{9\ 990N = 348\ 774,}$$

and therefore N = $^{348\ 774}/_{9\ 990}$.

Sometimes a peculiar thing happens with this method. Consider the repeating decimal N = 0.099 99... Then

$$\begin{array}{r} 10N = 0.9999\ldots \\ \underline{N = 0.0999\ldots} \\ 9N = 0.9 \end{array}$$

equivalently (moving the decimal)

$$\mathbf{90N = 9}$$

and therefore N = $^9/_{90} = {}^1/_{10}$.

But don't we usually just write $^1/_{10}$ as 0.1?

As a consequence, we are led to conclude that $0.0\overline{9} = 0.1$.

The previous example shows that representations of numbers as decimals are not unique. Any terminating decimal number like 0.1 can also be represented as a decimal that ends in repeating 9's.

## Exercises: Section 4.6

Convert the following fractions into decimal form.

| | | | | | |
|---|---|---|---|---|---|
| 1. | 5/8 | 2. | 7/16 | 3. | 9/11 |
| 4. | 13/15 | 5. | 17/21 | 6. | 55/65 |

Convert each decimal into a fraction in its lowest terms.

| | | | | | |
|---|---|---|---|---|---|
| 7. | 0.6 | 8. | 0.7 | 9. | 0.85 |
| 10. | 0.115 | 11. | 0.834 | 12. | 0.798 6 |

For #13 to #18, convert each repeating decimal into a fraction in its lowest terms.

13. $0.\overline{3}$ 14. $0.\overline{41}$ 15. $2.\overline{36}$

16. $1.\overline{12}$ 17. $0.5\overline{3}$ 18. $0.0\overline{5}$

19. $1.23\overline{45}$ 20. $98.7\overline{6543}$

21. a) Find the decimal that represents 1/6.

b) Find the decimal that represents 5/6.

c) Add the two decimal expressions found in a) and b) to obtain a decimal expression for 1/6 + 5/6 = 6/6 = 1. Explain this result.

22. It is a fact that 567 895 678 956 789 × 123 451 234 512 345 = 70 107 422 641 441 946 362 742 060 205. Use this fact to compute:

a) 56 789 567 895 678 900 × 1 234 512 345.123 45

b) $\dfrac{701\ 074\ 226\ 414\ 419\ 463\ 627.420\ 602\ 05}{1\ 234\ 512\ 345.123\ 45}$

23. Add $3.\overline{218} + 1.\overline{407}$, and write the sum of both as a repeating decimal and as a fraction reduced to lowest terms.

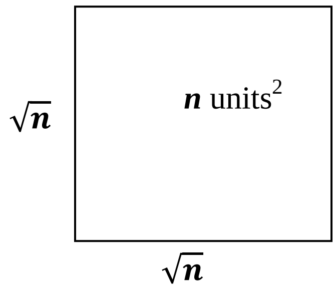
# 4.7 Irrational Numbers and Roots

One simple consequence of our discussion in the previous section is that not all numbers are integers or rational numbers. For example, what type of number is 0.01001000100001…? It is clearly a non-repeating decimal. Non-repeating decimals are called **irrational numbers.** We have just seen that any fraction can be written as a repeating decimal. It follows that irrational numbers are precisely those that cannot be written in the form $p/q$, where $p$ and $q$ integers and $q$ is not equal to zero.

One familiar irrational is π, the ratio of the circumference of a circle to its diameter. π is approximately 3.141 592 653 589 793 238 46. Although it is not obvious, the decimal expansion never repeats, no matter how far we take it. Since the proof that π is irrational is difficult, let us turn instead to other familiar irrationals.

## Roots

Recall from Unit 2 that the square root of a positive integer $\boldsymbol{n}$ is the number $\boldsymbol{m}$ for which $\boldsymbol{m^2 = n}$. The terminology makes sense, because the square root of $\boldsymbol{n}$ is the length of one side of a square whose area is $\boldsymbol{n}$ square units.




This representation of the square root as the length of a line segment shows that the square root of a positive number is an actual quantity, i.e. the square root of a positive number is a positive number.

We can also represent $\sqrt{\boldsymbol{n}}$ as the power $\boldsymbol{n}^{½}$, which makes sense because:

$$\boldsymbol{n}^{½} \times \boldsymbol{n}^{½} = (\boldsymbol{n}^{½})^{\mathbf{2}} = \boldsymbol{n}^{(½\times\mathbf{2})} = \boldsymbol{n}^{1} = \boldsymbol{n}.$$

As with square roots, we can think of $m^{\text{th}}$ roots as fractional powers:

$$\sqrt[m]{\boldsymbol{n}} = \boldsymbol{n}^{\mathbf{1}/m}.$$

The representation of a number by a root may not be unique. For example, $\sqrt{50}$ represents the same number as $5\sqrt{2}$, because $\sqrt{50} = \sqrt{5^2 \times 2} = \sqrt{5^2} \times \sqrt{2} = 5 \times \sqrt{2} = 5\sqrt{2}$. The representation of this number as $5\sqrt{2}$ is preferred by convention, since all perfect square factors involved have been reduced so that the smallest possible number remains under the root. Such a representation of a number expressed as a root is said to be **simplified**.

To reduce an arbitrary root to its simplified form, we use the prime factorization, and reduce the pattern $\sqrt[m]{q^m}$ to $q$ to simplify whenever possible. For example,

$\sqrt[3]{108} = \sqrt[3]{2^2 \times 3^3} = \sqrt[3]{2^2} \times \sqrt[3]{3^3} = 3\sqrt[3]{2^2} = 3\sqrt[3]{4}$,

$\sqrt[5]{96} = \sqrt[5]{2^5 \times 3} = \sqrt[5]{2^5} \times \sqrt[5]{3} = 2\sqrt[5]{3}$,

$\sqrt{6615} = \sqrt{3^3 \times 5 \times 7^2} = \sqrt{3^3} \times \sqrt{5} \times \sqrt{7^2} = \sqrt{3^2 \times 3} \times \sqrt{5} \times 7 = 3 \times \sqrt{3} \times \sqrt{5} \times 7$

Some other useful patterns with exponents are:

$$\boldsymbol{n^a} \times \boldsymbol{n^b} = \boldsymbol{n^{a+b}},\ (\boldsymbol{n^a})^{\boldsymbol{b}} = \boldsymbol{n^{a\times b}},\ \text{ and }\ \boldsymbol{n^{-1}} = \frac{\mathbf{1}}{\boldsymbol{n}}.$$

**Roots as Irrationals**

Returning to our discussion of irrational numbers, it turns out that the square roots of positive integers that are not perfect squares are always irrational numbers. For example, consider $\sqrt{2}$. By a sequence of estimations, we can determine that $\sqrt{2}$ lies between 1.414 and 1.415:

$$(1.4)^2 = 1.96 < 2 < (1.5)^2 = 2.25$$
$$(1.41)^2 = 1.9881 < 2 < (1.42)^2 = 2.0164$$
$$(1.414)^2 = 1.999396 < 2 < (1.415)^2 = 2.002225.$$

This sequence of estimations may lead one to believe that $\sqrt{2}$ can be represented as a fraction. To show that $\sqrt{2}$ is not a fraction, we use an indirect argument:

If $\sqrt{2}$ can be represented by a fraction, then this fraction can be written in lowest terms, so we would have that $\sqrt{2} = a/b$, where *a* and *b* are relatively prime integers. After squaring both sides, we get 2 = $a^2/b^2$. But if $a/b$ is in lowest terms, then $a^2/b^2$ is also in lowest terms, because the only primes that divide a perfect square integer $\boldsymbol{m^2}$ are the primes that divide ***m.*** Since 2 = $2/1$ is the lowest terms representation of this number, it follows from $2/1 = a^2/b^2$ that 2 = $a^2$ and 1 = $b^2$. But we know that 2 = $a^2$ is impossible, because 2 is not a perfect square integer. Therefore, the assumption that $\sqrt{2}$ is rational leads to something impossible. We then have to conclude by indirect reasoning that $\sqrt{2}$ must be an irrational number.

Similar arguments can be used to show that whenever an $m^{th}$ roots of an integer is not an integer, then it will be irrational.

## Exercises: Section 4.7

Identify whether each of the following numbers are rational or irrational.

1. $4/13$
2. $\sqrt{27}$
3. 0.121 121 112...
4. 3.14
5. $0.0\overline{61}$
6. $\sqrt[3]{10}$

Explain why the following numbers are irrational.

7. $\sqrt[3]{4}$

8. $\sqrt{8}$

9. 123.465 789 101 112…

10. $\sqrt{2}/5$

11. $\sqrt[4]{56}$

12. 5.646 646 664 666 646…

Explain why the following numbers are rational.

13. 21/29

14. $0.\bar{3}$

15. $\sqrt{16}$

16. $25/\sqrt{4}$

17. $\sqrt[3]{8/27}$

18. 1.234 512 345 123 451…

Find the following. Express your answer as a radical in simplified form.

13. $\sqrt{2} \times \sqrt[5]{8}$

14. $\sqrt[3]{16} \times \sqrt[4]{8}$

15. $\sqrt[3]{108} \times \sqrt[3]{54}$

16. $\sqrt[3]{100} \times \sqrt[6]{100\,000}$

17. $\sqrt[6]{250} \div \sqrt[9]{6250}$

18. $\sqrt{30} \times \sqrt{45} \div \sqrt[6]{90}$

Decide which of the following statements are True and which statements are False. Support your answer with a short justification (when true) or a simple counterexample (when false).

19. The sum of two irrational numbers is always an irrational number.

20. The sum of a rational number and an irrational number is never a rational number.

21. The product of two irrational numbers is always an irrational number.

22. The product of an irrational number and a rational number is always an irrational number.

23. Prove that $\sqrt{3}$ is irrational.

24. Prove that $\sqrt[3]{2}$ is irrational.

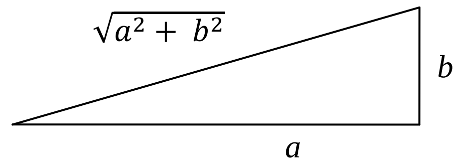
25. Prove that $\sqrt{16}$ is rational.

26. We know that $\sqrt{2}$ is irrational. What about $\sqrt{\sqrt{2}}$ ? Explain your answer.

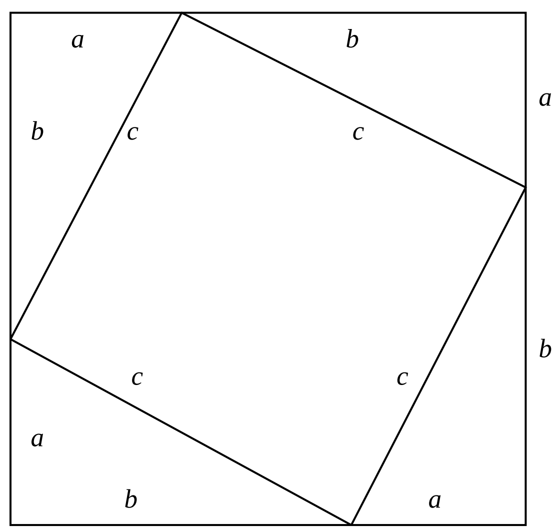
# 4.7 Appendix: The Pythagorean Theorem

Irrational square roots often arise as lengths in real-world measurement problems as a result of the *Pythagorean theorem*:

In any right-angled triangle with side lengths $a$ and $b$, the length of the hypotenuse is $\sqrt{a^2 + b^2}$.




There are many geometric proofs of the Pythagorean Theorem. One is to draw 4 copies of the triangle to make a square of side length $a + b$:




From this picture, by calculating the area of the outside square in two ways, we can see that $(a + b)^2 = [4 \times \frac{1}{2}(ab)] + c^2$, so $a^2 + 2ab + b^2 = 2ab + c^2$, and hence $a^2 + b^2 = c^2$, as the theorem says.

Triples of integers (***n,m,h***) for which $\boldsymbol{n^2 + m^2 = k^2}$ are called ***Pythagorean triples***. The three sides of integral-length right triangles are always the numbers in a Pythagorean triple. The first Pythagorean triple we encounter is the smallest: (3,4,5). Finding Pythagorean triples was important to ancient engineers, who kept valuable scrolls and tablets that gave lists of the Pythagorean triples they could use in the construction of buildings and temples.

## Exercises: Section 4.7 Appendix

1. Find all six Pythagorean triples whose side lengths are all less than or equal to 20.

2. Suppose two of the numbers in a Pythagorean triple are divisible by a prime number. Show that the third number in the triple is also divisible by this prime.

3. Let $a$ and $b$ be positive integers with $a > b$. Show that $(a^2 - b^2, 2ab, a^2 + b^2)$ is always a Pythagorean triple.

# 4.8 Unit Review Exercises

What is the prime factorization for each of the following?

| | | | | | |
|---|---|---|---|---|---|
| 1. | 504 | 2. | 3 015 | 3. | 1 258 |

Check if each of the following is divisible by 2, 3, 4, 5, 6, 8, 9, 11, or 12.

| | | | | | |
|---|---|---|---|---|---|
| 4. | 1 288 | 5. | 8 493 969 | 6. | 409 311 |

Find the number of divisors for each of the following given numbers:

| | | | | | |
|---|---|---|---|---|---|
| 7. | 108 | 8. | 2 000 | 9. | $2^3 3^1 5^2$ |

Find the GCD for each of the following pairs of numbers:

| | | | | | |
|---|---|---|---|---|---|
| 10. | 180 and 216 | 11. | 60 and 328 | 12. | 56 and 144 |

Find the LCM for each of the following sets of numbers:

| | | | | | |
|---|---|---|---|---|---|
| 13. | 27 and 75 | 14. | 14 and 36 | 15. | 12, 14, and 18 |

Use long division to convert the following fractions to decimals.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 16. | $\frac{1}{3}$ | 17. | $\frac{3}{16}$ | 18. | $\frac{3}{11}$ | 19. | $\frac{13}{25}$ |

Convert the following from decimals to fractions in their lowest terms.

20. 0.6 21. 0.721 22. 0.43 23. 0.95

Convert the following repeating decimals to fractions in their lowest terms.

24. $0.\overline{85}$ 25. $1.0\overline{2}$ 26. $0.\overline{54}$ 27. $3.7\overline{12}$

28. Express $\frac{1}{6}+\frac{1}{10}-\frac{1}{7}$ as a single fraction in lowest terms.

29. Convert $0.\overline{17}+0.\overline{64}$ to a fraction in lowest terms.

30. Add $54.7\overline{16}$ to $63.\overline{243}$ in two ways:

a) as a repeating decimal.

b) convert to fractions and add the fractions.

31. Joanne and Larry work at the movie theatre together. Larry gets every 6th night off and Joanne gets every 8th night off. They both had March 6th off, so they went to a movie together. When is the next time that they will both have the same night off again?

32. Prove that p + 7 can never be a prime if p is a prime. What about p + 3?

33. Explain why the statement "$n^2 + 4n + 3$ is a prime" is not true for any positive integer $n$.

34. You multiply three different prime numbers together and get a product that is greater than 2000. What is the smallest possible value of the largest of your primes?

35. Determine all possible values of the digits *x*, *y*, *z* for which

a) 64 537 84*x* is divisible by 4, b) 4 329 7*y*5 is divisible by 9,

c) 354 *z*78 is divisible by 11.

36. a) Is $213_7$ divisible by 2? The divisibility rules in different bases are often different from the rules for our base-10 numbers, so the answer is not obvious.

b) What is the rule that states when a number written in base 7 is even?

37. You are given two numbers whose product is 10 000, but neither number contains a zero. What is the sum of the two numbers?

38. Find the prime factorization of the **gcd** of 61 843 320 and 737 880.

39. a) Find the greatest common divisor of 150, 200, and 225.

b) You are given ribbons of length 150cm, 200 cm, and 225 cm. Your job is to cut them into equal pieces at least 13 cm long. How long should they be?

c) Three satellites appear together in the sky over the equator on January 1, 2000. If their orbits about the Earth last 150, 200, and 225 days respectively, in what year will they be together again?

40. At Molly's party, all of the 60 guests drink a cup of coffee. How many have their coffee black if $^1/_5$ have both cream and sugar, ¾ have no cream, and $^2/_3$ have no sugar?

41. Andrea works at the Co-op grocery store. She has 208 cans of tomato soup and 156 cans of mushroom soup that she is supposed to put in stacks. What is the largest number of cans she can put in each stack so that each stack has the same number of cans and only one kind of soup in it?

42. a) Find the least common multiple of 15 and 66.

b) Write the sum of 4/15 and 5/66 as a fraction reduced to lowest terms.

c) Two friends, who now live out of town, come to Regina periodically for business. The judge comes for one day every 15 days, while the accountant comes for one day every 66 days. They were together in Regina this year on January 11. On what day will they next be together here?

d) You are standing in a slow-moving line and see that 57/66 of the line is in front of you, while 2/15 of the line is behind you. How many people are in line?

e) We want to print 1000 copies of a single page to hand out to our students, and we use two printers to run them off. The faster printer takes 66 seconds to print 57 pages, while the slower printer takes 15 seconds to print 2 pages. To the nearest second, how long will it take the two printers working together to print 1000 pages?

43. Find the greatest common divisor and least common multiple of 1 528 835 and 1 106 028.

44. You are standing in a slow-moving line and notice that 5/9 of the line is in front of you while 5/12 of the line is behind you. How many people are in the line?

45. Find a pair of numbers whose least common multiple is 24 and whose greatest common divisor is 12. Can you find a second such pair?

46. a) Show that the sum of the divisors of 120 add up to 360.

b) What is the sum of the reciprocal of the divisors of 120?

47. If $n$ is an odd number with 35 factors, how many factors does $4n$ have?

48. Multiply the consecutive even numbers together until the product ($2 \times 4 \times 6 \times 8\ldots$) becomes divisible by 1995. What is the smallest integer you can multiply up to?

49. Find the value of $n$ if $n! = 2^{31} \times 3^{15} \times 5^{7} \times 7^{4} \times 11^{3} \times 13^{2} \times 17 \times 19 \times 23 \times 29 \times 31$. (Remember that $n!$ means multiplying all whole values from 1 to the value of $n$ together.)

50. The 5-digit number $32a1b$ is divisible by 156. What are the values for $a$ and $b$?

51. Here is an interesting arrangement of the counting numbers (excluding zero). What are the next two numbers in the sequence: 0, 1, 10, 2, 100, 11, 1000, 3, 20, 101, …? (Hint: we put this exercise in the factoring chapter for a reason!)

52. The First Nations people of the Plains decorated clothing, bags, containers, quivers, and many other items. Beading was not only decorative, but symbolic and spiritual as well. For example, certain colours and shapes have symbolic meanings, but the beadwork could also have a personal meaning. Many of the patterns we see today are based on quillwork designs used before the introduction of glass beads (by European traders).

| | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Y | Y | Y | B | B | B | R | R | R | Y | Y | Y | B | B | B | R | R | R |
| W | T | T | W | T | T | W | W | W | T | T | W | T | T | W | W | W | T |
| T | T | T | T | T | T | T | W | T | T | T | T | T | T | T | W | T | T |
| T | T | T | T | T | T | T | W | T | T | T | T | T | T | T | W | T | T |
| W | T | T | R | T | T | W | W | W | T | T | R | T | T | W | W | W | T |
| T | T | T | T | T | T | T | W | T | T | T | T | T | T | T | W | T | T |
| T | T | T | T | T | T | T | W | T | T | T | T | T | T | T | W | T | T |
| W | T | T | W | T | T | W | W | W | T | T | W | T | T | W | W | W | T |
| Y | Y | Y | B | B | B | R | R | R | Y | Y | Y | B | B | B | R | R | R |

a) Flora is designing a bead bracelet in the above pattern. The flower pattern repeats every eight beads. The border pattern repeats every nine beads. If she needs the bracelet to be between 60 and 100 beads in length in order to fit her wrist, how many beads long should she make it so that the bracelet ends at a point where both patterns are complete?

b) How many times in the bracelet does the red centre of a flower line up with a red bead on the border?

53. Flora changes the design of her bracelet so that the border design has a repeat of six beads:

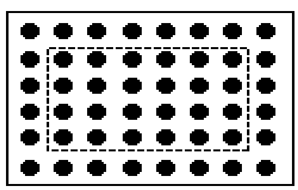
| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Y | Y | B | B | R | R | Y | Y | B | B | R | R |
| W | T | T | W | T | T | W | W | W | T | T | W |
| T | T | T | T | T | T | T | W | T | T | T | T |
| T | T | T | T | T | T | T | W | T | T | T | T |
| W | T | T | R | T | T | W | W | W | T | T | R |
| T | T | T | T | T | T | T | W | T | T | T | T |
| T | T | T | T | T | T | T | W | T | T | T | T |
| W | T | T | W | T | T | W | W | W | T | T | W |
| Y | Y | B | B | R | R | Y | Y | B | B | R | R |

a) If the new bracelet is to be between 60 and 80 beads in length how many beads long should she make it so that the bracelet ends at a point where both the flower pattern and the border pattern are complete?

b) How many times in the bracelet does the red centre of a flower line up with a red bead on the border?

c) Make up your own bracelet design that has a middle pattern and a border pattern which repeat at different lengths. Determine how long (number of beads) to make your bracelet in order to have the designs match at the ends.

55. Jones's mother used to bake biscuits in a rectangular pan which held 8 by 6 biscuits, as shown.



Jones had often noticed that exactly half of the biscuits were crusty *"outside"* biscuits, and half were soft *"inside"* ones. One day Jones asked Smith if he knew any arrangement of biscuits into rows and columns in a rectangular pan which would produce half "outside" and half "inside". Smith thought a while, and said he could do it, but it would take a lot of biscuits, in fact more than 50. Jones, who, of course knew that 6 times 8 was 48, offered to give Smith $1 for every biscuit over 50, if Smith would give him $1 for every biscuit less than 50. Smith agreed, and eventually collected $10 from the unfortunate Mr. Jones. How did he do it?

Check to see if the following is divisible by 7 or 13.

56. 7 112 448 57. 1 291 836 58. 7 865

59. Recall that two numbers $m$ and $n$ are relatively prime when $\gcd(m,n)=1$.

a) Use the Euclidean algorithm to show that 195 and 58 are relatively prime.

b) Use the equations generated by the Euclidean algorithm in part a) to solve the congruence

$$58x \equiv 1 \bmod 195.$$

c) Do you think the congruence $nx \equiv 1 \bmod m$ will always have a solution when $\gcd(m,n)=1$?

60. A group of tourists decided to divide themselves evenly among the tour busses. If they had tried to sit 22 on each bus, one tourist would have been left over. But then with one fewer bus, the tourists were able to divide themselves evenly among the remaining busses. How many tourists were there?

61. Some teachers at a school decide to take all the grade 2 students on a field trip. They arrange the seating plan to have 9 students sit on each bus, but 3 of the students are left over. However, the teachers solve this problem by not using two of the busses, so now there is the same number of students on each bus with none left over. How many grade 2 students attend this school?

# Unit 5

# Ratios, Percent, and Probability

**Ratio, proportion, percent, quotient, fraction, rate** -- there are dozens of ways of describing the same thing. The reason for this abundance of terminology is that rational numbers arise naturally in daily life, so special terminology arose that was convenient for each special purpose. Ordinary people were able to master these concepts (without ever going to school!) long before the algebra was invented that would solve the problems we will be discussing here.

You should be able to go back and forth among the various terminologies:

"*a* and *b* are in the **ratio** of 2 to 5"

is also written

$a : b = 2 : 5$ (which reads, "*a* is to *b* as 2 is to 5"),

which means the same as

$${}^{a}/_{b} = {}^{2}/_{5} = {}^{4}/_{10} = 0.4 = 40\%,$$

which means the same as

*a* and *b* are in the **proportion** 2 to 5.

However you say it, you probably want to switch to fractions if you need to compute by hand, or to decimals if you want to compute by calculator. But for most problems, you can deduce the answer by using common sense. Some examples will make the situation clear.

## Advice

We should train ourselves to understand these problems and their solutions using "common sense". First appearances are often misleading. We must not apply any formula until we clearly understand the problem and have made certain that our formula applies to the problem. We also need to make sure that our answers "make sense". As we work through the examples in the following sections, we will see why we must analyze problems before trying formulas.

# 5.1 Ratio and Proportion

One place where we use ratios and proportions naturally is when talking about speed or velocity. A person who travels 100 km in 4 hours has an average speed of 25 km/h.

Here 100 km : 4h = 25 km : 1h,

i.e. 100km/4h = 25 km/h

1. You are travelling at 100 km/h and the sign on the road says that your destination is 75 km away. How long will it take to get there?


If you go 100 km in 1 h, you go ${}^{3}/_{4}$ of that distance in ${}^{3}/_{4}$ of an hour, or 45 minutes. That is 100 : 1 = 75 : ${}^{3}/_{4}$. Algebraically, we could let the time required be $x$ hours. Then we have ${}^{100}/_{1} = {}^{75}/_{x}$, or $x = {}^{75}/_{100} = 0.75$ hours (= 45 minutes).

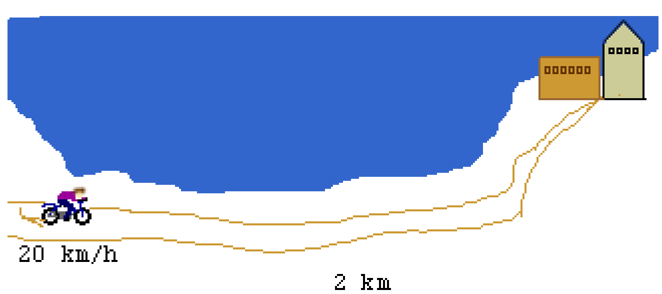
**Remark:** In high school, we were taught to cross multiply to solve proportions such as

$$\frac{63}{x} = \frac{9}{5}$$

That would mean computing $5(63) = 9x$, or $x = {}^{315}/_{9} = 35$. What a waste of time! In examples such as this, surely it is easier to note that to get 63 from 9 we must multiply by 7, so to keep the proportion we have to multiply 5 by 7 and get 35; that is,

$${}^{63}/_{x} = {}^{9}/_{5} \text{ means } {}^{7\times 9}/_{x} = {}^{9}/_{5}, \text{ thus } x = 7 \times 5 = 35$$

2. If you average 20 km/h to ride a bicycle 2 km to the university, how long does a typical journey take?




6 minutes. (You are travelling 1/10 of the 20 km, so it takes 1/10 of an hour.) In the language of ratio and proportion, ${}^{20\text{ km}}/_{1\text{ h}} = {}^{2\text{ km}}/_{0.1\text{ h}}$.

3. Allen makes salad dressing by mixing one part vinegar with four parts oil. Harley makes salad dressing by mixing two parts vinegar with five parts oil. Judi makes salad dressing by mixing equal parts of Allen's and Harley's dressing. What is the proportion of oil to vinegar in Judi's salad dressing?

Think about preparing a batch of dressing using a particular measurement, for example a tablespoon.

| | Batches | Vinegar | Oil | Dressing |
|---|---|---|---|---|
| Allen's recipe | 1 | 1 tablespoon | 4 tablespoons | 5 tablespoons |
| Harley's recipe | 1 | 2 tablespoons | 5 tablespoons | 7 tablespoons |

To get equal amounts to mix, Judi could make 7 batches using Allen's recipe and 5 batches using Harley's recipe.

| | Batches | Vinegar | Oil | Dressing |
|---|---|---|---|---|
| Allen's recipe | 7 | 7 tablespoons | 28 tablespoons | 35 tablespoons |
| Harley's recipe | 5 | 10 tablespoons | 25 tablespoons | 35 tablespoons |
| Judi's mixture | | 17 tablespoons | 53 tablespoons | 70 tablespoons |

Thus, the proportion of oil to vinegar in Judi's recipe is 53:17. Of course, a cook would notice that 53:17 is almost the same as 3:1.

If a recipe intended for 4 persons is being used for a party of 8 persons, you do not have to be a rocket scientist to know that you multiply the amount of each ingredient by 2. But this is an artificial problem. More often the cook has to stretch a recipe intended for 4 people to feed 5 persons. Here common sense and experience are more useful than formulas -- how do you multiply an egg by 5/4?

4. When the exchange rate for the US dollar is 0.75, it means that a one-dollar item in Canada will cost only 75 US cents; that means that a US citizen shopping in Canada gets a 25¢ bonus (think about whether these are US or Canadian cents). A Canadian shopping in the US meanwhile has to pay $\$1.33^1/_3$ Canadian for something that costs \$1 US, which is a $33^1/_3$¢ premium. Is that fair?

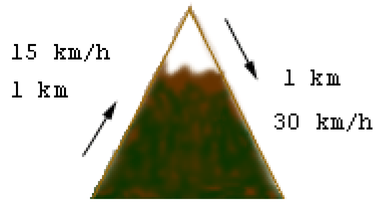
Of course it is, since

$$\frac{1.00\,Canadian}{0.75\,US} = \frac{100}{75} = \frac{4}{3} = \frac{133\frac{1}{3}}{100}$$

In other words, in this situation, a US dollar is worth 133 $^1/_3$ cents Canadian.

5. The distance up a mountain is 1 km. I drive up at 15 km/h and down at 30 km/h. What is my average speed for the round trip?

Think before you answer!



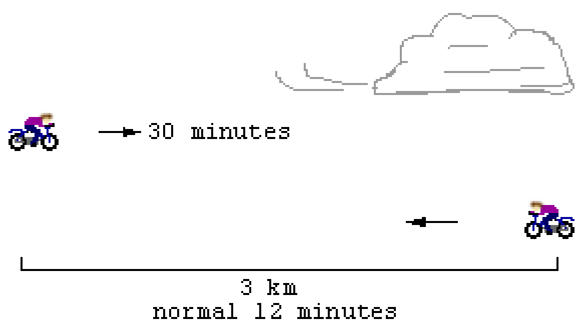

The average speed is the distance per time elapsed.

I drive 2 km.

It takes $^1/_{15}$ h (= 4 minutes) to go up the mountain and $^1/_{30}$ h (= 2 minutes) to go down the mountain. This is a total of $^1/_{10}$ h (= 6minutes).

Thus, the average speed is 2 km in $^1/_{10}$h, or $^{2\ km}/_{0.1\ h}$ = 20 km/h. Common sense tells us that the answer must be between 15 and 30, but closer to 15 because more of the time is consumed at the slower speed.

6. It normally takes 12 minutes to ride my bike 3 kilometres when there is no wind. It took 30 minutes today with the wind in my face. For the return trip, with the wind at my back, will I save 30 - 12 = 18 minutes?




(I doubt it! I would arrive 6 minutes before I start!)

My average speed without any wind is 15 km/h.

My average speed going into the wind is 6 km/h.

That means that the wind subtracted 9 km/h from my speed, so that when it is at my back it will add 9 km/h to my speed, increasing my normal speed to 24 km/h. Returning the 3 km at 24 km/h will take $^3/_{24}$ = $^1/_8$ of an hour = 7.5 minutes. So even though I lost 18 minutes against the wind, I gain only 4.5 minutes with the wind.

7. If Smith takes 8 hours to paint and room and Jones takes 12 hours to paint a room, how long does it take Smith and Jones to paint the room working together?

(Certainly not $(8 + 12) \div 2 = 10$ hours! You should see immediately why 10 hours cannot be correct.)

Imagine Jones and Smith start painting the room. How much of the room will be painted in 1 hour?

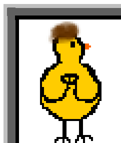
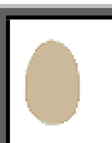
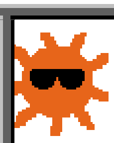
$$^{1}/_{8} + {}^{1}/_{12} = {}^{3}/_{24} + {}^{2}/_{24} = {}^{5}/_{24}.$$

Thus, after 2 hours it's ${}^{10}/_{24}$; after 3 hours it's ${}^{15}/_{24}$; after 4 hours it's ${}^{20}/_{24}$; after 5 hours they have finished and gone to the pub.

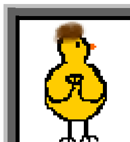
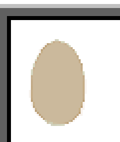
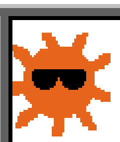
We can now see that the way to approach the problem is to add their painting speeds. Their combined rate is ${}^{3}/_{24} + {}^{2}/_{24} = {}^{5}/_{24}$ room per hour. Thus, it takes them ${}^{24}/_{5}$ hours (which is 4h 48min) to paint one room. (In detail, ${}^{5}/_{24}$ rooms per hour means 5 rooms in 24 hours, and therefore 1 room in ${}^{24}/_{5}$ hours.)

8. A chicken and a half lays an egg and a half in a day and a half. How long will it take for 6 chickens to lay 24 eggs?

Let's put the information in a table to help us understand.

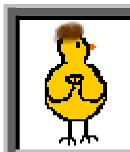
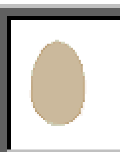
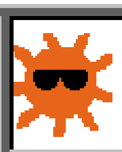
|  |  |  |
|---|---|---|
| $1^{1}/_{2}$ | $1^{1}/_{2}$ | $1^{1}/_{2}$ |

Now let's think. How many eggs would 3 chickens lay in a day and a half?

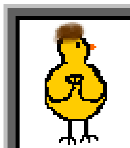
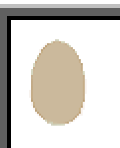
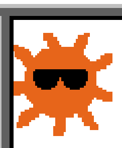
|  |  |  |
|---|---|---|
| $1^{1}/_{2}$ | $1^{1}/_{2}$ | $1^{1}/_{2}$ |
| 3 | 3 | $1^{1}/_{2}$ |

We multiplied the chickens by 2 and therefore the eggs by two, but this is still happening in a day and a half.

Now how many eggs would 6 chickens lay in a day and a half?

|  |  |  |
|---|---|---|
| 3 | 3 | $1^{1}/_{2}$ |
| 6 | 6 | $1^{1}/_{2}$ |

Now we have our 6 chickens, so we just need to know how many days it takes for them to lay 24 eggs.

|  |  |  |
|---|---|---|
| 6 | 6 | $1^{1}/_{2}$ |
| 6 | 24 | 6 |

Since we multiply $1^{1}/_{2}$ days by 4 to get 6 days, we multiply 6 eggs by 4 to get 24 eggs.

## Exercises: Section 5.1

1. Give three ratios that equal 5 to 3.

For exercises #2 to #5, convert the following ratios to fractions in their lowest terms.

2. 40 to 30

3. 72 to 220

4. 24 to 120

5. 192 to 40

Solve the equations in exercises #6 to #9 (try to do it without cross multiplying):

6. $\frac{z}{4} = \frac{175}{20}$

7. $\frac{49}{56} = \frac{x}{8}$

8. $\frac{y}{6} = \frac{18}{4}$

9. $\frac{k}{80} = \frac{20}{100}$

10. Four carrots contain 15 calories. How many carrots would contain 50 calories?

11. If sales tax on a \$16.99 compact disc is \$2.72, how much tax will there be on a \$129.99 compact disc player? (Answers are rounded to the nearest penny when dealing with money.)

12. A recipe for oatmeal chocolate chip cookies calls for $1^2/_3$ cups of flour to make four dozen cookies. How many cups of flour would you need if you wanted to make six dozen cookies?

13. Which would be the best buy for each of these three items?

| **Garbage bags** | **Ketchup** | **Frozen peas** |
|---|---|---|
| Box of 20: \$3.09 | 14-ounce bottle: \$0.89 | 8-ounce bag: \$0.45 |
| Box of 30: \$4.59 | 32-ounce bottle: \$1.19 | 16-ounce bag: \$0.49 |
| | 64-ounce bottle: \$2.95 | 50-ounce bag: \$1.59 |

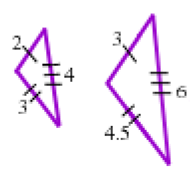
14. If a building casts a shadow 5 feet long at the same time a building 32 feet high casts a shadow of 2 feet, how tall is the first building?

15. If $1^1/_2$ chickens lay $1^1/_2$ eggs in $1^1/_2$ days, then how many eggs will six chickens lay in a week?

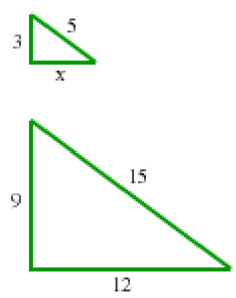
16. Two turtles run a race. The first turtle runs at 2 centimetres per second while the second turtle runs at 80 metres per hour. Evaluate their speed in kilometres per day. Which is faster?

17. Two triangles are said to be similar if they have the same shape (but not necessarily the same size). As in the diagram, corresponding sides of similar triangles have the same ratio, meaning

$^2/_3 = {}^4/_6 = {}^3/_{4.5}$




Use this information to find side *x* of the smaller triangle of the two similar triangles shown below.



18. In a race from point X to point Y and back, Jack averages 30 km per hour to point Y and 10 km per hour back to point X. Sandy averages 20 km per hour in both directions. Between Jack and Sandy, who finishes first?

19. A steamer plying between the river ports of St. Louis and Hannibal makes the trip up the river in 12 hours and the return trip in 8 hours. How long would it take a log floating past Hannibal to reach St. Louis? (Note: St. Louis is downstream from Hannibal).

# 5.2 Percent

99% of lawyers give the rest a bad name.
42.7% of all statistics are made up on the spot.

Percentages provide a convenient mathematical tool that makes it easy to compare fractions. For example, which is better, 7 out of 10 on quiz 1, or 12 out of 15 on quiz 2? The first has 70% correct answers, while the second has 80%. What we are doing here is finding a common denominator; in particular, we write each with a denominator of 100.

$$\frac{7}{10} = \frac{70}{100} \text{ while } \frac{12}{15} = \frac{4}{5} = \frac{80}{100}$$

$$\frac{70}{100} \text{ is 70\% and } \frac{80}{100} \text{ is 80\%.}$$

An alternative approach might be to find the least common denominator:

$$\frac{7}{10} = \frac{21}{30} \text{ while } \frac{12}{15} = \frac{24}{30}.$$

It is clear that 24 out of 30 is better than 21 out of 30. The advantage of using percentages is that they go well with our base 10 number system.

Once again, the terminology is designed to tell us how to proceed without any formulas: "**per cent**" means "out of 100". Thus 7:10 is a rate of 70 per 100. The basic skill required here is passing effortlessly among fraction, decimal, and percent. Given one we should be able to write the others: $^1/_4$ = 0.25 = 25%.

To go from a decimal to the corresponding percent, we multiply the decimal by 100 (i.e. slide the decimal point two places to the right) and stick in the percent sign. Thus, 32.579 = 3 257.9%, while 0.4% = 0.004. You might like to do this as

$$\frac{32.579}{1} = \frac{3\,257.9}{100}\text{, so } 32.579 = 3\,257.9\%\text{; and } \frac{0.4}{100} = \frac{0.004}{1}\text{, so } 0.4\% = 0.004$$

Switching between fractions and decimals was discussed earlier in this unit. Remember that

$$\frac{1}{3} = 0.\bar{3} = 0.333... = \frac{33\frac{1}{3}}{100} = 33\tfrac{1}{3}\%\text{, while } 33\% = \frac{33}{100} = 0.33\ (\neq \tfrac{1}{3}).$$

We must be careful when doing arithmetic with percentages!

**Never multiply percents!**

If a product is called for, first change any percentages to decimals. So 20% of 40 is $(^{20}/_{100})(40) = (0.2)(40) = 8$.

Also, adding or subtracting percentages is seldom automatic. We can add only if

1. we are adding percentages of the same thing, and
2. we are not counting something more than once (as when using Venn diagrams to count in Unit 5).

Thus, if 10% of the students are math majors while 15% are computer science majors, we need more information to compute the total number of students. It may be that some of these students are combined math and CS majors.

We might likewise need additional information when computing taxes. In Saskatchewan, the cost of some items we buy is increased by a 7% federal tax and a 6% provincial tax. The bill for such an item is increased by (7 + 6)% = 13%: the tax on a \$30 item is 0.07(\$30) = \$2.10 federal and 0.06(\$30) = \$1.80 provincial, which makes a total tax of 0.13(\$30) = \$3.90.

Sometimes (as with income tax) the second tax applies to the item after the federal tax has already been applied. To make the comparison clear, let's use the different system with the previous numbers: A \$30 item with a 6% provincial tax applied to the total after the 7% federal tax has already been applied. Thus, the 6% second tax is applied to 0.07(\$30) + \$30 = \$32.10, so that the provincial tax would be 0.06(\$32.10) = \$1.93, and the total tax paid is [0.07 + 0.06 + (0.07)(0.06)](\$30) = 0.1342(\$30) = \$4.03. The moral: think before you compute.

The three basic types of problems are as follows.

1. Find 32% of 50:
   $^{32}/_{100}(50) = 0.32(50) = 16$.

2. 15 is what percent of 60?

$^{15}/_{60}$ = 0.25= 25%. (Using common sense: 15 out of 60 is a fourth, which is 25%.)

3. 38 is 10% of what number?
   $^{38}/_{x} = {}^{10}/_{100}$, thus $x$ = 380 or 10% means a tenth, so the original number must have been ten times greater.

When using percentages, the general rule is that we must have a percent of something. The word 'percent' has no meaning unless it is clear what the 100% item happens to be.

For example, it makes no sense for a hockey player to give 110%. You cannot give more than everything you have to give. Even giving 100% fails to stand up to mathematical precision: any player who gave 100% would have to drop dead during the game. He would probably help the team more by giving his best effort.

In more practical situations, a 25% markup means that the invoice price (that a store has to pay to the supplier) is multiplied by 0.25 to compute the markup. Thus, a book that costs the store $40 marked up by 25% or $10, would sell for $40 + $10 = $50.

**Examples:**

1. A store that buys an item for $30 and sells it for $40 has marked it up by $40 - $30 = $10, which represents a markup of $^{10}/_{30} = 33^{1}/_{3}$%.

   Sometimes the selling price is marked on the item by the manufacturer, which sells it to the retailer at a discount. Thus, a $40 item that the retailer buys for $30 gives the retailer $10 off, which represents a discount of $^{10}/_{40}$ = 25%.

   It is important to note how this example illustrates the rule that a percent has to be applied to the original cost. The same $10 profit that the store makes in this example is $33^{1}/_{3}$% of the lesser amount ($30), and 25% of the greater amount ($40). Which number we use as the base price depends on the information we are given. It should always be clear which is the original price.

2. A store is selling a fur coat for $600 and advertises that this price represents 40% off the original price. That means that the original price $x$ has been discounted by $0.4x$ to arrive at $600. What is the original price?
   We could use algebra here, but it is easier to notice that 40% of $1 000 is $400, so that the original price was $1 000, the discount was $400, which produces a sale price of $600.

   Algebraically we have 600 = $x$ - 40% of $x$, which means $x - 0.4x = {}^{6}/_{10}x$; i.e. $600 = {}^{6}/_{10}x$, and $x$ = $1 000.

   (Notice the difference between the words "of" and "off." The question, "600 represents 40% of what number?" gets the answer 1500.)

3. 25% of the males in Engineering take Math 212 while 60% of the females take it. Does this mean that 85% of the Engineering students take Math 212?

Of course not. It makes no sense to add the 25 to the 60. The numbers apply to different groups of students. There is no way of knowing what percent of the Engineering students take Math 212 without knowing what percentage of the Engineering students are female or male.

For example, if 10% of the students are females, then 60% of the 10% = 6% of all the students are females taking the course and 25% of the 90% = 22.5% are males taking the course, which together makes a total of 6 + 22.5 = 28.5% of the Engineering students in Math 212. On the other hand, if 50% of the Engineering students are females that would mean 42.5% of the Engineering students take Math 212 (0.5(0.6) + 0.5(0.25) = 0.5(0.85) = 0.425).

4. In a fine restaurant one typically tips 15% of the cost of the meal (computed before the tax has been added). Suppose the $80.04 total on the bill already includes a tax of 16%. How much should the tip be?

First compute the cost of the meal, say $x$ dollars. Thus $x + 0.16x = 80.04$, where $x + 0.16x$ is equivalent to $1.16x$, thus $x = {}^{80.04}/_{1.16}$ = $69. (Check the value: 16% of 69 is $11.04, and 11.04 + 69 = 80.04 as desired.) The tip should be 15% of $69, which is $10.35.

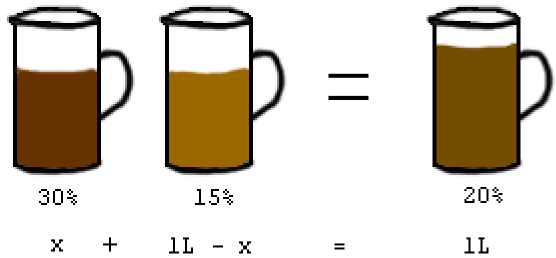
5. Sears advertises a sale of a regularly priced $10 000 Sony Home Entertainment Centre at 15% off. The Bay, which already had a sale at 10% off, advertises an extra 10% off its sale price in order not to lose customers. Sears retaliates with an extra 5% off its sale price. Are the two final sale prices the same? If they are not, which one is the better deal?

The thing to be aware of here is that the second discount is **off the sale price**!

The Sears first sale price is (0.85)(10 000).
Its final sale price is (0.95)(0.85)(10 000).

On the other hand, the Bay's first and final sale prices are (0.90)(10 000) and (0.90)(0.90)(10 000) respectively. Thus we need only compare (0.95)(0.85) = 0.807 5 to (0.90)(0.90) = 0.810 0. Therefore, Sears' final sale price is better (but only by $25 on a $10 000 item).

6. We have two jugs of iced tea. One is 30% flavour crystals and the other is 15% flavour crystals. How much of each mixture would we need to make 1 L of iced tea with 20% flavour crystals?




Drawing a picture helps in this problem. If we add $x$ amount of one mixture, then we have to add 1L - $x$ of the other to make 1L.

Think of the amount of crystals. We have $0.30x$ L crystals from the first jug and $0.15(1 - x)$ L crystals from the second jug. Thus,

$$0.30x + 0.15(1 - x) = 0.20(1)$$
$$0.30x + 0.15 - 0.15x = 0.20$$
$$0.15x = .05$$
$$3x = 1$$
$$x = {}^{1}/_{3}\text{L}$$

We would need ${}^{1}/_{3}$L of the 30% mixture and ${}^{2}/_{3}$L of the 15% mixture.

## Exercises: Section 5.2

Convert the following decimals to a percent.

1. 0.86
2. 0.763
3. 0.0059
4. 7.9

Convert the following fractions to percents.

5. ${}^{3}/_{5}$
6. ${}^{3}/_{50}$
7. ${}^{5}/_{6}$
8. ${}^{7}/_{4}$

Answer the following percent exercises.

9. 6% of 100 is __.
10. 25% of __ is 6.
11. 0.7% of 50 is __.
12. __ % of 14 is 2.
13. Find 42% of 16.
14. Find 37.2% of 17.
15. What percent of 52 is 16?
16. 15% of what number is 47 121?
17. 85.34 is what percent of 251?

18. On a recent shopping trip, Jennifer bought three t-shirts at $9.99 each, two picture frames at $5.99 each, and 5 packs of gum at $0.69 each. With a 7% sales tax, what is her total bill?

19. Suppose an item originally costs $500.00 and then it is marked down 10%. At a later date it is marked up 10%, is the resulting price once again $500.00? Why or why not?

20. If there is a 60% chance of rain tomorrow, what fraction (in lowest terms) represents the chance that it will not rain?

21. The price of the car Ken bought last year has been increased by 1.3% from $15 795. What is the price of the car this year?

22. Steve and Jodi originally bought their house for $85 000 and sold it for $103 000 a few years later. What was the percent increase in the value of their house?

23. A 100 kg bag of potatoes is 99% water. It dries out to the point that it is 98% water. How much does it weigh now? (Hint: it is not 99 kg.)

24. The Brick advertises a sale of a regularly priced $700 sofa at 15% off. Bob's Furniture Barn, which already had a sale at 10% off of the same regularly priced sofa, advertises an extra 15% off its sale price in order to not lose its customers. The Brick retaliates with an extra 10% off its sale price. Are the two final sale prices the same? If the prices are not the same, which one is the better deal?

25. You have a cup of coffee that is 12% cream and another cup which is 22% cream. How much would you have to take from each cup to make a cup of coffee that is 15% cream?

26. I received a score of 16 marks out of 20 marks on my first math midterm. On the second math midterm, I received a score of 23 marks out of 46 marks.

a) Which exam did I do better on?

b) Convert each score to a percent to verify your answer in a).

27. a) In the Chemistry Lab I had to mix some $33^1/_3$% alcohol with some $12^1/_2$% alcohol to get a 50 litre jug of 25% alcohol. How many litres of the $33^1/_3$% and of the $12^1/_2$% alcohol did I use?

b) Harry Hacker is trying to quit smoking. He has reduced his habit to $^1/_8$ of a carton per week except when he's studying for exams and, being all tensed up, then he smokes $^1/_3$ of a carton per week. Over the last 50 weeks, Harry has smoked $^1/_4$ of a carton per week on average. During those 50 weeks, how many weeks did Harry study for exams, and how many weeks did he not study for exams?

c) Explain how parts a) and b) are related.

28. Express the following items as percentages to the nearest whole percent.
(a) 7/10 (b) 0.814 (c) 12÷3 (d) 3/5 (e).9

[*Remark*. This problem comes from Edward MacNeal's *Mathsemantics*. MacNeal was a consultant for the American airline industry, and this question appeared on a quiz for his firm's prospective secretarial and clerical employees. 196 people took the quiz. The percentages of people who got the correct answers were 45% for (a), 35% for (b), 25% for (c), 48% for (d), and 43% for (e). The figures so alarmed him that throughout the book he used expressions like "84 out of 196"{ rather than "43 percent".

29. Comment on the following statement, and give a more accurate comment than the minister.

Podunk University authorities are concerned. They found that thirty percent of the male students and twenty percent of the female students have cheated on an exam at one time or another. The minister of education said "We will no longer fund an institution where 50 percent of the students are cheaters."

30. Match each percent with its fraction equivalent.

| | | | |
|---|---|---|---|
| a) | 50% | i) | $1/10$ |
| b) | 10% | ii) | $3/5$ |
| c) | 25% | iii) | $1/3$ |
| d) | 75% | iv) | $1/2$ |
| e) | $33^1/_3$% | v) | $1/4$ |
| f) | 110% | vi) | $11/10$ |
| g) | 60% | vii) | $3/4$ |

# 5.3 Variation

The hardness of butter is proportional to the softness of the bread.
The severity of the itch is proportional to the reach.

We say "***y* varies directly as *x***," or equivalently, "***y* is proportional to *x***," when there exists a constant $k$ such that $\boldsymbol{y} = k\boldsymbol{x}$ ($k$ is the **constant of variation**.)

Intuitively this means that if $x$ doubles, so does $y$; if $x$ triples, so does $y$; .... Since $^{y}/_{x} = k$, you should see that this is yet another way of saying $y : x = k : 1$, but here we are thinking that $x$ and $y$ are various values while they maintain the constant ratio $k$.

**Examples of direct variation:**

1. If $y$ varies directly as $x$, and $x = 3$ when $y = 18$, find $x$ when $y = 42$.

   We can determine the constant ($k$) by dividing $x$ into $y$. Thus, the constant is $^{18}/_{3} = 6$. Since we know the value of $y$, the value of the constant, and that $y = kx$, we substitute the values of $y$ and $k$ into the equation.

   $$42 = 6x, \text{ so } {}^{42}/_{6} = x. \text{ Thus } x = 7.$$

2. If $b^2$ varies directly as $a$, and $a = 24$ when $b = 12$, find $a$ when $b = 18$.

   We determine the constant ($k$) by dividing 24 into $12^2$ and we find the constant to be 6. Since $b^2 = ka$, and we know the value of $k$ and $b$, we substitute in the values.

   $$18^2 = 6a, \text{ so } {}^{324}/_{6} = a. \text{ Thus } a = 54$$

   Don't forget the exponent in your calculations.

3. Hooke's law states that the distance a spring stretches is proportional to the weight that stretches it. Thus, if a 2 kg weight stretches a spring 3 cm, a 3 kg weight stretches it 4.5 cm, a 4 kg weight stretches it 6 cm, and so on since $^{3}/_{2} = {}^{4.5}/_{3} = {}^{6}/_{4} = ....$

   Here the spring constant is 1.5 since

   $$^{3}/_{2} = {}^{\text{distance}}/_{\text{weight}} = {}^{k}/_{1},$$

   so within a suitable range of values the rule is that the distance stretched in cm is found by multiplying the weight in kg by 1.5 cm/kg.

   The constant $k$ in Hooke's law measures the stiffness of the spring: the stiffer the spring, the smaller the value of $k$ (and the spring is harder to stretch).

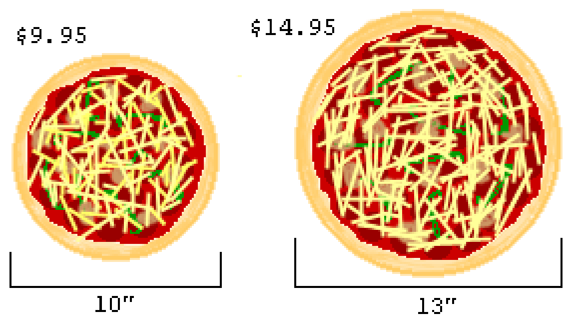
4. When dealing with circular mechanics in physics, we find, for a given circle, that the centripetal acceleration ($a$) directly proportional to the square of the speed ($v$) of an object going around the circle.

If object $A$ accelerates at a rate of 2m/s$^2$ when its speed is 8 m/s while object $B$ has a speed of 11m/s in a circle with the same radius, what is the acceleration of object $B$?

We calculate the constant ($k$) by dividing 2 into $8^2$, which shows that the constant is 32. We know that $ak = v^2$, $k = 32$, and $v = 11$, so we can substitute the values for the variables.

$$a(32) = (11)^2, \text{ so } a = {}^{121}/_{32}. \text{ Thus object } B \text{ is accelerating at 3.781 25 m/s}^2.$$

5. We know the amount of pizza is proportional to the square of its diameter. Which is the better deal, a 10" pizza for \$9.95 or a 13" pizza for \$14.95?




We are not told the amount of pizza in either case, but only that

$$\text{amount} = k \times (\text{diameter})^2 \text{ or } {}^{\text{amount}}/_{(\text{diameter})^2} = {}^{k}/_{1}.$$

Thus, the amount of the first is $10^2k$, and the amount of the second is $13^2k$. Hence, we know that the 13" pizza contains $13^2 \div 10^2 = 1.69$ times as much pizza as the 10" pizza. Since 1.69(\$9.95) = \$16.82, the selling price of \$14.95 amounts to a savings of almost \$2.

What we are doing here is the same as showing $13^2/10^2 > \$14.95/\$9.95$. Thus the big pizza is a good deal.

An alternative approach to the problem is to compare the ratio 9.95 : 100 = 0.0995 = cost per square inch (for the 10") to the ratio 14.95 : 169 = 0.088 (for the 13") and see that the latter is smaller, which translates into a better deal.

For the *inverse* situation where $y$ goes down as $x$ goes up we say "***y* is inversely proportional to *x***," or equivalently, "***y* varies inversely as *x***," to mean that there exists a constant $k$ for which $xy = k$. Equivalently,

$$y = {}^{k}/_{x} = k({}^{1}/_{x}),$$

"$y$ is inversely proportional to $x$" means that when $x$ doubles, $y$ is reduced to $^1/_2$ its value; when $x$ triples, $y$ is reduced to $^1/_3$ it value; etc.

**Examples of inverse variation:**

1. If $x$ varies inversely as $y$, and $x = 3$ when $y = 6$, find $x$ when $y = 10$.

   We know that $x = {}^k/_y$ and that $3 = {}^k/_6$, which implies that $k = 18$. We can substitute in the values since we know $y = 10$ and $k = 18$.

   $$x = {}^{10}/_{18}\text{, thus } x = 1.8.$$

2. If $f$ varies inversely as $e^3$, and $e = 3$ when $f = 3$, find $e$ when $f = 0.648$.

   We solve for the constant as $fe^3 = k$, thus $(3)(3)^3 = k = 81$. We can solve for the unknown value of $e$ by substituting the known values of $f$ and $k$ into the formula.

   $$\text{Since } fe^3 = k\text{, then } (0.648)e^3 = 81,$$
   $$\text{therefore } e^3 = {}^{81}\!/_{0.648}.$$
   $$\text{Thus } e^3 = 125\text{, so } e = \sqrt[3]{125} = 5.$$

3. The time it takes to travel a given distance is inversely proportional to the average speed. i.e. time = $^k/_{\text{avg. speed}}$ or $k$ = (time) × (avg. speed). Thus, if a trip takes $2^1/_2$ hours at 100 km/h, how long would it take at 80 km/h?

   Surely we do not need a formula to determine that it takes $^{100}/_{80}$ times as long. Nevertheless, the rules state that the constant $k$ (in this case, the distance traveled) is $k = 2.5(100) = 250$. Thus, if $t$ is the required time at 80 km/h,

   $$250 = t(80) \text{ or } t = 3^1/_8 \text{ hours} = 3 \text{ hours and } 7½ \text{ minutes.}$$

4. The intensity of light on an object varies inversely as the square of the distance from the light source. Even if the units have no meaning for you, the words tell you that the product of (intensity) × (distance)$^2$ is a constant.

   If the intensity from a lamp 2m away is 10 units, what would be the intensity of the same lamp at a distance of 8 m?

   Algebraically we have $k = 10(2^2) = 40$, thus at 8m, so
   $$40 = (\text{intensity}) \times 8^2 \text{ or the intensity is } {}^{40}/_{64} = 0.625 \text{ units.}$$

## Exercises: Section 5.3

1. If $a$ varies directly as $b$, and $a = 30$ when $b = 8$, find $a$ when $b = 4$.

2. If $x$ varies directly as $y^3$, and $x = 48$ when $y = 4$, find $x$ when $y = 6$.

3. If $m$ varies inversely as $n^2$, and $m = 9$ when $n = {}^2/_3$, find $m$ when $n = {}^5/_4$.

4. The more gasoline you pump, the more you will have to pay.
   Is this an example of direct or inverse variation?

5. The faster you drive, the less time it takes to get somewhere.
   Is this an example of direct or inverse variation?

6. For a given base, the area of a triangle varies directly as its height. Find the area of a triangle with a height of 6 cm, if the area is 10 cm$^2$ when the height is 4 cm.

7. For a constant area, the length of a rectangle varies inversely as the width. The length of a rectangle is 27 meters when the width is 10 meters. Find the length of a rectangle with the same area if the width is 18 meters.

8. Nat and Sheila canoed around Wascana Lake for 30 days covering 222 kilometres. Assuming a constant rate, what distance would they canoe in 65 days?

9. The force required to compress a spring varies directly as the change in length of the spring. If a force of 5.6 kg is required to compress the spring 8 cm, how much force is required to compress the spring 12 cm?

10. The illumination produced by a light source varies inversely as the square of the distance from the source. If the illumination produced 4m from a light source is 75 units, find the illumination produced 8 meters from the same source.

11. The real number $\pi$ is defined to be the ratio of the circumference to the diameter of *any* circle.

    a) If a circle has diameter 10 cm, what will its circumference be?
    b) What will the diameter be of a circle with circumference 10 cm?

# 5.4 Probability

**Probability theory** is the branch of mathematics that is concerned with situations involving chance processes. In these situations, more than one thing can happen, and we wish to measure the likelihood of each possibility.

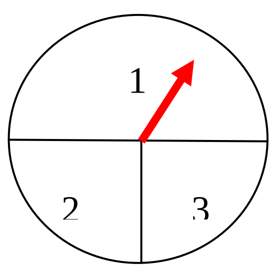
A chance process that can be repeated a large number of times is called a probability **experiment**. The act of carrying out the experiment one time is called a **trial**. The things that can result from one trial of a probability experiment are called its **outcomes**, and the set of all possible outcomes of an experiment is its **sample space**. This perspective of a chance process allows us to apply the ideas of set theory directly to probability experiments. Subsets of outcomes in the sample space are called **events,** with events consisting of just one outcome being the **simple events**. The **probability** of an event in an experiment is a measure of the likelihood that a random outcome of the experiment will be an element of the subset representing that event.
Probabilities can often be represented as a fraction that lies between 0 and 1. For example, in an experiment for which each of 8 possible outcomes is equally likely for one trial, then the probability of each of its simple events will be $^1/_8$.

Typical probability experiments arise from familiar games of chance, which are useful for illustrating these concepts. Some examples are:

1. Rolling an ordinary 6-sided die. The outcomes for this experiment are the numbers 1, 2, 3, 4, 5, and 6 that appear on the top of the die when it stops rolling. So the sample space is a set of size 6. The simple events are "a 1 is rolled", "a 2 is rolled", etc. "An even number is rolled" is an example of a more complicated event consisting of more than one outcome. When we assume the die is "fair", it means that each simple event is equally likely, so the probability of each simple event in the die-rolling experiment is $^1/_6$.

2. Tossing a coin. The outcomes are "heads" or "tails", depending on which side of the coin lands facing up. Assuming the coin is "fair", each of the two simple events in this experiment has probability $^1/_2$.

3. Taking a spin. If we spin the given wheel and look at where the arrow points when it stops, the possible outcomes will be 1, 2, or 3. This time we have an example of a probability experiment whose simple events are not equally likely.




Since the area of region 1 is twice the areas of regions 2 and 3, we can predict that the probabilities of the simple event of "spinning a 1" in this experiment is $^1/_2$, while the probability of both the other two simple events is $^1/_4$.

These examples illustrate about the probabilities of simple events in the sample space of a probability experiment with *n* distinct outcomes: ***the sum of the probabilities all the simple events in the sample space is always 1.***

**Theoretical probability vs. Experimental Probability**

Probability experiments, being chance situations, have an element of unpredictability. Even in the straightforward examples given above, saying that the probability of a simple event is $^1/_n$ does not mean that the event will occur exactly once during the first *n* trials. For example, if you toss the same coin 5 000 times, the number of times the outcome is a head is not likely to be exactly 2500, but if the coin is fair you would expect the number of heads to be close to 2500. When we say that the probability of "getting a head" is $^1/_2$ in the coin-toss experiment, we mean that

(the number of times "getting a head" occurs) ÷ (the number of times the coin is tossed)

should get closer and closer to $^1/_2$ as the number of times we toss the coin gets larger and larger. If we toss the coin a very large number of times, we should expect this fraction to be very close to $^1/_2$. This kind of probability is called **theoretical probability**. It has to do with our understanding and expectations of the experiment.

On the other hand, suppose we actually carry out the coin toss experiment 5 000 times and get a head 2 515 times. Then we can say that the **experimental probability** for the event of getting a head is

$$^{2515}/_{5000} = 0.503.$$

We call this the experimental probability of an event because it is based on data collected by actually carrying out the experiment a large number of times.

Data collected through experiment is often reported in statistical tables. The most common type of statistical table is a Two-way table, which is also known as a **Carroll diagram**. Here is an example of a problem where you are asked to determine experimental probabilities of events based on statistical data presented in a Carroll diagram.

**Problem**: A research study for a new test for a virus sampled 1000 patients and collected the following data:

| | Has the virus | Does not have the virus | Total |
|---|---|---|---|
| Tested Positive | 49 | 199 | 248 |
| Tested Negative | 4 | 748 | 752 |
| Total | 53 | 947 | 1000 |

Based on this experimental data, what is the probability that a randomly selected patient

a) will test positive?
b) does not have the virus and tests negative?

Answers: You can find the experimental probabilities using ratios of entries from the table:
a) $^{248}/_{1000} = 0.248.$ b) $^{748}/_{1000} = 0.748.$

**Calculating Probabilities of Events using a Venn diagram model**

Suppose $S$ is the sample space of a probability experiment with $n$ distinct outcomes. If $E$ is an event in S, we would like to be able to calculate the probability of the event $E$ given the probabilities of the $n$ simple events in $S$. We will write P($E$) for the probability of the event $E$.

Earlier we saw that the sum of the probabilities of the simple events in $S$ is 1. This tells us that when $E_1, E_2, \ldots, E_n$ are the $n$ distinct simple events in $S$, then

$$P(E_1) + P(E_2) + \ldots + P(E_n) = 1.$$

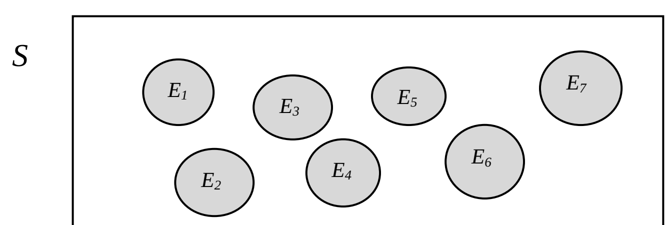
Since these simple events are precisely all of the single-element subsets of $S$, and we know that exactly one of these simple events occurs each time the experiment is carried out, we can determine that the probability of the event $S$ must also be 1.

This will also be true for an arbitrary event $A$ in $S$. If the distinct simple events in $A$ are $E_1, E_2, \ldots, E_r$ after a suitable renumbering, with $r \leq n$, then

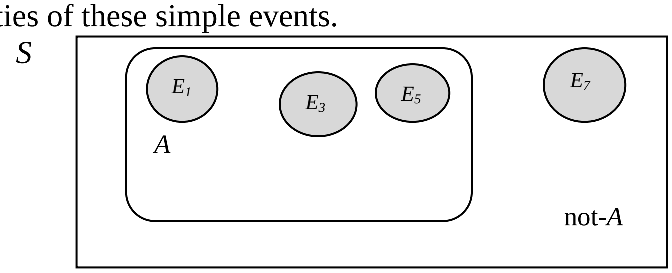
$$P(A) = P(E_1) + P(E_2) + \ldots + P(E_r)$$

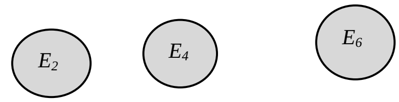
This includes the extreme case when $A$ is the empty set Ø. Since the empty set contains no simple events, we have P(Ø) = 0. When an event has probability 0, then it is impossible for the event to occur in the experiment.

This view of the probability of events as being the sum of the probabilities of the simple events they contain makes it easy to use a Venn diagram to model the probability of events in a given sample space. We view the sample space $S$ as our universal set.



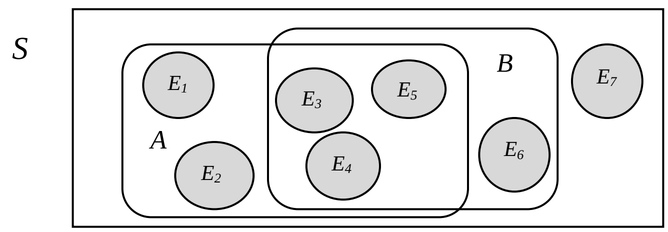

For each element of S, we have a simple event $E_i$. Each simple event has a probability $P(E_i) \geq 0$, and sum of these probabilities is 1. An arbitrary event $A$ in the sample space is determined by the simple events that it contains, and the probability P($A$) is the sum of the probabilities of these simple events.






For any event *A*, there is a complementary event not-*A* that is defined to be the event that occurs when *A* does not occur. As a subset of S, not-*A* = *S* – *A*. Since the sum of the probabilities of all the simple events in *S* is 1, we have that P(not-*A*) = 1 – P(*A*). For example, in our die-rolling experiment, P(we do not roll a 1 or 3) = 1 – P(we roll a 1 or 3) = 1 – ⅓.

If we introduce a second event *B*, we can use our Venn diagram to determine the probabilities of the events P(*A* or *B*) or P(*A* and *B*) that are determined by the union and intersection of the subsets *A* and *B*.




From this picture, we can see using the inclusion-exclusion principle that we will have a similar rule for probabilities:

P(*A* or *B*) = P(*A*) + P(*B*) – P(*A* and *B*).

For example, again using our die-rolling experiment, if we let *A* be the event that we roll an odd number, and *B* be the event that we roll a number 3 or less, then

P(roll odd or $\leq$ 3) = P(roll odd) + P(roll $\leq$ 3) – P(roll odd and $\leq$ 3)
= ½ + ½ – ⅓ = ⅔

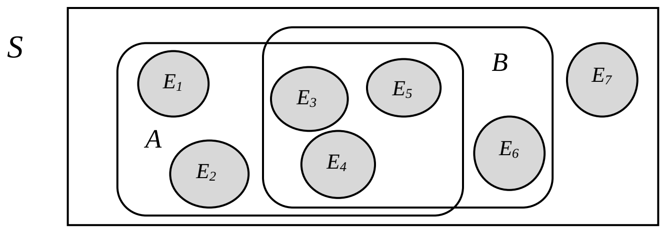
The easiest situation for calculating P(*A* or *B*) occurs when P(*A* and *B*) = 0. If *A* and *B* are two events for which P(*A* and *B*) = 0, then we say that the events A and B are **mutually exclusive**. It is easy to see that any pair of distinct simple events in a probability experiment will be mutually exclusive, as is the pair consisting of an event *A* and its complementary event not-*A*.

**Conditional Probability**

As we have in logic, in probability theory there is also a conditional, and again it is a bit more complicated (and interesting!) to understand than the straightforward "and/or/not" constructions. Given two events *A* and *B* in our sample space, the **conditional probability** for *A* given *B* is the probability that A will occur given that B occurs on the same trial of the experiment. The conditional probability of $A$ given $B$ is computed as a ratio:

$$P(A|B) = \frac{P(A \text{ and } B)}{P(B)}.$$

To understand this idea in our the Venn diagram model above, when we say $A$ given $B$ it means that we assume B has occurred, so we restrict our sample space to just the simple events that lie in $B$, and calculate the ratio of the sum of the probabilities of the simple events in $B$ that also lie in $A$ to the sum of the probabilities of the simple events in $B$. In our previous picture,




we would have P($B$) = P($E_3$)+P($E_4$)+P($E_5$)+P($E_6$) and P($A$ and $B$) = P($E_3$)+P($E_4$)+P($E_5$). Since the probabilities of the simple events are nonnegative numbers, P($A$ and $B$) will always be less than or equal to P($B$), so the ratio of these formed by the conditional probability P($A$|$B$) will always be a nonnegative number between 0 and 1.

Taking an example from our die-rolling experiment, if A is the event that the roll is odd and B is the event that the roll is ≤ 3, then P($A$|$B$) = $\frac{P(A \text{ and } B)}{P(B)} = \frac{1/3}{1/2} = {}^{2}/_{3}$.

Conditional probability gives us a way to determine if the occurrence of one event can influence the occurrence of another. We say that event $A$ is **independent** of event $B$ when P($A$|$B$) = P($A$). This makes sense because saying event $A$ is **dependent** on event $B$ should mean that the probability that $A$ occurs will change when event $B$ occurs.

For an example of dependent events, go back to our virus-testing data. If $A$ is the event that a patient tests positive and $B$ is the event that they have the virus, then we calculate P($B$)= 0.053, P($A$) = 0.248, and P($A$ and $B$) = 0.049.

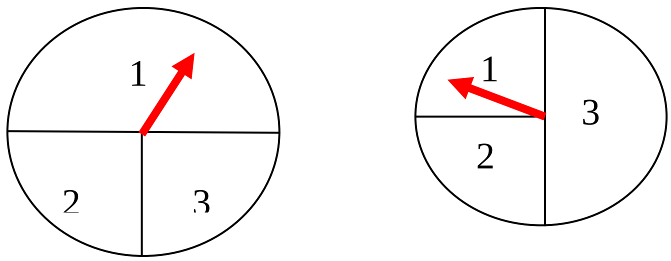
| | Has the virus | Does not have the virus | Total |
|---|---|---|---|
| Tested Positive | 49 | 199 | 248 |
| Tested Negative | 4 | 748 | 752 |
| Total | 53 | 947 | 1000 |

Therefore, P($B$|$A$) = ${}^{0.049}/_{0.053} = {}^{49}/_{53} \approx 0.925$.
So we can conclude that the probability that a patient has the virus increases a lot when the result of this test is positive. So event $B$ is dependent on event $A$.

When event B is dependent on event A, it is also true that event A is dependent on event B. This can be seen in our example by calculating $P(A|B) = {}^{0.049}/_{0.248} = {}^{49}/_{248} \approx 0.198$.

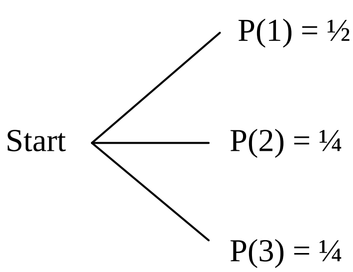
**Multistep Probability Experiments and Tree diagram models**

In more complicated probability experiments, a single trial consists of a sequence of smaller experiments. For example, suppose we invent a probability experiment in which each trial consists of spinning the two spinners shown in succession and recording the 3 results:

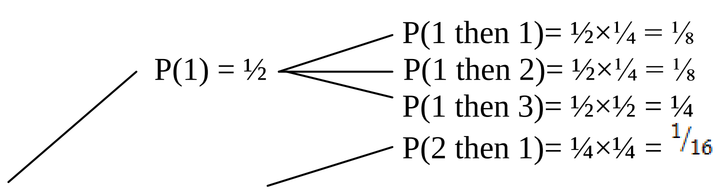



Spinner #1 Spinner #2

We can model experiments for which each trial is a sequence of chance events using a model called a tree diagram. To create a tree diagram for this experiment, we first draw 3 branches coming out of a starting point for the 3 possible outcomes from Spinner #1. Label the ends of each branch with the probability of the corresponding simple event for Spinner #1.




Next, we treat all of the outcomes from Spinner #1 as being a starting points for a tree diagram corresponding to Spinner #2, and *multiply* the probability of the simple events from Spinner #1 with those of Spinner #2 to get the probability of the sequence of these events.



Start — P(2) = ¼ — P(2 then 2)= $\frac{1}{4}\times\frac{1}{4} = \frac{1}{16}$

P(2 then 3)= $\frac{1}{4}\times\frac{1}{2} = \frac{1}{8}$

Start — P(3) = ¼ — P(3 then 1)= $\frac{1}{4}\times\frac{1}{4} = \frac{1}{16}$

P(3 then 2)= $\frac{1}{4}\times\frac{1}{4} = \frac{1}{16}$

P(3 then 3)= $\frac{1}{4}\times\frac{1}{2} = \frac{1}{8}$

This is the completed tree diagram for the probability experiment consisting of the sequence of 2 spins. There are 9 simple events, which can be listed by giving the possible pairs of outcomes from each spinner. We can see that the sum of the probabilities of the simple events is 1, which it must be in order that these are all the simple events in the sample space. Again, now that we know the probabilities of the simple events, it is possible to answer any question about the probability of events or conditional probabilities of any pair of events in this sample space.

## Exercises: Section 5.4

1. A spinner has 8 equally spaced regions numbered 1 to 8. Find the probability of

   a) the result of a spin being more than 3 and less than 7;
   b) the result of a spin being more than 3 or less than 7;
   c) the result of a spin being an even number and a perfect cube;
   d) the result of a spin being an even number or a perfect cube;
   e) the result of two spins in a row being 6 or more.
   f) the result of spinning 4 odd numbers in a row.

2. The weather forecast in Moose Jaw for tomorrow says there is a 10% chance of rain, and the weather forecast for Prince Albert says a 90% chance of rain. Which of the following are correct? (Find all correct answers.)

   a) the chance that it rains in both cities tomorrow is more than 10%;
   b) the chance that it does not rain in either city tomorrow is closer to 90% than 10%;
   c) if it rains in Prince Albert, then it will most likely rain in Moose Jaw also;
   d) if it rains in Moose Jaw, then it will most likely rain in Prince Albert also;
   e) the chance that it will rain in both places is exactly 50%;
   f) it is most likely to rain in just one of the two cities.

3. Find the probability of each simple event that can occur when two dice are rolled and the outcome is the total.

4. Suppose you have drawn 3 Aces in a row from a standard deck of 52 playing cards. Find the probability of randomly drawing an Ace on your next draw
   a) if you have replaced the drawn Ace each time previously,
   b) if you have kept the first 3 Aces.

5. Suppose that E and F are independent events.
Show that P(*E* and *F*) = P(*E*) × P(*F*).

6. The following information on the number of male and female students enrolled in each faculty was collected in the course of the University of Regina's Academic Program Review in 2010.

| | Arts | Business | Education | Engineering | Fine Arts | Kinesiology | Science | Social Work | Total |
|---|---|---|---|---|---|---|---|---|---|
| Male | 1430 | 831 | 458 | 779 | 254 | 212 | 765 | 151 | 4880 |
| Female | 2152 | 919 | 1698 | 232 | 330 | 316 | 634 | 1023 | 7304 |
| Total | 3582 | 1750 | 2156 | 1011 | 584 | 528 | 1399 | 1174 | 12184 |

Based on this data, what is the probability (to 3 decimal places) that a randomly selected University of Regina student is:

a) female and in Arts?

b) male, or in Business, Engineering, or Science?

c) female, given that they are in Education?

d) in Science, given that they are male?

e) in Education or Social Work, given that they are female?

7. *The Monty Hall problem*. Monty Hall was host to a live game show in the early years of television. Participants from the audience were randomly chosen to play a game in which they were shown 3 doors, with a grand prize behind one of the doors. The player selected one door at the start, then Monty would remove one of the remaining doors, which Monty knew did not have the prize behind it. Then the player was given the option of switching their door for the remaining one, or to keep the door they had originally chosen.

a) Find the probability that the player finishes with the door covering the grand prize given that the player switches doors.

b) Does the strategy of switching doors double the player's chance of winning? Explain your answer.

8. Probabilities in gambling are often reported in the form of **odds**. If the odds in favour of winning a bet are reported as the ratio *a : b*, then this means the expected probability of winning is $^{a}/_{a+b}$. Similarly, the expected probability of losing the bet would be odds against winning same bet would be $^{b}/_{a+b}$, so the odds against would be reported as *b : a*.

Odds for betting at the horse racing track are set by the house depending on how much is bet on each horse. Minimum bets are most often \$2, and the house keeps a percentage of the total amount bet. The odds for each horse are set according to the ratio

$$\frac{\text{total amount to be paid out}}{\text{amount bet on the horse}},$$

rounded to the nearest fraction below that has denominator 1, 2, or 5.
For example, if \$1 500 in total is bet, with \$550 of it on one horse, then the odds are determined by the fraction $^{1500}/_{550} \approx 2.72$, which is just above $2.6 \approx {}^{13}/_{5}$, so the odds on this horse would be set at 13 to 5, meaning that a winning bet receives \$13 in return for every \$5 bet on this horse.

Suppose there are five horses in a race, and just before the race these are the amounts bet on each horse:

Horse #1: \$1 000
Horse #2: \$ 200
Horse #3: \$ 800
Horse #4: \$ 490
Horse #5: \$ 10

a) How would the house set the odds for this race?

b) If a person made a \$10 on each horse, which winners would result in them making a profit?

c) How much of this money would the house get if Horse #3 wins? If Horse #1 wins?

d) How can we interpret odds given at the race track in terms of odds for and against?

9. A slot machine at the Casino has 4 rollers. Each roller has 4 pictures.

The first three rollers have 2 Queens, 1 Crown, and 1 Treasure Chest.
The fourth roller has 2 Skunks, 1 Queen, and 1 Treasure Chest.

Each play of the slot machine costs 25¢. Every second quarter paid into the machine goes into the Jackpot. Matching 4 Queens wins a free play, Matching 4 Treasure Chests wins all the money in the Jackpot.

a) Calculate the probability of winning the Jackpot on one spin.

b) Calculate the probability of winning a free play or the Jackpot on one spin.

c) Suppose you play the slot machine 400 times. What is the probability that the house has at least half of the money you have put into the machine?

# 5.4 Appendix -- Blood Types

In 1930 Karl Landsteiner, an Austrian living in the U.S., received a Nobel Prize for achievements that included his discovery thirty years earlier of the human blood groups. He classified blood type according to the presence of three protein substances (called antigens) that carry the labels A, B, and Rh. According to the Red Cross, 46% of all Canadians have blood that contains the A antigen, 12% have the B antigen, 85% have the Rh antigen (called the Rh factor), 3% have both A and B, 38.5% both A and Rh, 10% both B and Rh, and 2.5% have all three antigens. To answer the exercises below you should summarize this data with a Venn diagram whose three circles are labelled A, B, and Rh.

Landsteiner introduced the terminology that we still use today. A person's blood is classified in group A if it contains the A antigen but not the B; that is, group A blood belongs to the set $A \cap B'$. (You should check in your diagram that this implies 43% of Canadians have group A blood.) Group B blood contains the B antigen but not the A, while group AB contains both antigens and group O contains neither. The group name is assigned a plus sign if the blood contains the Rh factor (and is termed "Rh positive"), and a negative sign if the factor is absent (and is termed "Rh negative"). Thus each circle of your Venn diagram includes four different blood types; for example, the "A circle" contains types $A^+$, $A^-$, $AB^+$, $AB^-$. Do not confuse the set we are calling A (which refers to the antigen A) with the blood group A (which does not contain the B antigen).

## Exercises: Section 5.4 Appendix

1. Make a list of the eight blood types in both Landsteiner's notation and the corresponding set notation. Compute in each case the percentage of Canadians having each blood type. In particular, what percentage has $O^-$ blood?

**Rule**: A person cannot accept an antigen that is not already in his or her blood. So, for example, a person without the B antigen can safely receive a transfusion only if the blood contains no B antigen -- he or she can receive either group A or O blood, not group B or AB.

2. Which blood type can be accepted by any recipient? What percentage of Canadians are such universal donors?

3. Describe in set notation the set of blood types that can receive blood from a donor classified $A^+$. What percentage of Canadians can safely receive $A^+$ blood?

4. Describe in set notation the set of blood types that are acceptable to a person classified $A^+$. What percentage of Canadians can donate blood to a person classified $A^+$?

# 5.5 Unit Review Exercises

Write the following ratios as fractions in lowest terms.

1. 16 teachers per 250 students

2. 600 km of driving per 48 L of gas

3. 213 songs per 18 CD's

4. 117 pages of notes per 26 students

Solve the equations in exercises #19 to #22 (without cross multiplying).

5. $\frac{z}{6} = \frac{54}{18}$

6. $\frac{x}{15} = \frac{30}{75}$

7. $\frac{y}{32} = \frac{3}{4}$

8. $\frac{k}{6} = \frac{28}{4}$

9. If 48 L of gas costs $28.56, how much can you get for $20?

10. You can buy iced tea mix in a 350g container, a 600g container or a 1kg container. Looking at the prices for each container, which is the best buy?

350g costs $3.99
600g costs $5.49
1kg costs $8.99

Convert each of the decimals to a percent.

11. 0.316

12. 0.102

13. 0.36

14. 3.14

Convert each of the fractions to a percent.

15. $^{1}/_{5}$

16. $^{7}/_{8}$

17. $^{4}/_{6}$

18. $^{9}/_{20}$

19. What is 27% of 480?

20. What percent of 480 is 27?

21. 27% of what is 480?

22. A leather couch is regularly priced $2 500.00. It is marked down 10% and then another 5%. What is the final sale price?

23. When 24 litres of gasoline are put into a car's tank, the indicator goes from ¼ of a tank to $^5/_8$ of a tank. What is the tank's capacity?

24. Looking back to Unit 4 review, exercise 52. Flora is designing a bead bracelet and she determines that the bracelet should be 72 beads in length. The flower pattern repeats every eight beads. The border pattern repeats every nine beads. Each flower pattern requires 1 red, 40 turquoise, and 15 white beads. Each border pattern requires 3 yellow, 3 blue, and 3 red beads. How many beads of each colour does she need to make the bracelet?

25. How many beads of each colour are needed for the 2$^{nd}$ bracelet design (Unit 4 review, exercise 53), which has a border of 2 yellow, 2 blue, and 2 red beads?

26. The kid is growing really fast: this year his height is 10% more than it was last year, and last year his height was 20% more than the year before. By what percentage has his height increased during the last two years?

27. If your salary of $40 000 increases by 5% next year, increases by 2% the following year, and decreases by 5% the year after that, what will you annual salary be in three years? (to the nearest dollar)

28. In 1993, sales of Brand X detergent increased by 20% over the previous year. In 1994, they decreased 30% from the previous year, and in 1995 sales increased 10% over the previous year. The sales in 1995 were what percent of the sales in 1992?

29. If you have two mugs of hot chocolate, one that is 32% chocolate and the other 20% chocolate, how much of each would you need to make one cup that is 25% chocolate?

30. A school board plans to merge two schools into one school of 1 000 students in which 42% of the students will be out of province. One of the schools has a 10% out of province student body and the other has a 90% out of province student body. What is the student population in each of the two schools?

31. If 10 children can eat a small box of candy in 5 minutes, how many of these boxes can 20 children eat in 10 minutes?

32. If ten people can paint 60 houses in 120 days, how many days will it take five people to paint 30 houses?

33. Joey and his brother Nathan dig up potatoes to fill up 100lb bags. It takes Joey 36 minutes to fill one bag, and it takes Nathan 50 minutes to fill up one and a half bags. How many bags can they fill up completely in 5 hours if they work together?

34. The interest on an investment varies directly as the rate of interest. If the interest is \$48 when the interest rate is 5%, find the interest when the rate is 4.2%.

35. The ancient Babylonians guessed that the area of a round plane figure might be directly proportional to the square of its circumference. When they measured the area using an example with a circumference of 3.5, they found the area to be close to 1. If they were correct, what would the area be when they tried a figure with a circumference of 10?

36. Over a specified distance, speed varies inversely with time. If a car goes a certain distance in one-half hour at 30km/hr, what speed is needed to go the same distance in three-quarters of an hour?

37. At the end of the Revolutionary War, the currency of the United States was in a deplorable condition. The unit was the pound (₤), but the pound varied considerably from state to state. Thus the first American arithmetic (Pike, 1786) contains dozens of rules from which we select two:

    i) To reduce Newhampshire currency to Southcarolina currency, multiply the Newhampshire by 7 and divide the product by 9, and the quotient is the Southcarolina.
    Example: (₤108 Newhampshire) × 7 = 756; 756/9 = ₤84 Southcarolina

    ii) To reduce Newyork currency to Southcarolina currency, multiply the Newyork by 7 and divide the product by 12, and the quotient is the Southcarolina.

    Formulate an equitable rule for reducing Newyork to Newhampshire. The Newhampshire currency was what percentage of the Newyork?

38. A woman collects antique snuffboxes. She bought two, but found herself short on money and had to sell them quickly. She sold them for \$600 each. On one she made 20% and she lost 20% on the other. Did she make or lose money on the whole deal? How much?

39. a) The price of an item is increased 12% to \$124. What was the original price?

    b) Realizing that \$124 is now too high for the market to bear, the merchant decides to reduce the price to \$100; by what percentage must the price now be decreased?

40. If your bill at a restaurant is \$28.90 including a 7% tax, how much should you leave for a tip if you want to leave 15% of the cost of the food before tax?

41. After a 45% reduction, a VCR sold for \$147.50 including 13% sales tax. What was the price before the reduction and before the taxes?

42. Two trains start at 7 a.m., one from Regina going to Winnipeg, the other from Winnipeg to Regina. The faster train makes the trip in 8 hours, the slower in 12 hours. At what hour of the day will the two trains pass each other?

43. The following exercise comes from *The Scholar's Guide to Arithmetic* by Bonnycastle ($6^{th}$ edition, 1795). If a cardinal can pray a soul out of purgatory, by himself, in an hour, a bishop in three hours, a priest in five hours, and a friar in seven hours, how long would it take them to pray out three souls from purgatory, if they all pray together?

44. A commuter reaches the railroad station nearest his home at 5 p.m., where his wife meets him with the family car. One day he unexpectedly arrived at the station at 4 p.m. and, instead of waiting for his car at the station, he started out for home on foot. After a certain length of time he met his wife and traveled the rest of the way in the car. He reached home sixteen minutes ahead of the usual time. How long did he walk?

45. The provincial government is to increase nursing salaries by 14% over the next three years. This increase will come from yearly increases of 4% the first year, 4% the second year, and 6% the third year. If a nurse has a $30 000 salary now, what will this nurse's salary become in three years?

46. The proprietor began telling me how difficult it was to compute the discounted shelf price of a bottle on the basis of the wholesale cost. To make his point, he demonstrated the problem with a case of burgundy that cost him $37.50. Dividing by 12 he arrived at a price of $3.125 for each bottle. He computed that the 20% mark-up would be $0.625, which he added to the base price to get $3.75. He then computed the federal tax of 11% and added it on. Then he computed the provincial tax of 7% on top of the federal tax and added it on. Finally he computed a 10% discount and subtracted it from the price with the taxes included, arriving at a shelf price for the burgundy. "Oh," he lamented, "what I would not give for a magic formula that would enable me to do this in one step!"

   a) Provide a formula for the distraught proprietor so that all he has to do is plug in the price of any case of wine and out comes the shelf price.

   b) Use your formula to find the shelf price for the burgundy.

47. A cookie recipe calls for $^1/_4$ tsp of cinnamon and $^1/_3$ tsp of nutmeg. If you accidentally put in $^1/_3$ tsp of cinnamon and $^1/_4$ tsp of nutmeg, how much nutmeg should you add so that these two ingredients are in the correct proportion?

48. The resistance, $R$, in a wire, measured in Ohms, varies inversely as the square of its diameter, $d$. Knowing that a 0.5 cm thick piece of wire has a resistance of 48 Ohms, find the resistance in a 1.0 cm thick wire.

49. Kepler's Third Law of planetary motion states that if $T$ is the time it takes a planet to orbit the Sun and $D$ is the average distance from the planet to the Sun, then $T^2$ varies directly as $D^3$.

We know that the Earth has $T = 1$ year and $D = 1$ AU (Astronomical Unit). The average distance between Uranus and the Sun is about 19.18 times that of the Earth (i.e. $D = 19.18$ AU). How many Earth years does it take for Uranus to orbit the Sun?

50. Among the 50 guests at a party, 80% drink alcoholic beverages. Of those, 40% drink vodka, 30% drink whiskey, and 20% drink both. How many drink only vodka?

51. In a community with a total population of 50, each person is classified as male or female, as adult or young, and as superstitious or otherwise. The male population, the adult population, and the superstitious population each account for 40% of the total population. There are 5 superstitious adult males, 10 superstitious adults, 8 superstitious males, and 7 adult males. What per cent of the total population consists of non-superstitious young females?

52. Suppose $b/a$ is a fraction of positive integers in lowest terms, with $b > a$. Show that $(a+b)/b$ and $a/(b-a)$ will both be in lowest terms, and conclude that $(a+b)/b \neq a/(b-a)$.

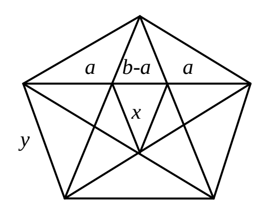
53. Two positive lengths $b > a$ are said to be in the *golden ratio* if

$$b/a = b + a/a = a/b - a.$$

Use exercise #52 to show that the golden ratio is an irrational quantity.

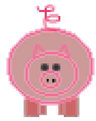
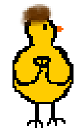
54. The Pythagoreans constructed lengths in golden ratio using the diagonals of the regular pentagon. When the diagonal of a regular pentagon is intersected by another diagonal, the diagonal is separated into two pieces of length $b$ and $a$, with $b > a$.

Using similar triangles appearing in the diagram, show that $x = a$, $y = b$, and $b/a$ is the golden ratio.



## Answers: Introduction

1.

Thus there are 30 pigs  and 40 chickens . 

2. Yes, you can order exactly 217 Chicken McNuggets®.

3. There are 89 nine ways for the rabbit to climb ten stairs.

4. The area of the rectangle is $16 \times 9 = 144$m, so the area of the square must be 144m. The square root of 144 is 12, one side of the square. Therefore each side is 12m. $12 \times 4 = 48$m.

## Answers: Section 1.1

1. 30 2. 16 3. 51 4. 152 5. 1 110

6. The common feature of all the sequences in #1 - #5 is that you add the same number to each term in a sequence to obtain the next term.

1) 5 2) 3 3) 7 4) 35 5) 111

7. 20 and 13 8. 52

9. 132 10. 292 11. 143 12. 388 13. 49

14. 486 15. 64 16. 50 421 17. 3.5

18. 0.000 000 000 9

19. The common feature in all of the sequences in #15 - #19 is that the terms in each sequence are multiplied by the same term to obtain the next term.

20. 14) 3 15) 2 16) 7 17) $^1/_2$ or 0.5 18) $^1/_{100}$

21. The common difference is 5 in the sequence.

22. The first term is 8 and the common ratio is $^3/_2$.

23. This is the sequence of prime numbers. The next term is 17.

24. This is the sequence of consecutive squares; the difference between consecutive terms is increasing by 2. The next term is 36.

25. You add the previous two terms to obtain the next term. The next term is 89.

26. You add the previous two terms to obtain the next term. The next term is 123.

27. You multiply the first term by 2 to obtain the second, the second term by 3 to obtain the third, etc. The next term is 5040.

28. The sum of any six consecutive terms is 0, so the sum of the first 96 terms is 0. The sum of the first 100 terms is therefore 2 + 5 + 3 + (-2) = 8.

29. The next symbol is  , 7 written backward and forward.

## Answers: Section 1.3

Generally, we have only given one solution for each of the questions.

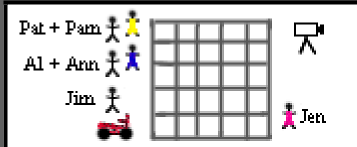
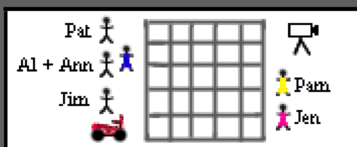
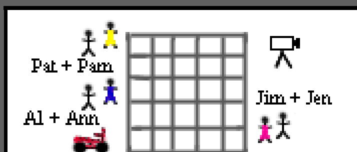
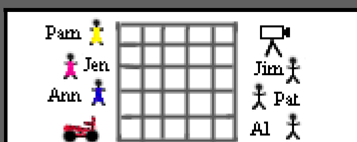
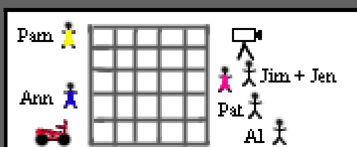
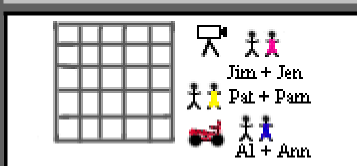
| | | | |
|---|---|---|---|
| 1. | 5 cows and 13 ducks. | 2. | 48 cars and 52 bikes. |

3. 16 single chocolate bars, 23 bags of 2 pieces of fruit, and 11 bags with one chocolate bar and 4 fruits.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 4. | 6 | 5. | 7 | 6. | 4 | 7. | 9 |
| 8. | 3 | 9. | 9 | | | | |
| 10. | 8 | 11. | 3 | 12. | 25 | 13. | 31 |

14. Let's name our three couples Jim and Jen, Pat and Pam, and Al and Ann.

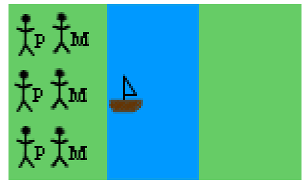
Here is one way that they could all get to the movie, with at most two riding on the scooter, and without leaving any of the girls with another guy without their date present:

| Step # | Who Rides to the Movies | Who Rides Back | Resulting Picture |
|---|---|---|---|
| **1** | Jim + Jen | Jim |  |
| **2** | Pat + Pam | Pat |  |
| **3** | Jim + Pat | Pat + Pam |  |
| **4** | Pat + Al | Jen |  |
| **5** | Jen + Pam | Pam |  |
| **6** | Ann + Pam | nobody |  |

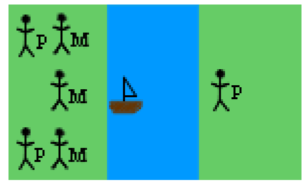
\

15. Here is one way you could get all six across without having the physicists outnumber the mathematicians at any time:

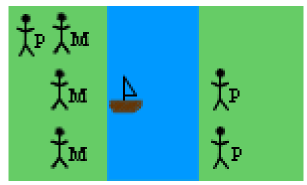
Here is how they start out:



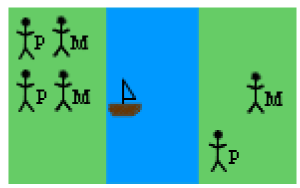
A mathematician and physicist cross together and the mathematician returns.



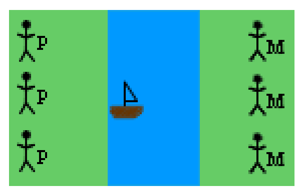
Now send two physicists across, and one returns.



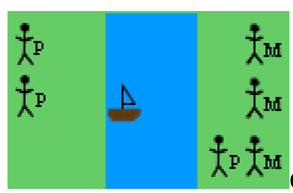
Now two mathematicians cross, and a mathematician and physicist return.



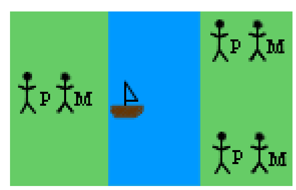
The two mathematicians cross, And the physicist returns.



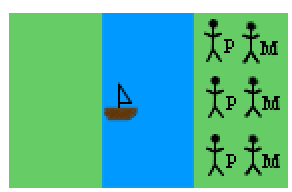
Then two physicists cross, and a mathematician or a physicist returns.



or

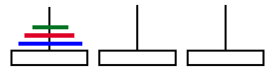


Now the last two cross, and everyone is across safely.

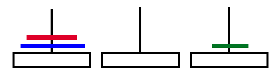


16. Tower of Hanoi

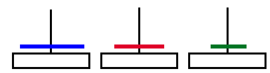
a) Here is how you start, with all the rings on the leftmost tower.



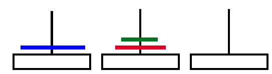
Now move the smallest ring on to the rightmost tower.



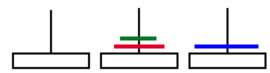
Then move the middle sized ring to the middle tower,



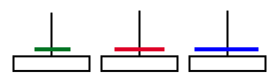
and then the smallest back onto the middle sized ring.



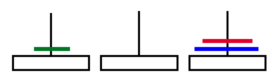
Then you can move the largest ring to the rightmost tower.



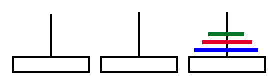
Now move the smallest ring to the leftmost tower.



Then move the middle sized ring onto the largest ring,



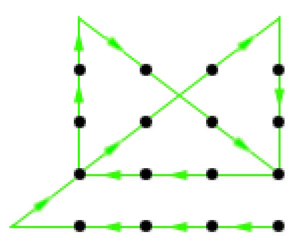
and the smallest onto the middle sized ring.



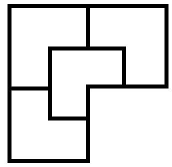
b) 15 moves. 31 moves.

c) $2^{64} - 1$ moves.

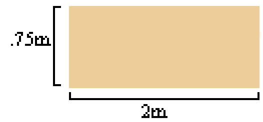
17. Here is one possible answer:



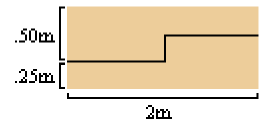
18. If this land is divided like so, all four children's land is the same size and shape.



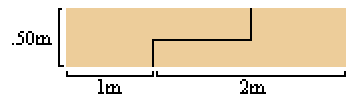
19. If you have a board,



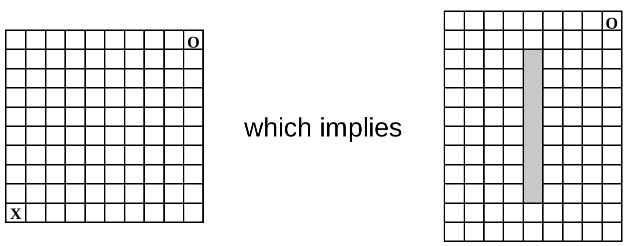

and you cut it once like so,



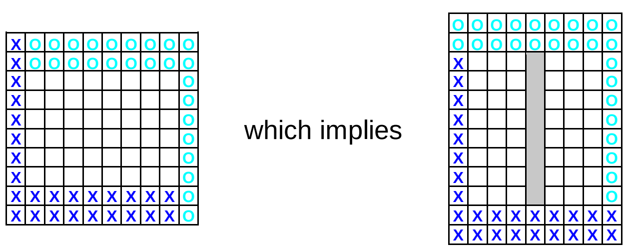

then it can be fitted together like this to cover the hole.




20. The argument is in several steps. We label one of the sides of the cut with X's, and the other side with O's. In the 10 by 10, opposite corners must be in opposite pieces because we are limited to 9 across.




Since we can only go down 10 (and we are using a symmetric pattern), the bottom two rows are X's and the top two rows are O's. Since we can only go nine across, the leftmost column must be X's, and the rightmost column must be O's.



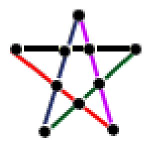
By starting in one of the corners of the inner rectangle formed in the 10 by 10 piece, you can determine which piece of the cut it must be part of.  Working one row or column at a time from the corner you start with, you automatically arrive at the solution.

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| X | O | O | O | O | O | O | O | O | O |
| X | O | O | O | O | O | O | O | O | O |
| X | X | O | O | O | X | O | O | O | O |
| X | X | O | O | O | X | O | O | O | O |
| X | X | X | O | O | X | X | O | O | O |
| X | X | X | O | O | X | X | O | O | O |
| X | X | X | X | O | X | X | X | O | O |
| X | X | X | X | O | X | X | X | O | O |
| X | X | X | X | X | X | X | X | X | O |
| X | X | X | X | X | X | X | X | X | O |

which implies

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| O | O | O | O | O | O | O | O | O |
| O | O | O | O | O | O | O | O | O |
| X | O | O | O | | O | O | O | O |
| X | O | O | O | | O | O | O | O |
| X | X | O | O | | X | O | O | O |
| X | X | O | O | | X | O | O | O |
| X | X | X | O | | X | X | O | O |
| X | X | X | O | | X | X | O | O |
| X | X | X | X | | X | X | X | O |
| X | X | X | X | | X | X | X | O |
| X | X | X | X | X | X | X | X | X |
| X | X | X | X | X | X | X | X | X |

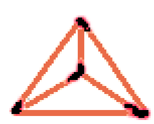
21. If the coins are arranged like this, they make five rows of four.



22. There are lots of answers to this problem.  You just have to overcome the belief  barrier that each child's portion must be in one piece.  Here is one way it could be done, assigning a letter to each child to show their portions of land.

| | | |
|---|---|---|
| C | E | A |
| A | D | C |
| B | C | A |
| E | E | B |
| D | D | B |

23. To solve this problem, we have to realize that the answer can be 3D to form the four equilateral triangles.  Doing so makes the problem easy to solve.



24. 60 years old.

25. 81 CD’s

26. 19

27. Gwen was 24 and Bill was 21.

28. 33 boxes.

29. Two full boxes weigh 40 oz.

30. Cook them simultaneously – seven minutes.

31. 310 kg of oranges.

32. 100 m

33. $16.00

34. 40 posts.

35. 22.5 seconds.

36. 11 seconds.

37. 24 pages

38. 21

39. 111

40. 69

41. Washington High won the high jump (and all other events except the shot put).

42. 4 095

43. 615

44. 4 234

45. 14 820

46. When we write the sequence forward and backward, then add them together, we do not get the same sums along the bottom, so Gauss' method will not work.

```
      1 +   2 +   4  +  8 + 16 + 32 + 64
+    64 +  32 +  16  +  8 +  4 +  2 +  1
----------------------------------------
     65 +  34 +  20  + 16 + 20 + 34 + 65
```

47. 65 150

48. There can be 1 or 3 red cards in the first pile

49. Move E left 1, P down 2, X right 1, A down 1, O left 3, B up 1, C up 2, X exits right 5.

## Answers: Section 1.4.1

1.

| p | q | p → q | not-q | (p → q) and (not-q) | not-p | [(p → q) and (not-q)] → (not-p) |
|---|---|---|---|---|---|---|
| T | T | T | F | F | F | **T** |
| T | F | F | T | F | F | **T** |
| F | T | T | F | F | T | **T** |
| F | F | T | T | T | T | **T** |

This truth table shows the indirect argument statement is always true, so it is a tautology.

**2.**

| p | q | r | p → q | q → r | (p → q) and (q → r) | p → r | [(p → q) and (q → r)] → (p → r) |
|---|---|---|---|---|---|---|---|
| T | T | T | T | T | T | T | **T** |
| T | T | F | T | F | F | F | **T** |
| T | F | T | F | T | F | T | **T** |
| T | F | F | F | T | F | F | **T** |
| F | T | T | T | T | T | T | **T** |
| F | T | F | T | F | F | T | **T** |
| F | F | T | T | T | T | T | **T** |
| F | F | F | T | T | T | T | **T** |

3.

| p | q | p or q | not-p | (p or q) and (not-p) | [(p or q) and (not-p)] → q |
|---|---|---|---|---|---|
| T | T | T | F | F | **T** |
| T | F | T | F | F | **T** |
| F | T | T | T | T | **T** |
| F | F | F | T | F | **T** |

4.

| p | q | p and q | (p and q) → p | p or q | p → (p or q) |
|---|---|---|---|---|---|
| T | T | T | **T** | T | **T** |
| T | F | F | **T** | T | **T** |
| F | T | F | **T** | T | **T** |
| F | F | F | **T** | F | **T** |

5. i) If you build it, then you will see him.

6. ii) He got a bonus.

7. ii) Thus, if R, then T.

8. i) You studied.

9. ii) So $1 + 1 \neq 3$

10. If the Tragically Hip comes to town, then I will be broke until payday.

11. Randy Ferbey curls.

12. Beggars aren't choosers.

13. Therefore, Joel likes cartoons.

14. If he is your son, then he is not fit to serve on jury.

15. If it is a guinea pig, then it does not appreciate Beethoven.

16. If one is a pawnbroker, then one is honest.

17. If Fang is Arbu he would have said, "I am Arbu."
Thus, Gang would be Arbu and Hang would be Bosnin.

If Fang is Bosnin he would have said, "I am Arbu."
Thus Gang would be Arbu and Hang would be Bosnin.

So, Gang is Arbu and Hang is Bosnin.

## Answers: Section 1.4.2

| | | | | |
|---|---|---|---|---|
| 1. True | 2. True | 3. False | 4. True | 5. True |
| 6. True | 7. False | 8. True | 9. False | 10. False |

11. Some people do not like coffee.

12. All zebras do have stripes.

13. For all integers, $49 - x \not\equiv -1$ or $49 - x \neq 0$.

14. For some real numbers $x$, there does not exist a real number $y$ such that $xy = 1$.

15. Some dogs do not go to heaven.

16. valid
17. invalid
18. valid

19. invalid
20. invalid

21. "Your sons are not fit to serve on jury".
(or "If he is your son, then he is not fit to serve on jury".)

22. "Guinea pigs don't appreciate Beethoven".
(or "If it is a guinea pig, then it does not appreciate Beethoven.")

23. "All pawnbrokers are honest".
(or "If one is a pawnbroker, then one is honest".)

24. "Opium-eaters do not wear white kid gloves".
(or "If he is an opium-eater, then he does not wear white kid gloves".)

25. "I can't read any of Mr.Brown's letters".
(or "If it is written by Mr. Brown, then I can't read it.")

## Answers: Section 1.4.3

1. Sue's mother, Jo, has a garter snake named Belle.

2. Dana owns a red Dodge, Sarah owns a white Ford, and Brad owns a blue Chev.

## Answers: 1.4 Appendix A

1. **(f → s) and (s → f)**
2. **(not-s) and (not-b)**
3. **(not-s) and f**
4. **(f → w) and (not-b)**
5. **w or b**
6. **[f and (not-w)] → b**
7. **f → w**
8. **(s → w) or ((not-s) → (not-w))**
9. **s → b**
10. **(s → w) ∧ (not-b)**

11. Let **P** be the statement "The defendant is guilty."
Let **Q** be the statement "He had an accomplice."
In order for **P → Q** to be false, **P** has to be true.

12.

| p | q | p and q | not-(p and q) |
|---|---|---|---|
| T | T | T | F |
| T | F | F | T |
| F | T | F | T |
| F | F | F | T |

13.

| p | q | not-p | not-q | (not-p) or (not-q) |
|---|---|---|---|---|
| T | T | F | F | F |
| T | F | F | T | T |
| F | T | T | F | T |
| F | F | T | T | T |

14.

| p | q | $q \rightarrow p$ |
|---|---|---|
| T | T | T |
| T | F | T |
| F | T | F |
| F | F | T |

15.

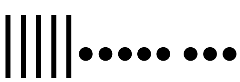
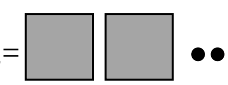
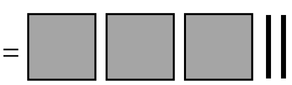
| p | q | not-q | (not-q) or p |
|---|---|---|---|
| T | T | F | T |
| T | F | T | T |
| F | T | F | F |
| F | F | T | T |

16. **not-(p and q)** ≡ **(not-p) or (not-q)** and **$q \rightarrow p$** ≡ **(not-q) or p**
**$q \rightarrow p$** is not logically equivalent to **$p \rightarrow q$** because their truth tables do not match.

## Answers: Section 2.1

1. 90 063 041

2. a) $(1 \times 10^4) + (2 \times 10^3) + (3 \times 10^2) + (4 \times 10^1) + (5 \times 10^0)$
   b) $(1 \times 10^6) + (9 \times 10^5) + (8 \times 10^2) + (3 \times 10^0)$
   c) $(5 \times 10^9) + (7 \times 10^6) + (8 \times 10^4) + (2 \times 10^3)$

3. a) 58 =  b) 202=  c) 330 = 

## Answers: Section 2.2

1. 116  2. 80  3. 130  4. 133  5. 633  6. 1600
7. 1 119  8. 8 817  9. 7 912  10. 383 352  11. 2 216

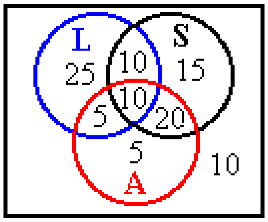
## Answers: Section 2.3

1. 44  2. 32  3. 54  4. 35  5. 285  6. 48
7. 423  8. 2 235  9. 44 295  10. 6891

11. 222 235 12. 80 467 13. 92 273
14. – 92 273 15. – 6 213 16. – 1 881

17. a) 65 b) 12 c) 25

18. a)




b) 10 c) 35 d) 45

## Answers: Section 2.4

1. 108 2. 581 3. 6 622 4. 42 480 5. 1 896 6. 52 575
7. 6 072 8. 2 726

9. Estimation shortcut: (700 – 8) × 28 = 19 600 – 224 = 19 376.
10. Doubling and halving: 2 563 × 64 = 5 126 × 32 = 10 252 × 16 = 20 504 × 8
= 164 032
11. General method: (876 × 900) + (876 × 9) 9 = 788 400 + 7884 = 796 284
12. General method: (9 555 × 200)+ (9 555 × 20)+ (9 555 × 2)
= 1 911 000 + 191 100 + 19 110 = 2 102 100 + 19 110 = 2 121 210
13. Doubling and halving: 4.95 × 0.13 = 9.10 × 0.065 = .5460 + .0455 = 0.5915
14. Estimation shortcut: – 37 × (60 + 1) = – (222 + 37) = – 259
15. General method: 206 × 27 = 5 400 + 162 = 5 562. (Answer is positive.)
16. Doubling and halving: 168.922 × 16 = 337.844 × 8 = 2702.752

17. a) 22.94 $cm^2$ b) 3159 $mm^2$ c) 3640 $m^3$ d) 268 $cm^3$

18. There are 90 different cars available.

19. a) The possible last digits of the squares of single digits are 0, 1, 4, 9, 6, and 5. The last digit of any perfect square will be the last digit of the square of its last digit.

b) A negative times a negative is a positive. A positive times a positive is a positive. 0 times 0 is 0. Therefore, *n* times *n* cannot be negative.

c) All of them. 625 = $25^2$, 729 = $27^2$, 1 024 = $32^2$, 6 241 = $79^2$, and 10 000 = $100^2$.

d) 521.

20. $(x5)^2$ is a 4-digit number ending in 25. Its first two digits are $x^2 + x$. This can be seen from the first two estimates in the general multiplication strategy: The first estimate is $x^2 \times 100$, to get the next estimate we would add (x × 50) twice.

21. a) 0, 1, 8, 27, 64, 125, 216, 343, 512, and 729.
b) $10^n$ is a perfect cube when *n* is a multiple of 3.
c) Our answer in a) tells us this will be the cube of a number in the 70's. The last digit 3 tells us the last digit of the cube root has to be a 7, since 7 is the only digit whose cube ends in a 3. So the cube root has to be 77.

## Answers: Section 2.5

1. 141 2. 7 515 3. 1 544 4. 15 544 R 2 5. 3 120 R 6

6. (÷13)→ 56 ÷ 3 = 18 R 2, so 728 ÷ 39 = 18 R 26.
7. (÷9) → 1104 ÷ 9 = 122 R 6, so 9936 ÷ 81 = 122 R 54.
8. (÷11) → 109 ÷ 8 = 13 R 5, so 1199 ÷ 88 = 13 R 55.
9. (÷8) → 9594 ÷ 108, (÷2) → 4797 ÷ 54, (÷9) → 533 ÷ 6 = 88 R 5,
so 76 572 ÷ 864 = 88 R (5×9×16) = 88 R 720.

10. Estimation shortcut: 9 935 ÷ 45 = (9 000 ÷ 45) + (900 ÷ 45) + (35 ÷ 45)
= 200 + 20 + (35 ÷ 45) = 220 R 35
11. Common divisor shortcut: 14840 ÷ 424, (÷8) → 1855 ÷ 53 = 35.
12. Common divisor shortcut: 13 296 ÷ 240, (÷12) → 1108 ÷ 20 = 55.4
Since 0.4 x 20 x 12 = 96, the answer is 55 R 96.
13. General method: 282 R 2630.

14. a) Answers will vary. Almost any nonequal pair of integers will do.
b) i) True ii) False iii) True

## Answers: Section 2.6

| | | | | | |
|---|---|---|---|---|---|
| 1. | $2\ 165_7$ | 2. | $50_6$ | 3. | 578 |
| 4. | $22\ 002_3$ | 5. | $1\ 001\ 001_2$ | 6. | $2A_{16}$ |
| 7. | $23_4$ | 8. | $11\ 112_3$ | 9. | $1225_6$ |
| 10. | $430_5$ | 11. | $\{x, y, z\} = \{5, 6, 7\}$ | 12. | $x + y + z = 22$, so $\{x, y, z\} = \{6, 7, 9\}$ <u>or</u> $\{5, 8, 9\}$ |

13. $3 \le (x+y+z) \le 21_7$, so $x+y+z = 12_7$. But this would make the total $132_7$.

14. Oct 31 = Dec 25 because $25_{10} = 31_8$.

15. $3\ 111_5$
16. $10\ 334_8$
17. $14\ B73_{12}$
18. $5\ 256_8$
19. $1\ 1011_2$
20. $2\ D7F_{16}$

21. $100\ 1101_2$ 22. $2\ 122\ 202_4$ 23. $28\ AA0_{12}$
24. $50_{12}$ R $B_{12}$ 25. $110_2$ R 1 26. $321_5$ R $21_5$

## Answers: Section 2.6 Appendix

1.
$$\begin{array}{r} 2y+3 \\ \times\ \ 6y+1 \\ \hline 2y+3 \\ 12y^2+18y \phantom{+3} \\ \hline 12y^2+20y+3 \end{array}$$

2.
$$\begin{array}{r} 6n^2-2n+12 \\ \times\ \ 4n-7 \\ \hline -42n^2+14n-84 \\ 24n^3-8n^2+48n \phantom{-84} \\ \hline 24n^3-50n^2+62n-84 \end{array}$$

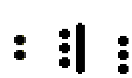
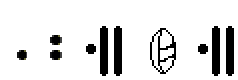
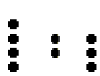
3.
$$\begin{array}{r} 5z^2 \phantom{+0z} -6 \\ \times\ \ 3z+11 \\ \hline 55z^2+0z-66 \\ 15z^3+0z^2-18z \phantom{-66} \\ \hline 15z^3+55z^2-18z-66 \end{array}$$

4. $2y + 3$ 5. $4x – 7$ 6. $z – 2$

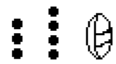
## Answers: Section 2.7

1. 612
2. 1 976
3. 1 776
4. CLXXXIX
5. DCCCCXLVIII or C$\overline{\text{M}}$XLVIII
6. MMCXX
7. MMMXXVIII
8. MMMMMDCCCLVII or M$\overline{\text{V}}$DCCCLVII
9. DCXXII
10. CCLX
11. $\overline{\text{V}}$MMDCCCLXXXI
12. MDCCXLII

## Answers: Section 2.8

1. 4 897
2. 16 228
3. 2 299 920
4. 
5. 
6. 
7. 
8. 
9. 

10. 

11. 

12. 

## Answers: Section 3.1

1. $83 \equiv 1 \pmod{2}$
2. $2^3 \equiv 3 \pmod{5}$
3. $26 \equiv 2 \pmod{12}$
4. $78 \equiv 1 \pmod{7}$
5. $95 \equiv 7 \pmod{11}$
6. $16 \equiv 0 \pmod{2}$
7. $15 \equiv 3 \pmod{6}$
8. $81 \equiv 0 \pmod{9}$
9. $30 \equiv 6 \pmod{8}$
10. $-5 \equiv 3 \pmod{8}$
11. $-3 \equiv 6 \pmod{9}$
12. $60 \equiv 0 \pmod{12}$
13. F
14. F
15. T
16. F
17. T
18. F

## Answers: Section 3.2

1. 1
2. 3
3. 15
4. 0
5. 4
6. 0
7.

| + | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| **0** | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
| **1** | 1 | 2 | 3 | 4 | 5 | 6 | 0 |
| **2** | 2 | 3 | 4 | 5 | 6 | 0 | 1 |
| **3** | 3 | 4 | 5 | 6 | 0 | 1 | 2 |
| **4** | 4 | 5 | 6 | 0 | 1 | 2 | 3 |
| **5** | 5 | 6 | 0 | 1 | 2 | 3 | 4 |
| **6** | 6 | 0 | 1 | 2 | 3 | 4 | 5 |

| × | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| **0** | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| **1** | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
| **2** | 0 | 2 | 4 | 6 | 1 | 3 | 5 |
| **3** | 0 | 3 | 6 | 2 | 5 | 1 | 4 |
| **4** | 0 | 4 | 1 | 5 | 2 | 6 | 3 |
| **5** | 0 | 5 | 3 | 1 | 6 | 4 | 2 |
| **6** | 0 | 6 | 5 | 4 | 3 | 2 | 1 |

8. 4 9. 2 10. 2 11. 4

12. $x = 2$

13. $z = 6$

14. {-17, -10, -3, 4, 11, 18}

15. {-20, -8, 4, 16}

16. $\{2, 2 \pm 11, 2 \pm 22, ...\}$

17. There is no solution.

18.. $\{4, 5, 4 \pm 9, 5 \pm 9, 4 \pm 18, 5 \pm 18, \ldots\}$

19. $\{2, 2 \pm 5, 2 \pm 10, \ldots\}$

## Answers: Section 3.3

1. We are given January 1, 2010 was a Friday. January 1, 2000 was 10 years earlier, and 3 of those years were leap years (2000, 2004, and 2008).
$(10 \times 365) + 3 \equiv (3 \times 1) + 3 \bmod 7$
$\equiv 6 \bmod 7$
6 days before a Friday is a Saturday. So January 1, 2000 was a Saturday.

2. a) We know January 1, 2000 was a Saturday. June 30, 1912 is 87 and a half years earlier. Between 1913 and 1999 the leap years were 1916, 1920, ..., 1996. By the Fence post trick this is 21 leap years. By adding up the number of days in each month, June 30, 1912 was (1+31+31+30+31+30+31) days before January 1, 1913.
$[(87 \times 365) + 21 + (1+31+31+30+31+30+31)]$
$\equiv [(3\times1)+0+(1+3+3+2+3+2+3)] \bmod 7$
$\equiv 6 \bmod 7$.
6 days before a Saturday is a Sunday. So June 30, 1912 was a Sunday.

b) $(90 \times 365) + 22$ (= # of leap years) $\equiv (6 \times 1) + 1 \bmod 7 \equiv 0 \bmod 7$.

3. a) September 1, 1905 was a Friday.
   b) September 1, 2055 will be a Wednesday.

4. The United States celebrated its 200th birthday on a Sunday.

5. The explosion occurred on a Saturday. $11 \equiv 4 \pmod 7$. The clouds stopped emerging on a Wednesday.

6. February 15, 1965 was a Monday.

7. a) January 1, 2001 was a Monday.
   b) Julian Day 2 455 555 will be a Friday.

8. Friday.

9. Answers will vary depending on the year. Consult this year's calendar.

## Answers: Section 3.4

1. 127 (86 – 29)

2. $123456789 \equiv 0 \pmod 9$

3. $7267 \equiv 1 \pmod 3$; $7267 \equiv 4 \pmod 9$

## Answers: Section 3.5

1. a) 167 b) 167 c) It's the same problem with different words

2. $s = 19$ 3. $s = -65$ 4. No solution

5. They could have started with 67 books.

## Answers: Section 4.1

1. Note that we only have to check for divisibility up to 13, since the root of 200 is less than 15, and 14 and 15 are covered by multiples of 2 and 3.

   The table should look like this:

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| **101** | ~~102~~ | **103** | ~~104~~ | ~~105~~ | ~~106~~ | **107** | ~~108~~ | **109** | ~~110~~ |
| ~~111~~ | ~~112~~ | **113** | ~~114~~ | ~~115~~ | ~~116~~ | ~~117~~ | ~~118~~ | ~~119~~ | ~~120~~ |
| ~~121~~ | ~~122~~ | ~~123~~ | ~~124~~ | ~~125~~ | ~~126~~ | **127** | ~~128~~ | ~~129~~ | ~~130~~ |
| **131** | ~~132~~ | ~~133~~ | ~~134~~ | ~~135~~ | ~~136~~ | **137** | ~~138~~ | **139** | ~~140~~ |
| ~~141~~ | ~~142~~ | ~~143~~ | ~~144~~ | ~~145~~ | ~~146~~ | ~~147~~ | ~~148~~ | **149** | ~~150~~ |
| **151** | ~~152~~ | ~~153~~ | ~~154~~ | ~~155~~ | ~~156~~ | **157** | ~~158~~ | ~~159~~ | ~~160~~ |
| ~~161~~ | ~~162~~ | **163** | ~~164~~ | ~~165~~ | ~~166~~ | **167** | ~~168~~ | ~~169~~ | ~~170~~ |
| ~~171~~ | ~~172~~ | **173** | ~~174~~ | ~~175~~ | ~~176~~ | ~~177~~ | ~~178~~ | **179** | ~~180~~ |
| **181** | ~~182~~ | ~~183~~ | ~~184~~ | ~~185~~ | ~~186~~ | ~~187~~ | ~~188~~ | ~~189~~ | ~~190~~ |
| **191** | ~~192~~ | **193** | ~~194~~ | ~~195~~ | ~~196~~ | **197** | ~~198~~ | **199** | ~~200~~ |

multiples of 2
remaining multiples of 3
remaining multiples of 5
remaining multiples of 7
remaining multiples of 11
remaining multiples of 13

2. Composite
3. Prime
4. Composite
5. Prime
6. Prime
7. Composite

8. a) 15 is not a prime number and thus is not needed.
   b) The square root of 713 is approximately 26, and 37 is larger than 26
   c) {2, 3, 5, 7, 11, 13, 17, 19, 23}
   d) 713 is not a prime as 23 divides into it evenly.
   e) 733 is the prime.

9. If $m$, $m+2$, $m+4$ are three consecutive odd integers, then at least one of them will be $\equiv 0 \bmod 3$.

## Answers: Section 4.2

1. $120 = 2^3 \times 3 \times 5$
2. $630 = 2 \times 3^2 \times 5 \times 7$
3. $1\ 800 = 2^3 \times 3^2 \times 5^2$
4. $168 = 2^3 \times 3 \times 7$
5. $88 = 2^3 \times 11$
6. $49 = 1 \times 49$

7. 0, 2, 4, 6, 8

8. 1, 4, 7

9. None

10. 0, 1, 2, 3, 4, 5, 6, 7, 8, 9

11. 0, 2, 4, 6, 8

12. 0, 3, 6, 9

13. 2, 5, 8

14. 2

15. None

16. 840 17. 120 18. 60 19. 360

| divisibility | a) 2 | b) 3 | c) 4 | d) 5 | e) 6 | f) 8 | g) 9 | h) 11 | b) 12 |
|---|---|---|---|---|---|---|---|---|---|
| 20. 3156 | yes | yes | yes | no | yes | No | no | no | yes |
| 21. 572 | yes | no | yes | no | no | No | no | yes | no |
| 22. 6240 | yes | yes | yes | yes | yes | yes | no | no | yes |
| 23. 123 789 | no | yes | no | no | no | No | no | no | no |
| 24. 50 193 | no | yes | no | no | no | No | yes | yes | no |
| 25. 4704 | yes | yes | yes | no | yes | yes | no | no | yes |
| 26. 1001 | no | no | no | no | no | No | no | yes | no |
| 27. 360 360 | yes | yes | yes | yes | yes | yes | yes | yes | yes |

28. a) The number must be divisible by both 3 **and** 5.
b) **400447770**. It must end in 0 to be divisible by 5. Every digit must appear three times to be divisible by 3 – make sure you understand why. The order is chosen to make the number as small as possible.

29. 21

| divisibility | a) 7 | c) 13 |
|---|---|---|
| 30. 3156 | no | No |
| 31. 572 | no | Yes |
| 32. 6240 | no | Yes |
| 33. 123 789 | no | No |
| 34. 50 193 | no | Yes |
| 35. 4704 | yes | No |
| 36. 1001 | yes | Yes |
| 37. 360 360 | yes | Yes |

38. 1 831 019 is divisible by 17, but not by 19, 23, 29 or 79.

39. Checking divisibility by 7:

```
  360 360
 -350 350
   10 010
  - 7 007
    3 003
  - 2 800
      203
    - 140
       63
    -  63
        0
```

Yes, divisible by 7.

Checking divisibility by 13:

```
  360 360
- 260 260
  100 100
  -91 091
    9 009
  - 7 800
    1 209
  - 1 170
       39
    -  39
        0
```

Yes, divisible by 13.

## Answers: Section 4.3

| | | | | | |
|---|---|---|---|---|---|
| 1. | 64 | 2. | 36 | 3. | 10 |
| 4. | 8 | 5. | 3 | 6. | 32 |
| 7. | 8 | 8. | 7 | 9. | 10 |
| 10. | 3 | 11. | 2 | 12. | 6 |

13. The $1^{st}$, $4^{th}$, $9^{th}$, $16^{th}$, $25^{th}$, $36^{th}$, $49^{th}$, $64^{th}$, $81^{st}$, and $100^{th}$ lockers will be locked.

## Answers: Section 4.4

| | | | | | |
|---|---|---|---|---|---|
| 1. | $2^2 5^1$ | 2. | $2^2$ | 3. | $3^3$ |
| 4. | $2^1$ | 5. | $2^2 3^1$ | 6. | $5^1 13^1$ |
| 7. | 12 | 8. | 5 | 9. | 10 |
| 10. | 2 | 11. | 12 | 12. | 35 |

13. $2^1 3^2$ 14. $2^1 3^1$ 15. $3^1$

16. 23 17. $x^b y^c z^c$ 18. 48 coins. 19. 12 inches.

20. a) Look at $n$ (mod $p$) and ($n$ +1) (mod $p$) for each prime $p$ that divides $n$. In other words, except for 1, any divisor of $n$ is not a divisor of ($n$ + 1)

b) By taking 101 out of 200 terms, two terms must have the $n$ and ($n$ + 1) relationship, as a minimum of two numbers must be consecutive.

21. 72 22. 84

23. 630 24. 72

25. 126 26. 126

27. 180 28. 108

29. 840 30. 138

| | **Numbers** | **gcd** | **Calculation** | **lcm** |
|---|---|---|---|---|
| 31. | 12 and 18 | 6 | (12)(18) = (6)(lcm) | 36 |
| 32. | 72 and 270 | 18 | (72)(270) = (18)(lcm) | 1 080 |

33. The 108th truck. 34. It will take 360 seconds

## Answers: Section 4.4 Appendix

1. 25 people.

2. There must be 325 travellers.

## Answers: Section 4.5

1. $\frac{18}{103}$ 2. $\frac{1}{9}$ 3. $\frac{64}{209}$

4. $\frac{31}{46}$ 5. $\frac{117}{235}$ 6. $\frac{13}{121}$

7. $^{3}/_{8}$

8. $^{2}/_{8} = ¼$

9. $^{4}/_{8} = {^{1}/_{2}}$

10. ½

11. 10

12. a) $^{4}/_{9}$ b) $^{43}/_{63}$ c) $^{7}/_{180}$

## Answers: Section 4.6

1. 0.625
2. 0.4375
3. $0.\overline{81}$
4. $0.8\overline{6}$
5. $0.\overline{809523}$
6. $0.\overline{846153}$
7. $^{3}/_{5}$
8. $^{7}/_{10}$
9. $^{17}/_{20}$
10. $^{23}/_{200}$
11. $^{417}/_{500}$
12. $^{3993}/_{5000}$
13. $^{1}/_{3}$
14. $^{41}/_{99}$
15. $^{26}/_{11}$
16. $^{37}/_{33}$
17. $^{8}/_{15}$
18. $^{1}/_{18}$
19. $^{679}/_{550}$
20. $^{548\ 642}/_{5\ 555}$
21. a) 0.166 666 ...
    b) 0.833 333 ...
    c) 1/6 + 5/6 = 6/6 = 1
    = 0.166 666 ... + 0.833 333 ...
    = 0.999 999 ...

22. a) 70 107 422 641 441 946 362 742 060.205 b) 567 895 678 956.789

23. $4.\overline{625}$ and $^{4621}/_{999}$

## Answers: Section 4.7

| | | | | | |
|---|---|---|---|---|---|
| 1. | rational | 2. | irrational | 3. | irrational |
| 4. | rational | 5. | rational | 6. | irrational |

7. 4 is not the cube of an integer

8. 8 is not the square of an integer

9. There is no repeating block of numbers, despite the existing pattern

10. $\sqrt{2}$ is an irrational number, and any multiplication or division involving a rational and an irrational number will always result in an irrational answer.

11. 56 cannot be reduced to a single number raised to any power, let alone raised to 4.

12. There is no repeating block in the number, despite the existing pattern

| | | | | | |
|---|---|---|---|---|---|
| 13. | $2\sqrt[10]{2}$ | 14. | $4\sqrt[12]{2}$ | 15. | 18 |
| 16. | $10\sqrt{10}$ | 17. | $\sqrt[9]{20}$ | 18. | $3\sqrt[6]{37500}$ |

19. False. 0.121121112… + 0.212212221… = 0.333333333…, or $-\sqrt{2}+\sqrt{2}=0$

20. True. Let $i$ be the irrational number and $^{p}/_{q}$ (where $p$ and $q$ are integers) be the rational. If you could write $i + {}^{p}/_{q}$ as a fraction, say $^{m}/n$ (where $m$ and $n$ are integers), then you would have $i + {}^{p}/_{q} = {}^{m}/_{n}$. The rules for subtracting fraction leads to $i = {}^{m}/_{n} - {}^{p}/_{q} = {}^{mq-np}/_{nq}$, which means that you would be able to write $i$ as a fraction contrary to the definition of <u>rational number</u>.

21. False. $\sqrt{2}$

22. True. Let $i$ be the irrational number and $^{p}/_{q}$ (where $p$ and $q$ are integers) be the rational. If you could write $i \times {}^{p}/_{q}$ as a fraction, say $^{m}/n$ (where $m$ and $n$ are integers), then you would have $^{ip}/_{q} = {}^{m}/_{n}$. Multiply both sides of the equation by $q$ and divide both sides by $p$ to get $i = {}^{mq}/_{np}$, which means that you would be able to write $i$ as a fraction, contrary to the definition of <u>rational number</u>.

23. Assuming $\sqrt{3}$ is rational and written in lowest terms leads to $3 = {}^{a^2}/_{b^2}$ and the right hand side is a fraction in lowest terms. So 3 would have to be the square of the integer *a*, which is impossible.

24. Assuming $\sqrt[3]{2}$ is rational and written in lowest terms leads to $2 = {}^{a^3}/_{b^3}$, with the right hand side in lowest terms. But then 2 has to be the cube of an integer, which it is not.

25. $\sqrt{16} = {}^4/_1$, which is rational. Note that if you follow the argument in the text with 16 in place of 2, you arrive at $16q^2 = p^2$, or $(4q)^2 = p^2$ so that $4q = p$. All you have to do is pick $p$ to be any multiple of 4.

26. $\sqrt{\sqrt{2}}$ is irrational. The square of a rational number would be a rational (because the square of a fraction is a fraction), but the square of $\sqrt{\sqrt{2}}$ is $\sqrt{2}$ which is ***not*** rational.

## Answers: Section 4.7 Appendix

1. (3,4,5), (5,12,13), (6,8,10), (8,15,17), (9,12,15), and (12,16,20).

2. If two of *n*, *m*, and *k* are divisible by a prime *p*, then it follows from $n^2 + m^2 = k^2$ that the prime *p* will divide the square of the third. Looking at the prime factorization shows that p must then divide the third.

3. $(a^2 - b^2)^2 + (2ab)^2 = a^4 - 2a^2b^2 + b^4 + 4a^2b^2 = a^4 + 2a^2b^2 + b^4 = (a^2 + b^2)^2$.

## Answers: Section 5.1

1. 10 to 6, 50 to 30, 2.5 to 1.5 (there are infinitely more possibilities).

| | | | | | |
|---|---|---|---|---|---|
| 2. | 4 : 3 | 3. | 18 : 55 | 4. | 1 : 5 |
| 5. | 1 : 5 | 6. | $z = 35$ | 7. | $x = 7$ |

8. $y = 27$   9. $k = 16$   10. $13^1/_3$ carrots (or 14).

11. \$20.81   12. $2^1/_2$ cups

13. The best buys are the box of 30 garbage bags, the 32-ounce ketchup bottle, and the 16-ounce bag of frozen peas.

14. 80 feet   15. 28 eggs

16. The second turtle wins the race as it travels at 1.920 km/day while the first turtle only travels at 1.728 km/day.

17. $x = 4$

18. Sally finishes first   19. 48 hours

## Answers: Section 5.2

1. 86%   2. 76.3%

3. 0.59%   4. 790%

5. 60%   6. 6%

7. $83^1/_3$%   8. 175%

9. 6   10. 24   11. 0.35

12. $14^2/_7$%   13. 6.72   14. 6.324

15. $30^{10}/_{13}$%   16. 314 140   17. 34%

18. Jennifer's total bill comes to \$45.40 + \$3.18 = \$48.58.

19. No, the resulting price is not \$500.00. The item is originally marked down to \$450.00, but 10% of that is \$45.00, which gives a total marked up price of \$495.00.

20. $^2/_5$   21. \$16 000.34

22. Approximately 21%

23. 50 kg

24. Final sale price is the same

25. You need 0.7 c. from the 12% cream and 0.3 c. from the 22% cream to make a cup of 15% cream coffee.

26. a) I scored better on the first math midterm.
    b) First math midterm : 16 ÷ 20 = 0.8 = 80 %
    Second math midterm: 23 ÷ 46 = 0.5 = 50 %

27. a) 30 litres of the $33^1/_3$% alcohol, and 20 litres of the $12^1/_2$% alcohol
    b) Harry studied for 30weeks, but he did not study for 20 weeks.
    c) Parts a) and b) are identical questions, only part a) uses fractions while part b) uses percents.

28. (a) 70%, (b) 81%, (c) 400%, (d) 60%, (e) 90%

29. Without knowing the percentage of female students, all one can say is that between twenty and thirty percent of the students have cheated on an exam at one time or another.

30. a) iv) b) i) c) v) d) vii)
    e) iii) f) vi) g) ii)

## Answers: Section 5.3

1. 15
2. 162
3. $^{64}/_{25}$
4. direct variation
5. inverse variation
6. 15 $cm^2$
7. 15 meters
8. 481 km
9. 8.4kg
10. 18.75 units
11. a) 10π cm b) 10/π cm

## Answers: Section 5.4

1. a) $\frac{3}{8}$ b) 1 c) $\frac{1}{8}$ d) $\frac{5}{8}$ e) $^{9}/_{64}$ f) $^{1}/_{16}$

2. d) and f) are the only correct answers.

3.

| Roll | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Probability | $^{1}/_{36}$ | $^{1}/_{18}$ | $^{1}/_{12}$ | $^{1}/_{9}$ | $^{5}/_{36}$ | $^{1}/_{6}$ | $^{5}/_{36}$ | $^{1}/_{9}$ | $^{1}/_{12}$ | $^{1}/_{18}$ | $^{1}/_{36}$ |

4. a) $^{1}/_{13}$ b) $^{1}/_{49}$

5. P( $E$ and $F$) = P($E|F$) × P($F$). Since $E$ and $F$ are independent events,
P($E|F$) = P($E$).

6. a) 0.177 b) 0.547 c) 0.788 d) 0.547 e) 0.373

7. a) $\frac{2}{3}$ b) Yes. The chance of winning is $\frac{1}{3}$ if they do not switch doors.

8. a) #1: 5 to 2; #2: 25 to 2; #3: 3 to 1; #4: 5 to 1; #5: 250 to 1.

   b) Horses #2 and #5 c) $100 out if Horse #3 wins, $0 if #1 wins.

   d) Odds given at the track are in the form $a+b$ to $b$ when the odds in favour are $a$ to $b$. So if a bet at the track pays out at 5 to 2, then the odds in favour are 2 : 3.

9. a) $^{1}/_{196}$ b) $^{9}/_{196}$ c) 1

## Answers: Section 5.4 Appendix

1.

| | | |
|---|---|---|
| $A^-$ | A ∩ B' ∩ Rh' | 7 % |
| $A^+$ | A ∩ B' ∩ Rh | 36 % |
| $B^-$ | B ∩ A' ∩ Rh' | 1.5 % |
| $B^+$ | B ∩ A' ∩ Rh | 7.5 % |
| $AB^-$ | A ∩ B ∩ Rh' | 0.5 % |
| $AB^+$ | A ∩ B ∩ Rh | 2.5 % |
| $O^-$ | A' ∩ B' ∩ Rh' | 6 % |
| $O^+$ | Rh ∩ A' ∩ B' | 39 % |

2. $O^-$ can be accepted by any recipient, thus, 6 % of Canadians are universal donors.

3. Those that can receive blood from a donor classified $A^+$ are $A \cap Rh = \{A^+, AB^+\}$ = 36 % + 2.5 % = 38.5 % of Canadians can safely receive $A^+$ blood.

4. The set of blood types that are acceptable to a person classified $A^+$ is $B' = \{A^-, A^+, O^-, O^+\}$ = 7 % + 36 % + 39 % + 6 % = 88 %.

# Index