# Seminar Notes: Algebraic Aspects of Association Schemes and Scheme Rings

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## 1 The Algebraic Definition of Coherent Configurations and Association Schemes

Let $X$ be a finite set of size $n$. If $r$ is a relation on $X$ (i.e. $r \subset X \times X$), then the *adjacency matrix* $\sigma_r$ is the $n \times n$ $(0,1)$-matrix whose $(i,j)$ entries are 1 if $(i,j) \in r$ and 0 otherwise. If $S$ is a set of relations on $X$, then we define the *complex adjacency algebra* of $S$ to be the subalgebra of $M_n(\mathbb{C})$ generated by the adjacency matrices of relations in $S$. Algebraic properties of and relationships between the relations in $S$ can be detected in the adjacency algebra. We will particularly interested in cases when the adjacency matrices define a basis for an associative algebra with nonnegative integer structure constants. Further properties arise when the adjacency matrices commute, or when the eigenvalues of the adjacency matrices lie in a fixed subring of $\mathbb{C}$. Observe that $S$ is a partition of $X \times X$ if and only if $\sum\limits_{r \in S} \sigma_r = J$, the all 1's matrix.

**Definition 1.1.** *Let $X$ be a set (not empty and not necessarily finite), and let $S$ be a set of relations on $X$. The pair $(X,S)$ is a coherent configuration (CC) if*

*(a) $S$ is a partition of $X \times X$;*

*(b) for all $s \in S$, $s^* = \{(y,x) \in X \times X : (x,y) \in s\} \in S$;*

*(c) there is a subset $\Delta \subset S$ such that $\bigcup_{\delta \in \Delta} \delta = 1_X$ (the equality relation on $X$); and*

*(d) for all $p,q,r \in S$, there is a cardinality $a_{pqr}$ such that for all $(x,z) \in r$, $|\{x \in X : (y,x) \in p, (x,z) \in q\}|$ has cardinality $a_{pqr}$.*

When $(X,S)$ is a CC, we will sometimes say that $S$ is a CC on $X$. By convention, when $S$ is a CC on $X$ we insist that $\emptyset \notin S$. The relations in the set $\Delta$ in the definition are called the *fibres* of $S$. When the set $X$ is a finite ordered set (usually taken by default to be $\{1,2,\dots,n\}$ for some positive integer $n$), any CC $S$ on $X$ is a finite set of relations $\{s_1,\dots,s_d\}$, and one can define adjacency matrices $\sigma_{s_i}$ for $i = 1,\dots,d$. The CC definition is equivalent to the set of adjacency matrices satisfying the following five properties:

(a) for each $s \in S$, $\sigma_s$ is a $(0,1)$-matrix;

(b) $\sum\limits_{s\in S} \sigma_s = J$ (the all 1's matrix);

(c) for each $s \in S$, $(\sigma_s)^T = \sigma_{s^*}$ for some $s^* \in S$;

(d) there is a subset $\Delta \subset S$ such that $\sum\limits_{\delta\in\Delta} \sigma_\delta = I$; and

(e) there is a set of non-negative integers $\{a_{pqr} : p, q, r \in S\}$ such that for all $s, t \in S$;

$$\sigma_p \sigma_q = \sum_{r\in S} a_{pqr} \sigma_r.$$

The non-negative integers appearing in the definition are the *intersection numbers* of the CC. When $X$ is a finite set of size $n$ then we say that $(X, S)$ has order $n$. When $(X, S)$ is a CC of finite order, its intersection numbers are the structure constants for the adjacency algebra $\mathbb{C}S$. The dimension of the adjacency algebra is precisely the size of $S$. Since the structure constants of this adjacency algebra are nonnegative integers, we can define the *integral adjacency ring* $\mathbb{Z}S$ in the obvious way, and this leads to adjacency algebras over any field and adjacency rings over any commutative ring with unity in the obvious fashion (i.e. the adjacency ring over the commutative ring with unity $\mathcal{R}$ is just $\mathcal{R} \otimes_{\mathbb{Z}} \mathbb{Z}S$.)

The following properties of CC's are immediate from the definition.

**Proposition 1.2.** *Let $(X, S)$ be a CC of order $n$, and let $\Delta = \{\delta_1, \dots, \delta_f\}$ be the set of fibres of $S$. Then*

*(a) the fibres provide a natural partition $\{X_1, \dots, X_f\}$ of $X$ for which each $\delta_i = \{(x, x) : x \in X_i\}$;*

*(b) for all $s \in S$, $s \subseteq X_i \times X_j$ for some $1 \le i, j \le f$;*

*(c) the structure constants of the CC have the property that for all $p, q, r, s \in S$,*

$$\sum_{t\in S} a_{qrt} a_{pts} = \sum_{t\in S} a_{pqt} a_{trs};$$

*(d) the adjacency algebra $\mathbb{C}S$ is a self-adjoint subalgebra of $M_n(\mathbb{C})$ that is closed under pointwise (Schur or Hadamard) multiplication and contains $J$ (i.e. $\mathbb{C}S$ is a* ***coherent*** *algebra).*

*(e) $\mathbb{C}S$ is a semisimple algebra.*

*Proof.* (a) Obvious.

(b) Let $s \in S$, and let $i, j \in \{1, \dots, f\}$. Then $\delta_i s \delta_j$ is an nonnegative integer linear combination of the $\{\sigma_r : r \in S\}$. If $\delta_i s \delta_j \neq 0$, then its only nonzero entries will be 1's that occur in the positions $(x, y)$ for which $(x, y) \in s \cap (X_i \times X_j)$. Since there is a unique relation in $S$ with a nonzero entry in any given position, this forces $\delta_i s \delta_j = s$, and so $s \subseteq (X_i \times X_j)$ for this $i$ and $j$.

(c) Since $\mathbb{C}S$ is an associative algebra, we have $(\sigma_p\sigma_q)\sigma_r = \sigma_p(\sigma_q\sigma_r)$, for all $p, q, r \in S$. Interpreting this in terms of structure constants gives the desired formula.

(d) This follows easily from the definition since the basis consisting of the adjacency matrices is closed under the transpose.

(e) It is well-known that self-adjoint $\mathbb{C}$-subalgebras of $M_n(\mathbb{C})$ are always semisimple.

□

There are several specific kinds of CC's and related algebras that we will deal with in these notes.

1. ***Association schemes*** are *homogeneous* CC's, those with $\Delta = \{I = s_0\}$. Since these are the central theme of these notes, we will often refer to them simply as "schemes" for short, and write $S = \{s_0, s_1, \ldots, s_d\}$ when the scheme has $d$ nonidentity relations (i.e. the scheme is of rank $d$).

2. *Commutative Association schemes* are schemes whose adjacency matrices commute.

3. *Symmetric association schemes* are schemes for which every $\sigma_s$ is a symmetric matrix. Such schemes are automatically commutative.

4. *Thin association schemes* are schemes in which every relation has exactly one 1 in every row and column. Note that the adjacency matrices of thin schemes form a finite group of order $n$, and conversely every finite group of order $n$ can be identified with a thin scheme by means of its left regular representation.

5. *Table Algebras* are a finite-dimensional algebras $A$ with a distinguished basis $\mathbf{B} = \{b_1 = 1, b_2, \ldots, b_r\}$ and an involution $*$ for which $\mathbf{B}^* = \mathbf{B}$, and the structure constants $\{\lambda_{ijk} : 0 \leq i, j, k \leq r\}$ determined by the basis $\mathbf{B}$ are nonnegative real numbers for which $\lambda_{ij1} > 0$ if and only if $b_j = (b_i)^*$.

**EXERCISES:**

**Exercise 1.1:** Prove that if the adjacency matrices of CC's are all symmetric, then the adjacency matrices commute.

**Exercise 1.2:** Suppose $(X, S)$ is a symmetric association scheme whose adjacency matrices form a group of order $n$. Characterize the isomorphism type of the group.

**Exercise 1.3:** Suppose $(X, S)$ is a non-symmetric commutative association scheme whose adjacency matrices form a group of order $n$. What can be said about the group?

**Exercise 1.4:** Prove that the only $(0, 1)$-matrices in the adjacency algebra of a CC $(X, S)$ are the adjacency matrices corresponding to the relations in $S$.

**Exercise 1.5:** The adjacency algebras of small coherent configurations can be conveniently presented in terms of a basic matrix. If $\{s_i : i = 0, 1, \ldots, d\}$ are the relations of $S$, then the basic matrix for $S$ is $\sum_{i=0}^{d} i\sigma_{s_i}$. For each of the following basic matrices, characterize the underlying configuration as being a CC, an association scheme, a commutative association scheme, a symmetric association scheme, a thin association scheme (i.e. a group), or a thin commutative association scheme (i.e. an abelian group).

$$(a)\begin{bmatrix}0&3&4\\5&1&6\\7&8&2\end{bmatrix}\quad (b)\begin{bmatrix}0&2&2\\3&1&4\\3&4&1\end{bmatrix}\quad (c)\begin{bmatrix}0&1&2\\2&0&1\\1&2&0\end{bmatrix}\quad (d)\begin{bmatrix}0&1&1&2&2\\1&0&2&1&2\\1&2&0&2&1\\2&1&2&0&1\\2&2&1&1&0\end{bmatrix}$$

$$(e)\begin{bmatrix}0&1&2&3&4&4&5&5\\1&0&3&2&4&4&5&5\\3&2&0&1&5&5&4&4\\2&3&1&0&5&5&4&4\\5&5&4&4&0&1&2&3\\5&5&4&4&1&0&3&2\\4&4&5&5&3&2&0&1\\4&4&5&5&2&3&1&0\end{bmatrix}\quad (f)\begin{bmatrix}0&1&2&2&3&3&4&4&5&5\\1&0&4&4&5&5&2&2&3&3\\2&5&0&3&2&3&4&5&1&4\\2&5&3&0&3&2&5&4&4&1\\3&4&2&3&0&2&5&1&5&4\\3&4&3&2&2&0&1&5&4&5\\5&2&5&4&4&1&0&3&2&3\\5&2&4&5&1&4&3&0&3&2\\4&3&1&5&4&5&2&3&0&2\\4&3&5&1&5&4&3&2&2&0\end{bmatrix}$$

**Exercise 1.6:** Calculate the intersection numbers for the CC's in part (b) and (d) of Exercise 1.5.

**Exercise 1.7:** Let $(X,S)$ be a scheme, and let $s \in S$. Show that the row and column sums of the adjacency matrix $\sigma_s$ are all equal to the same positive integer $n_s$. (This integer is called the *valency* of $s$.)

**Exercise 1.8:** Let $(X,S)$ be a scheme, and let $n_s$ be the valency of each $s \in S$. Show the following:

(a) for all $r, s \in S$, $a_{r1s} = a_{1rs} = \delta_{rs}$;

(b) for all $q, r, s \in S$, $a_{qrs} = a_{r^*q^*s^*}$;

(c) for all $s \in S$, $n_{s^*} = n_s$;

(d) if $1 \leq a_{pq^*1}$, then $p = q$;

(e) for all $q, r, s \in S$, $a_{qsr}n_r = a_{rs^*q}n_q$;

(f) for all $q, r \in S$,

$$\sum_{s\in S} a_{qsr} = \sum_{s\in S} a_{sqr} = \sum_{s\in S} a_{sq^*r} = n_q; \text{ and}$$

(g) for all $q, r \in S$,

$$\sum_{s\in S} a_{qrs}n_s = n_q n_r.$$

## 2 Connections between association schemes, graphs, permutation groups, and group rings.

Association schemes originated as a combinatorial objects in design theory that combines properties of graphs and permutation groups. The connection with graphs is quite clear. The elements in every CC of order $n$ can be viewed as an *algebraic* decomposition of the complete directed graph $\vec{K}_n$ (including loops at each vertex) with the property that for each subgraph in the decomposition, the subgraph formed by reversing its arrows is also one of subgraphs in the decomposition. Here when we say that it is an algebraic decomposition we mean that every edge appears exactly once among the subgraphs in the decompositon, and that the adjacency matrices of the subgraphs in the decomoposition form a basis for an algebra with nonnegative integer structure constants.

On the other hand, some symmetric association schemes arise naturally from distance-regular graphs. An (undirected and loopless) *graph* $\Gamma = (\Gamma_V, \Gamma_E)$ consists of a set of vertices $\Gamma_V$ and a set of edges $\Gamma_E \subseteq \Gamma_V \times \Gamma_V$ for which $(\gamma, \delta) \in \Gamma_E \implies (\delta, \gamma) \in \Gamma_E$ and $(\gamma, \gamma) \notin \Gamma_E$, for all $\gamma, \delta \in \Gamma_V$. If $\gamma \in \Gamma_V$, then $\Gamma_1(\gamma) = \{\delta \in \Gamma_V : (\gamma, \delta) \in \Gamma_E\}$ is the set of *neighbours* of $\gamma$. The vertices in $\Gamma_1(\gamma)$ are said to be at distance 1 from $\gamma$. The graph $\Gamma$ is *regular of valency* $k$ if $|\Gamma_1(\gamma)| = k$ for all $\gamma \in \Gamma_V$.

Inductively, we say that a vertex $\delta$ is at at distance $r$ from a vertex $\gamma$ if $r$ is the length of the shortest possible sequence of vertices

$$\gamma = \delta_0, \delta_1, \delta_2, ..., \delta_r = \delta,$$

for which $(\delta_{i-1}, \delta_i) \in \Gamma_E$ for all $i \in \{1, \dots, r\}$. We denote the set of vertices at distance $r$ from $\gamma$ by $\Gamma_r(\gamma)$. We also set $\Gamma_0(\gamma) = \{\gamma\}$. If $\Gamma$ is a connected graph with finitely many vertices, then it is easy to see that if we fix $\gamma \in \Gamma$, then there exists a positive integer $D$ such that $\Gamma$ will be the disjoint union of the $\Gamma_i(\gamma)$, for $i \in \{0, 1, 2, \dots, D\}$. This number is called the *diameter* of the graph $\Gamma$.

When $\Gamma$ is a regular graph of valency $k$ having $v$ vertices, we can define a set of $v \times v$ $(0,1)$-matrices $A_0, A_1, \dots, A_D$ by setting $(A_i)_{\gamma\delta} = 1$ if $\delta \in \Gamma_i(\gamma)$ and $(A_i)_{\gamma\delta} = 0$ otherwise. In this case we will have $A_0 = I$ (the $v \times v$ identity matrix), and $A_1 = A :=$ the adjacency matrix of $\Gamma$. The matrix $A_i$ is the *distance-i adjacency matrix* of the graph $\Gamma$. Since for every pair $(\gamma, \delta) \in \Gamma_E$, $\delta \in \Gamma_i(\gamma)$ for precisely one of the $i$'s, and so we will have that $A_0 + A_1 + \cdots + A_D = J$, the $v \times v$ identity matrix.

**Definition 2.1.** *A connected regular graph $\Gamma$ of valency $k$ and diameter $D$ is called **distance-regular** if for all $i = 0, 1, \dots, D$, there are fixed nonnegative integers $c_i$ and $b_i$ such that for every pair of vertices $\gamma, \delta$ for which $\delta \in \Gamma_i(\gamma)$,*

$$|\Gamma_1(\delta) \cap \Gamma_{i-1}(\gamma)| = c_i, \text{ and}$$

$$|\Gamma_1(\delta) \cap \Gamma_{i+1}(\gamma)| = b_i.$$

*We will always have $c_0 = b_D = 0$, $c_1 = 1$ and $b_0 = k$. Furthermore,*

$$|\Gamma_1(\delta) \cap \Gamma_i(\gamma)| = k - b_i - c_i := a_i$$

*is also constant for every pair of vertices $\gamma, \delta$ for which $\delta \in \Gamma_i(\gamma)$. The numbers $a_i, b_i, c_i$ are the intersection parameters of the distance-regular graph $\Gamma$.*

**Proposition 2.2.** *Let $\Gamma$ be a distance-regular graph of valency $k$, diameter $D$, having $v$ vertices, and let $a_0, \ldots, a_D, b_0, \ldots, b_D, c_0, \ldots, c_D$ be the intersection parameters of $\Gamma$. Then the distance adjacency matrices $A_0, A, A_2, \ldots, A_D$ are the adjacency matrices of a symmetric association scheme of rank $D+1$ whose intersection parameters are completely determined by*

$$AA_i = b_{i-1}A_{i-1} + a_iA_i + c_{i+1}A_{i+1},$$

*with the convention that $A_{-1} = A_{D+1} = 0$, and $b_{-1} = c_{D+1} = 0$.*

*Proof.* It is straightforward to use induction and the definition of the intersection parameters of a distance-regular graph to establish the formula. We can therefore conclude that $\{A_0, A_1, \ldots, A_D\}$ generates a $D+1$-dimensional algebra with nonnegative integer structure constants. The conclusion then follows from the fact that the $A_i$'s are all symmetric and $A_0 + A_1 + \cdots + A_D = J$. $\square$

If $A$ is the adjacency matrix of a distance-regular graph, then the left operator for the action of $A$ on the space spanned by the $A_i$ in the above proposition in the basis given by the $A_i$'s is the tri-diagonal matrix

$$[A] = \begin{bmatrix} 0 & k & 0 & & & 0 \\ 1 & a_1 & b_1 & 0 & & \\ 0 & c_2 & a_2 & b_2 & \ddots & \\ & \ddots & \ddots & \ddots & \ddots & 0 \\ & & & c_{D-1} & a_{D-1} & b_{D-1} \\ 0 & & & 0 & c_D & a_D \end{bmatrix}.$$

Symmetric association schemes of rank 3 arise from distance-regular graphs of diamter 2, which are known as *strongly regular* graphs. An interesting example of a strongly regular graph of valency 3 is the Petersen graph $\mathcal{P}$. It has vertex set $\mathcal{P}_V = \{1, 2, \ldots, 10\}$ and edge set

$$\begin{aligned} \mathcal{P}_E = \ & \{(1,2),(2,3),(3,4),(4,5),(5,1),(1,6),(6,8),(8,10), \\ & (10,7),(7,9),(9,6),(7,2),(3,8),(4,9),(5,10)\}. \end{aligned}$$

For the Petersen graph $\mathcal{P}$, we have $k = 3$, $D = 2$, $v = 10$, and the relevant equations are

$$AA_0 = A, A^2 = 3A_0 + A_2, \text{ and } AA_2 = 2A_1 + 2A_2.$$

So the left operator for $A$ is

$$[A] = \begin{bmatrix} 0 & 3 & 0 \\ 1 & 0 & 2 \\ 0 & 1 & 2 \end{bmatrix}$$

The reader can check that the only other nontrivial structure constants in the association scheme arise from the equation $A_2^2 = 6A_0 + 4A_1 + 3A_2$. Since $A_2 = A^2 - 3A_0 = A^2 - 3I$, the eigenvalues of all the adjacency matrices in this association scheme are completely determined by the spectrum of $A$. Indeed, this will be the case for all association schemes arising from distance-regular graphs in this way, because each of the matrices $A_i$ will be a polynomial of degree $i$ evaluated at the matrix $A$.

We now turn to connections between association schemes and permutation groups. Recall that for any permutation $\tau \in S_n$, the symmetric group on $n$ symbols, the permutation matrix $P_\tau$ denotes the image of $\tau$ under the permutation representation for the action of $S_n$ on $X = \{1, 2, \dots, n\}$. This means that if $e_1, \dots, e_n$ denotes the standard basis of column vectors for $\mathbb{C}^n$, then $\tau(i) = j \iff P_\tau(e_i) = e_j \iff (P_\tau)_{ji} = 1$, and all other entries of $P_\tau$ are 0. So the permutation matrices are precisely the $(0, 1)$-matrices with a single 1 in every row and column.

**Proposition 2.3.** *Let $(X, S)$ be an scheme of order $n > 1$, and let $n_s$ be the valency of each $s \in S$. Then each adjacency matrix $\sigma_s$ is the sum of exactly $n_s$ permutation matrices.*

*Proof.* Let $s \in S$. Then $\sigma_s$ has exactly $n_s$ 1's in every row and column, with all other entries being 0. This implies that $\sigma_s$ is of the form $n_s A$, where $A$ is a doubly stochastic matrix, a nonnegative matrix whose row and column sums are all equal to 1, and $A$ has exactly $n_s$ nonzero entries in every row and column. It suffices to show that for any doubly stochastic matrix $A$, there is a permutation matrix $P$ for which $A_{ij} = 0 \implies P_{ij} = 0$. For this, we use an argument for a theorem of Birkhoff due to Saunders and Schneider that is outlined in Horn and Johnson's book *Matrix Analysis*.

Induct on the number of non-zero entries of $A$. We may assume that there is a row of $A$ with more than one nonzero entry, for otherwise $A$ is a scalar multiple of a permutation matrix. Choose a row of $A$ with a maximal number of nonzero entries, and pick one of the nonzero entries $A_{i_0,j_1}$ in this row. Since $A$ is doubly stochastic, there will be another nonzero entry $A_{i_1,j_1}$ in the $j_1$-th column with $i_1 \neq i_0$. Similarly, there will be another nonzero entry $A_{i_1,j_2}$ in the $i_1$-th row with $j_2 \neq j_1$, and another nonzero entry $A_{i_2,j_2}$ with $i_2 \neq i_1$. It may be the case that the only way to choose $A_{i_1,j_2}$ and $A_{i_2,j_2}$ will result in $i_2 = i_0$. It may also be the case that it is not possible to choose the next nonzero entry $A_{i_2,j_3}$ with $j_3 \notin \{j_1, j_2\}$. If either of these happen, then we set our sequence of nonzero entries (which we need for the next step in the algorithm) to be $(A_{1,j_1}, A_{i_1,j_1}, A_{i_1,j_2}, A_{i_2,j_2})$. Otherwise, $i_2 \notin \{i_0, i_1\}$, and we continue adding pairs of nonzero entries $A_{i_{c-1},j_c}, A_{i_c,j_c}$ to our sequence until we are forced to choose either $i_c \in \{i_0, i_1, \dots, i_{c-1}\}$ or $j_{c+1} \in \{j_1, \dots, j_c\}$. Note that if the latter happens, the nonzero entry $A_{i_c,j_{c+1}}$ is not added to our sequence. This means that the process will always terminate when our sequence has even length and is of the form

$$(A_{i_0,j_1}, A_{i_1,j_1}, A_{i_1,j_2}, A_{i_2,j_2}, \dots, A_{i_{c-1},j_c}, A_{i_c,j_c}).$$

Let $\mu$ be the minimum of the $A_{ij}$'s appearing in this sequence. Let $L$ be the $n \times n$ matrix with the entry $+\mu$ in positions $(1, j_1)$, $(i_1, j_2), \dots, (i_{c-1}, j_c)$, $-\mu$ in the postions $(i_1, j_1)$, $(i_2, j_2), \dots, (i_c, j_c)$, and 0's elsewhere. Then the row and column sums of $L$ are all 0. Therefore, $A - L$ is a doubly stochastic matrix, satisfies $A_{ij} = 0 \implies (A - L)_{ij} = 0$, and has fewer nonzero entries than $A$. By induction, there exists a permutation matrix $P$ for which $(A - L)_{ij} = 0 \implies P_{ij} = 0$. The result follows. □

Using this, we observe that the adjacency algebra of any scheme is contained in the image of the standard permutation representation $\mathcal{P} : \mathbb{C}S_n \to M_n(\mathbb{C})$. This representation of $\mathbb{C}S_n$ is faithful on $S_n$, and has two irreducible constituents, the trivial representation and the reflection representation of degree $n - 1$. It follows that for any scheme $(X, S)$ of order $n$, $\mathbb{C}S$ is a subalgebra of $\mathcal{P}(\mathbb{C}S_n)$ that is generated by the images of certain sums of distinct elements of $S_n$. It also follows that the integral adjacency ring $\mathbb{Z}S$ a subring of a homomorphic image of an integral group ring $\mathbb{Z}G$ generated over $\mathbb{Z}$ by the images of certain sums of elements of the finite group $G$.

One case where such subrings of $\mathbb{Z}G$ can be constructed is in the case of a *Schur ring*.

**Definition 2.4.** *Let $G$ be a finite group of order $n$. Let $\mathcal{F}$ be a partition of the set $G$ with the property that for all $s = \{g_1, \ldots, g_k\} \in \mathcal{F}$, $s^* = \{{g_1}^{-1}, \ldots, {g_k}^{-1}\} \in \mathcal{F}$. For each $s \in \mathcal{F}$, let $\sigma_s = \sum_{g \in s} g$, and let $\mathbb{Z}\mathcal{F}$ be the free abelian subgroup of $\mathbb{Z}G$ generated by $\{\sigma_s : s \in \mathcal{F}\}$. We say that $\mathbb{Z}\mathcal{F}$ is a* ***generic Schur ring*** *if $\mathbb{Z}\mathcal{F}$ is a subring of $\mathbb{Z}G$ that is free over $\mathbb{Z}$ with basis $\mathcal{F}$. In other words, for all $q, r \in \mathcal{F}$, there are (automatically nonnegative) integers $a_{qrs}$, $s \in \mathcal{F}$, such that $\sigma_q \sigma_r = \sum\limits_{s \in \mathcal{F}} a_{qrs} \sigma_s$. $\mathbb{Z}\mathcal{F}$ is a* ***unital Schur ring*** *if it is a Schur ring with the property that $\{1\} \in \mathcal{F}$.*

If $\mathbb{Z}\mathcal{F}$ is the unital Schur ring defined by a partition $\mathcal{F}$ of a group of order $n$, then the left regular representation of $G$ will map each $\sigma_s$ for $s \in \mathcal{F}$ to a $(0, 1)$-matrix, and it is easy to verify that the Schur ring $\mathbb{Z}\mathcal{F}$ is the adjacency ring for an scheme whose adjacency matrices are the $\sigma_s$'s for $s \in \mathcal{F}$.

Actions of groups on finite sets always give rise to unital Schur rings. If $G$ is a group of order $n$ and a group $H$ acts on $G$ by automorphisms, let $\mathcal{F}$ be the partition of $G$ into distinct $H$-orbits. Choose representatives $g_0 = e, g_1, \ldots, g_d$ for the distinct $H$-orbits, and let $\sigma_i = \sum\limits_{g \in {g_i}^H} g$. Note that if any $\alpha = \sum\limits_{g \in G} \alpha_g g \in \mathbb{Z}G$ is fixed by all elements of $H$, then the coefficients of $g_i^h$ must be equal to the coefficient of $g_i$, for each $i \in \{0, \ldots, d\}$, so $\alpha$ lies in $\mathbb{Z}\mathcal{F}$. Conversely, each $\sigma_i$ is fixed by all elements of $H$, and so we can conclude that $\mathbb{Z}\mathcal{F}$ is equal to the fixed-point subalgebra $(\mathbb{Z}G)^H$. In particular, $\mathbb{Z}\mathcal{F}$ is a Schur ring.

A common occurrence of a Schur ring occurs in the case of a *cyclotomic* association scheme. This is an association scheme whose adjacency matrices are *circulant matrices*, which are linear combinations of the powers of a fixed permutation matrix $P_\tau$. For example, consider the following list of basic matrices of schemes of order 6 excluding those of the two groups of order 6. The basic matrices are as follows:

$$(S_a) \begin{bmatrix} 0&1&1&1&1&1\\ 1&0&1&1&1&1\\ 1&1&0&1&1&1\\ 1&1&1&0&1&1\\ 1&1&1&1&0&1\\ 1&1&1&1&1&0 \end{bmatrix}, \quad (S_b) \begin{bmatrix} 0&1&2&2&2&2\\ 1&0&2&2&2&2\\ 2&2&0&1&2&2\\ 2&2&1&0&2&2\\ 2&2&2&2&0&1\\ 2&2&2&2&1&0 \end{bmatrix}, \quad (S_c) \begin{bmatrix} 0&1&1&2&2&2\\ 1&0&1&2&2&2\\ 1&1&0&2&2&2\\ 2&2&2&0&1&1\\ 2&2&2&1&0&1\\ 2&2&2&1&1&0 \end{bmatrix},$$

$$(S_d) \begin{bmatrix} 0&1&2&3&3&3\\ 2&0&1&3&3&3\\ 1&2&0&3&3&3\\ 3&3&3&0&1&2\\ 3&3&3&2&1&0\\ 3&3&3&1&2&0 \end{bmatrix}, \quad (S_e) \begin{bmatrix} 0&1&2&2&3&3\\ 1&0&3&3&2&2\\ 2&3&0&2&1&3\\ 2&3&2&0&3&1\\ 3&2&1&3&0&2\\ 3&2&3&1&2&0 \end{bmatrix}, \quad (S_f) \begin{bmatrix} 0&1&2&2&3&3\\ 1&0&2&2&3&3\\ 3&3&0&1&2&2\\ 3&3&1&0&2&2\\ 2&2&3&3&0&1\\ 2&2&3&3&1&0 \end{bmatrix}.$$

Here $S_a$ is the trivial single-class association scheme of order 6. To view $S_a$ as a Schur ring, we can take $\tau = (1, 2, 3, 4, 5, 6)$ and use the partition $\{\{1\}, \{\tau, \tau^2, \tau^3, \tau^4, \tau^5\}\}$. The others can also be viewed as Schur rings. For $S_b$ we can take $\tau$ to be a permutation for which $\tau^3 = (1, 2)(3, 4)(5, 6)$, so taking $\tau = (1, 3, 5, 2, 4, 6)$ suffices, and the partition can be taken to be $\{\{1\}, \{\tau^3\}, \{\tau, \tau^2, \tau^4, \tau^5\}\}$.. For $S_c$, we can take $\tau = (1, 4, 2, 5, 3, 6)$, and use the partition $\{\{1\}, \{\tau^2, \tau^4\}, \{\tau, \tau^3, \tau^5\}\}$. For $S_d$, we can again take $\tau = (1, 4, 2, 5, 3, 6)$ and refine the previous partition to $\{\{1\}, \{\tau^2\}, \{\tau^4\}, \{\tau, \tau^3, \tau^5\}\}$. For $S_e$, take $\tau = (1, 5, 3, 2, 4, 6)$ and use the partition $\{\{1\}, \{\tau^3\}, \{\tau, \tau^5\}, \{\tau^2, \tau^4\}\}$. For $S_f$, take $\tau =$

$(1,3,5,2,4,6)$ and use the partition $\{\{1\},\{\tau^3\},\{\tau,\tau^4\},\{\tau^2,\tau^5\}\}$. So all of the non-thin association schemes of order 6 are Schur rings consisting entirely of circulant matrices, and thus these are all cyclotomic association schemes.

Often a Schur ring can be realized as a subring of $\mathbb{Z}G$ for more than one group $G$. Indeed, $S_a$, $S_c$, and $S_d$ above can also be viewed as Schur subrings of $\mathbb{Z}S_3$.

The above example also provides insight into another phenomenon that occurs for CC's and association schemes that is known as fusion. After suitable changes of bases, each of the above Schur rings can be viewed as a subring of the same group ring $\mathbb{Z}\langle\tau\rangle$, which is itself the adjacency algebra of a thin association scheme. When $T$ and $S$ are two association schemes or CC's of order $n$ and $\mathbb{Z}T$ is a unital subring of $\mathbb{Z}S$, then we say that $T$ is a ***fusion*** of $S$, and $S$ is a ***fission*** of $T$. For association schemes of order 6, the lattice of Schur subrings (fusion subrings) of $\mathbb{Z}\langle\tau\rangle$ is

$$\begin{array}{ccccccc} & & \mathbb{Z}[S_b] & - & \mathbb{Z}[S_f] & & \\ & / & & \backslash & & \backslash & \\ \mathbb{Z}[S_a] & & & & \mathbb{Z}[S_e] & - & \mathbb{Z}\langle\tau\rangle. \\ & \backslash & & / & & / & \\ & & \mathbb{Z}[S_c] & - & \mathbb{Z}[S_d] & & \end{array}$$

Another example of a Schur ring is a double coset algebra. Let $H$ be a proper subgroup of a finite group $G$ having index $n$. Let $X = G/H := \{H = x_1H, x_2H, \dots, x_nH\}$ be the set of left cosets of $H$ in $G$. Then $G$ acts on the set of pairs of left cosets $X \times X$ via $g(x_iH, x_jH) = (gx_iH, gx_jH)$. The orbits of this action of $G$ on $X \times X$ are called the 2-*orbits* for the action of $G$ on $X$. Each 2-orbit $G(xH, yH)$ can be represented in the form $G(H, gH)$ for some $g \in G$, so for later use we let $g^H := G(H, gH)$ denote this 2-orbit. It is easy to see that the collection of 2-orbits is in 1-to-1-correspondence with the collection of double cosets of $H$ in $G$ via the map $g^H \mapsto HgH$, for all $g \in G$. We claim that the collection of 2-orbits for the action of $G$ on $X$ is association scheme on $X \times X$. To see this, observe that the 2-orbits do indeed partition $X \times X$, that the 2-orbit $1^H$ is the identity relation on $X$ (since left multiplication by $G$ is a transitive action on the set of left cosets of $H$ in $G$), and that the transpose of a 2-orbit is a 2-orbit since

$$\begin{array}{rcl} (xH, yH) \in g^H & \iff & Hx^{-1}yH = HgH \\ & \iff & H(x^{-1}y)^{-1}H = Hg^{-1}H \\ & \iff & (yH, xH) \in (g^{-1})^H. \end{array}$$

It remains to show the intersection numbers of this scheme are well-defined. Given 2-orbits $p^H$, $q^H$, and $r^H$, the definition of the corresponding intersection number is

$$a_{p^H q^H r^H} = |\{(xH, zH) \in r^H : \exists yH \in X, (xH, yH) \in p^H \text{ and } (yH, zH) \in q^H\}|.$$

If $(xH, zH) \in r^H$ and there exists a $yH$ satisfying $(xH, yH) \in p^H$ and $(yH, zH) \in q^H$, then we have that $yH \subseteq xHpH$, and so $zH \subseteq yHqH \subseteq xHpHqH \cap xHrH$. Conversely, for any $zH \subseteq xHpHqH \cap xHrH$, we have that $(xH, zH) \in r^H$, and since $HpHqH = (HpH)(HqH)$, there exists a $y \in xHpH$ such that $(xH, yH) \in p^H$ and $(yH, zH) \in HqH$. So for a given $xH$, the left cosets $zH$ for which $(xH, zH)$ belongs to the set in question are those contained in $xHpHqH \cap xHrH$. The number of these left cosets is $\frac{1}{|H|}|xHpHqH \cap xHrH| = \frac{1}{|H|}|HpHqH \cap HrH|$, so it only depends on the choice of the double cosets $HpH$, $HqH$, and $HrH$, as required.

The scheme of 2-orbits for the action of a group $G$ on the set of left cosets of a proper subgroup $H$ of $G$ is known as a *Schurian*, or *group case* association scheme. It will be denoted by $(G/H, G/\!\!/H)$ or by 2-orb$(G, G/H)$. Its scheme ring over $\mathbb{Q}$ is canonically isomorphic to the *double coset algebra* $e_H\mathbb{Q}Ge_H$, where $e_H = \frac{1}{|H|}\sum\limits_{h\in H} h := \frac{1}{|H|}(H^+)$ is the idempotent corresponding to the trivial character of the group $H$. Note that this double coset algebra is an example of a Schur ring, but it will not be a unital Schur ring unless $|H| = 1$. If $[G : H] = n$, the isomorphism from $\mathbb{Q}[G/\!\!/H] \to e_H\mathbb{Q}Ge_H$ is given by

$$\sigma_{g^H} \mapsto |H|(e_H g e_H).$$

Verification of this point is easily reduced to checking that the intersection numbers of the scheme agree with the structure constants of the double coset algebra in this basis. For a given $p, q, r \in G$, we have

$$\begin{array}{rcl} (e_H p e_H)(e_H q e_H) & = & e_H p e_H q e_H \\ & = & \frac{1}{|H|^3} \sum\limits_{h_1,h_2,h_3\in H} h_1 p h_2 q h_3 \\ & = & \frac{1}{|H|^3} \sum\limits_{HrH\in H\backslash G/H} \alpha_{HpH,HqH,HrH}\, e_H r e_H, \end{array}$$

where $\alpha_{HpH,HqH,HrH} = |HpHqH \cap HrH| = |H| a_{p^H,q^H,r^H}$. So it follows that scaling each of the $(e_H g e_H)$'s in the natural basis of the double coset algebra by $|H|$ will produce a basis with the same structure constants as $\mathbb{Q}[G/\!\!/H]$.

The reader should be aware that the basis elements of a Schur ring do not always correspond to a Schurian scheme. Over rings of characteristic zero, these correspond to fusion subschemes of Schurian schemes, which are not always Schurian.

The Schurian property is defined more generally for CC's using the 2-orbit concept. The collection of 2-orbits for the action of a finite group $G$ on a finite set $X$ is a CC on the set $X$, denoted 2-orb$(G, X)$. The fibres of 2-orb$(G, X)$ correspond to the orbits of $G$ on $X$. So $2\text{-}orb(G, X)$ will be an association scheme if and only if the action of $G$ on $X$ is transitive (i.e. there is only one orbit). It will be a half-homogeneous CC if and only if the action of $G$ on $X$ is half-transitive (i.e. every every orbit of $G$ on $X$ has the same size), and a symmetric association scheme if and only if the action of $G$ on $X$ is generously transitive (i.e. for every $(x, y) \in X \times X$, there exists a $g \in G$ such that $g(x, y) = (y, x)$).

On the other hand, given a CC $S$ defined on a set $X$ of size $n$, we can define its *combinatorial automorphism group* $Aut(S)$ to be the subgroup the group of all permutations $\phi$ of $X$ for which $(x, y) \in s \implies (\phi x, \phi y) \in s$, for all $s \in S$.

The functors $Aut$ and $2\text{-}orb$ can be composed. It is fairly easy to see that if $G$ acts on $X$, then $G$ will be a subgroup of $Aut(2\text{-}orb(G, X))$. We say that $G$ is a 2-*closed* permutation group on $X$ when $G = Aut(2\text{-}orb(G, X))$. $S$ is always a fusion CC of $2\text{-}orb(Aut(S), X)$, because if $s \in S$, the 2-orbit of $(x, y)$ under the action of $Aut(S)$ will be contained in $s$, but it may be the case that $s$ contains more than one 2-orbit of $Aut(S)$). We say that $S$ is a *Schurian* CC when $S = 2\text{-}orb(Aut(S), X)$.

In order for $(X, S)$ to be a Schurian CC, it is necessary and sufficient that every scheme relation $s \in S$ be a single 2-orbit of $Aut(S)$. So whenever $s \in S$ and $(x, y), (x, z) \in s$, then there must be a $\phi \in Aut(S)$ that fixes $x$ and maps $y$ to $z$. In other words, $(X, S)$ is a Schurian CC if and only if for all $x \in X$, the stabilizer of $x$ in $Aut(S)$ always acts transitively on $\{y \in X : (x, y) \in s\}$. A Schurian scheme is a Schurian CC with the additional property that $Aut(S)$ acts transitively on $X$. This characterization of Schurian schemes follows from Theorem 6.3.1 in Zieschang's book *Theory of Association Schemes.*

**Theorem 2.5.** *Let $(X, S)$ be an association scheme, and let $G = Aut(S)$. Let $x \in X$, and let $H$ be the stabilizer of $x$ in $G$. Then the following are equivalent:*

*(a) $(X, S)$ is a Schurian scheme;*

*(b) $G$ acts transitively on $X$ and, for all $s \in S$, $H$ acts transitively on $\{y : (x, y) \in s\}$;*

*(c) there is a bijection $\varphi : X \to G/H$ which induces a combinatorial isomorphism between $(X, S)$ and the group-case association scheme $(G/H, G/\!\!/H)$.*

In practical terms, this characterization makes it possible to always check whether or not a given scheme is Schurian. Starting with $S$, one computes the automorphism group of $S$ (which these days can be done using the `nauty` software implemented by the `grape` package available in GAP), then checks that $Aut(S)$ acts transitively on $X$, and if it does then one checks that for all $x \in X$, for all $s \in S$, $Stab_{Aut(S)}(x)$ acts transitively on $\{y : (x, y) \in s\}$.

Although many small schemes turn out to be Schurian, once the order is beyond 30 there are usually more non-Schurian schemes than Schurian ones. The smallest non-Schurian scheme is an association scheme $(X, S)$ of order 15:

$$\begin{bmatrix}
0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
2 & 0 & 1 & 1 & 1 & 2 & 2 & 2 & 1 & 1 & 1 & 1 & 2 & 2 & 2 \\
2 & 2 & 0 & 1 & 1 & 1 & 2 & 2 & 1 & 2 & 2 & 2 & 1 & 1 & 1 \\
2 & 2 & 2 & 0 & 1 & 2 & 1 & 1 & 2 & 1 & 1 & 2 & 1 & 1 & 2 \\
2 & 2 & 2 & 2 & 0 & 1 & 1 & 1 & 1 & 1 & 2 & 1 & 2 & 2 & 1 \\
2 & 1 & 2 & 1 & 2 & 0 & 1 & 2 & 2 & 2 & 1 & 1 & 1 & 2 & 1 \\
2 & 1 & 1 & 2 & 2 & 2 & 0 & 1 & 1 & 2 & 1 & 2 & 2 & 1 & 1 \\
2 & 1 & 1 & 2 & 2 & 1 & 2 & 0 & 2 & 1 & 2 & 1 & 1 & 1 & 2 \\
1 & 2 & 2 & 1 & 2 & 1 & 2 & 1 & 0 & 2 & 1 & 1 & 2 & 1 & 2 \\
1 & 2 & 1 & 2 & 2 & 1 & 1 & 2 & 1 & 0 & 1 & 2 & 1 & 2 & 2 \\
1 & 2 & 1 & 2 & 1 & 2 & 2 & 1 & 2 & 2 & 0 & 1 & 1 & 2 & 1 \\
1 & 2 & 1 & 1 & 2 & 2 & 1 & 2 & 2 & 1 & 2 & 0 & 2 & 1 & 1 \\
1 & 1 & 2 & 2 & 1 & 2 & 1 & 2 & 1 & 2 & 2 & 1 & 0 & 1 & 2 \\
1 & 1 & 2 & 2 & 1 & 1 & 2 & 2 & 2 & 1 & 1 & 2 & 2 & 0 & 1 \\
1 & 1 & 2 & 1 & 2 & 2 & 2 & 1 & 1 & 1 & 2 & 2 & 1 & 2 & 0
\end{bmatrix}$$

In this case, $Aut(S)$ is a group of $S_{15}$ order 21 that has three orbits on $X = \{1, \ldots, 15\}$ (of sizes 7, 7, and 1), so the scheme is not Schurian.

There is a unique non-Schurian scheme of order 16 and rank 3 appearing in the exercises that has a transitive automorphism group, which is a fusion scheme of a Schurian scheme of rank 4.

## EXERCISES:

**Exercise 2.1**: Symmetric association schemes of rank 3 arise from distance-regular graphs of valency 2, which are known as *strongly regular* graphs. If $\Gamma$ is a strongly regular graph on $n$ vertices with valency $k$, show that the characteristic polynomial of $A$ is of the form $x^2 - (\lambda - \mu)x - (k - \mu)$, where $\lambda$ and $\mu$ are the constants given in the equation $A^2 = kI_n + \lambda A + \mu A_2$. (For this reason, an important invariant of a strongly regular graph is its type $(n, k, \lambda, \mu)$.

**Exercise 2.2**: Let $G$ be a cyclic group of order 7. Determine all of the Schur subrings of $\mathbb{C}G$.

**Exercise 2.3**: Let $G = S_3$. Find all of the Schur subrings of $S_3$.

**Exercise 2.4**: Let $G = S_3$, and let $H = \langle(1,2)\rangle$. Find the basis matrix presentation for the Schurian scheme defined by the ordinary Hecke algebra corresponding to $H$.

**Exercise 2.5**: Let $G = A_4$, and let $H = \langle(1,2,3)\rangle$. Calculate the basis matrix for the Schurian scheme defined by the double cosets of $H$ in $G$.

**Exercise 2.6**: Suppose $H$ is a normal subgroup of a group $G$ having index $n$. Show that the ordinary Hecke algebra $e_H\mathbb{C}Ge_H$ is isomorphic to $\mathbb{C}[G/H]$.

**Exercise 2.7**: One simple construction of symmetric association schemes from schemes is called *stratification*. Given an scheme $(X, S)$ of finite order, the stratification of $S$ is a new set of relations $\tilde{S}$ on $X$ given by $\tilde{S} = \{s \cup s^* : s \in S\}$.

(a) Show that if $S$ is a commutative association scheme on $X$, then its stratification $\tilde{S}$ will also be an association scheme on $X$.

(b) Show that the group $S_3$ defines a non-commutative scheme whose stratification is not an association scheme.

**Exercise 2.8**: Let $S$ be a CC. Show that $Aut(S)$ is the precisely the group of all permutations whose permutation matrices commute with every adjacency matrix in $S$.

**Exercise 2.9**: Show that the automorphism group of the scheme of order 16 and rank 3 whose basic matrix is shown is transitive, but the scheme is still non-Schurian.

$$\begin{bmatrix}
0 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
1 & 0 & 1 & 1 & 2 & 2 & 2 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 2 \\
1 & 1 & 0 & 2 & 1 & 2 & 2 & 1 & 2 & 2 & 1 & 1 & 2 & 2 & 2 & 2 \\
1 & 1 & 2 & 0 & 2 & 1 & 2 & 2 & 1 & 2 & 2 & 2 & 1 & 1 & 2 & 2 \\
1 & 2 & 1 & 2 & 0 & 2 & 1 & 2 & 2 & 2 & 1 & 2 & 1 & 2 & 1 & 2 \\
1 & 2 & 2 & 1 & 2 & 0 & 1 & 2 & 2 & 2 & 2 & 1 & 2 & 1 & 2 & 1 \\
1 & 2 & 2 & 2 & 1 & 1 & 0 & 2 & 2 & 1 & 2 & 2 & 2 & 2 & 1 & 1 \\
2 & 1 & 1 & 2 & 2 & 2 & 2 & 0 & 2 & 1 & 2 & 1 & 2 & 1 & 1 & 2 \\
2 & 1 & 2 & 1 & 2 & 2 & 2 & 2 & 0 & 1 & 1 & 2 & 1 & 2 & 2 & 1 \\
2 & 1 & 2 & 2 & 2 & 2 & 1 & 1 & 1 & 0 & 2 & 2 & 2 & 2 & 1 & 1 \\
2 & 2 & 1 & 2 & 1 & 2 & 2 & 2 & 1 & 2 & 0 & 1 & 1 & 2 & 2 & 1 \\
2 & 2 & 1 & 2 & 2 & 1 & 2 & 1 & 2 & 2 & 1 & 0 & 2 & 1 & 2 & 1 \\
2 & 2 & 2 & 1 & 1 & 2 & 2 & 2 & 1 & 2 & 1 & 2 & 0 & 1 & 1 & 2 \\
2 & 2 & 2 & 1 & 2 & 1 & 2 & 1 & 2 & 2 & 2 & 1 & 1 & 0 & 1 & 2 \\
2 & 2 & 2 & 2 & 1 & 2 & 1 & 1 & 2 & 1 & 2 & 2 & 1 & 1 & 0 & 2 \\
2 & 2 & 2 & 2 & 2 & 1 & 1 & 2 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 0
\end{bmatrix}$$

(Hint: Use GAP. Show that the automorphism group is transitive with order 192, and the stabilizer of 1 in $G$ has 4 double cosets, not 3, and hence the scheme is not Schurian.)

# 3 Characters of schemes

Since the adjacency algebras of coherent configurations are self-adjoint subalgebras of $M_n(\mathbb{C})$, it follows that they are semisimple. Therefore, there is a Wedderburn decomposition

$$\mathbb{C}S = \bigoplus_{\chi \in Irr(S)} \mathbb{C}Se_\chi$$

where the simple components $\mathbb{C}Se_\chi$ are the principal ideals generated by a primitive idempotent of the center of $\mathbb{C}S$. The number of simple components corresponds to the number of inequivalent irreducible matrix representations of $\mathbb{C}S$, which are distinguished by their irreducible characters $\chi$. Each simple component $\mathbb{C}Se_\chi$ is a simple $\mathbb{C}$-algebra, and hence of the form $M_{n_\chi}(\mathbb{C})$, where $n_\chi = \chi(I)$ is the degree of the irreducible character $\chi$ of $S$. Since $\mathbb{C}S$ has dimension equal to the rank $r$ of the CC, we automatically have

$$r = \sum_{\chi \in Irr(S)} n_\chi^2.$$

The left regular representation of the elements of the CC as left operators on the $r$-dimensional space $\mathbb{C}S$ affords the *regular* character $\rho$ of $S$, whose irreducible decomposition is

$$\rho = \sum_{\chi \in Irr(S)} n_\chi \chi.$$

The most natural matrix representation of an CC is its *standard* representation, which is the $n$-dimensional representation of the elements of the scheme in terms of their adjacency matrices. (The standard representation and the regular representation correspond only when $S$ is a group.) This affords the *standard* character of $S$, which we denote by $\gamma$. From the basic matrix for $S$, we see that for all $s \in S$,

$$\gamma(\sigma_s) = \begin{cases} |s|, & \text{if } s \text{ is a fibre of } S, \\ 0, & \text{otherwise.} \end{cases}$$

Here $|s|$ denotes the number of elements of $s$; i.e. the number of 1's occurring in $\sigma_s$.

For each irreducible character $\chi$, if $m_\chi$ denotes the multiplicity of $\chi$ in $\gamma$, then we have $\gamma = \sum m_\chi \chi$, so

$$n = \sum_{\chi \in Irr(S)} m_\chi n_\chi.$$

It is straightforward to obtain a formula for the $e_\chi$'s in terms of the irreducible character values $\{\chi(\sigma_s) : \chi \in Irr(S), s \in S\}$ and the multiplicities $m_\chi$. We have that for each $s \in S$, $\gamma(\sigma_{s^*} e_\chi) = m_\chi \chi(\sigma_{s^*})$. On the other hand, if $e_\chi = \sum_{t \in S} c_t \sigma_t$ where the $c_t$ coefficients all lie in $\mathbb{C}$, then $\gamma(\sigma_{s^*} e_\chi) = \sum_t c_t \gamma(\sigma_{s^*}\sigma_t) = c_s |s|$. Therefore,

$$e_\chi = m_\chi \sum_{s \in S} \frac{\chi(\sigma_{s^*})}{|s|} \sigma_s.$$

When $S$ is an scheme, then $|s| = n_s n$, so the formula becomes

$$e_\chi = \frac{m_\chi}{n} \sum_{s \in S} \frac{\chi(\sigma_{s^*})}{n_s} \sigma_s.$$

The $|Irr(S)| \times |S|$ matrix $(\chi(\sigma_s))_{\chi,s}$ is known as the *character table* of $S$. As with character tables of groups, it contains quite a lot of algebraic and representation-theoretic information about the structure of the CC.

There is a basic algorithm for computing the character table of an scheme which is analogous to well-known algorithms for group character tables, which we will now outline. The theoretical details of this algorithm will be dealt with later.

*Step 1.* Find a suitable basis of $Z(\mathbb{C}S)$. In the case of an association scheme of rank $d+1$, one can simply use the set of adjacency matrices $\{\sigma_s : s \in S\}$ themselves. For non-commutative schemes, one can use a maximal linearly independent subset of collection of "modified class sums"

$$\widehat{\sigma_s} = \sum_{t \in S} \frac{1}{n_t} \sigma_{t^*} \sigma_s \sigma_t.$$

as a basis for $Z(\mathbb{C}S)$. These modified class sums all have nonnegative integer coefficients in terms of the $\sigma_s$'s.

*Step 2.* Let $B = \{\widehat{\sigma_b}\}$ be the basis of $Z(\mathbb{C}S)$ obtained in Step 1, with $h = \dim(Z(\mathbb{C}S)) = |Irr(S)| = |B|$. For each element of $B$, calculate its matrix $M_h(\mathbb{N})$ as a left operator on $Z(\mathbb{C}S)$ in terms of the basis $B$.

*Step 3.* Find a set of $h$ eigenvectors common to all of the matrices $\widehat{\sigma_b}$ for $b \in B$. These must exist, because there are coefficients $\hat{p}_{\chi,b}$ (*eigenvalues*) and $\hat{q}_{b,\chi}$ (*dual eigenvalues*) for which

$$\widehat{\sigma_b} = \sum_{\chi} \hat{p}_{b,\chi} e_\chi \qquad \text{and} \qquad e_\chi = \sum_b \hat{q}_{\chi,b} \widehat{\sigma_b},$$

for all $b \in B$ and $\chi \in Irr(S)$. This implies that

$$\widehat{\sigma_b} e_\psi = \hat{p}_{b,\psi} e_\psi,$$

and so the $e_\psi$'s will be eigenvectors common to all of the $\widehat{\sigma_b}$'s. Furthermore, the entries of $e_\psi$ will be the coefficients $\hat{q}_{\psi,b}$ for $b \in B$ up to multiplication by a scalar, which is uniquely determined by normalizing in order to arrange that $e_\chi$ is an idempotent when it is represented in $\mathbb{C}S$. We can also find $m_\chi$ at this point, since the image of $e_\chi$ in its standard representation as an $n \times n$ matrix will be similar to a diagonal $(0,1)$-matrix with precisely $m_\chi n_\chi$ 1's on its diagonal.

*Step 4.* Determine the character table of the scheme from the $e_\chi$'s and the $m_\chi$'s. Solving for the eigenvalues in Step 3 produces the dual eigenvalues $\hat{q}_{\chi,b}$. We have computed the nonnegative integers $c_{b,s}$ for $b \in B$, $s \in S$ for which $\widehat{\sigma_b} = \sum\limits_s c_{b,s} \sigma_s$ in Step 2, so we have

$$e_\chi = \sum_b \hat{q}_{\chi,b} \sum_s c_{b,s} \sigma_s = \sum_s (\sum_b \hat{q}_{\chi,b} c_{b,s}) \sigma_s.$$

Now we can simply compare this to the formula

$$e_\chi = \frac{m_\chi}{n} \sum_{s \in S} \frac{\chi(\sigma_{s^*})}{n_s} \sigma_s$$

to determine the values of the $\chi(\sigma_s)$'s.

We will illustrate this algorithm with two examples:

**Example 3.1.** Let $\mathbb{C}S$ be the algebra with basis $S = \{1, \sigma_p, \sigma_q\}$ with multiplication given by

$$\begin{aligned}
\sigma_p^2 &= (57)1 + \sigma_q, \\
\sigma_p\sigma_q &= \sigma_q\sigma_p = (56)\sigma_p + (56)\sigma_q, \text{ and} \\
\sigma_q^2 &= (3192)1 + (3136)\sigma_p + (3135)\sigma_q.
\end{aligned}$$

Then $\mathbb{C}S$ is commutative. The $3 \times 3$ matrices corresponding to $\sigma_p$ and $\sigma_q$ are:

$$\sigma_p = \begin{bmatrix} 0 & 57 & 0 \\ 1 & 0 & 56 \\ 0 & 1 & 56 \end{bmatrix} \text{ and } \sigma_q = \begin{bmatrix} 0 & 0 & 3192 \\ 0 & 56 & 3136 \\ 1 & 56 & 3135 \end{bmatrix}.$$

The common eigenvectors for $[\sigma_p]$ and $[\sigma_q]$ are:

$$e_1 = [1, 1, 1]^T, e_2 = [1, \frac{7}{57}, \frac{-1}{399}]^T, \text{ and } e_3 = [1, \frac{-8}{57}, \frac{1}{456}]^T,$$

which satisfy

$$\begin{aligned}
\sigma_p e_1 &= 57e_1, & \sigma_p e_2 &= 7e_2, & \sigma_p e_3 &= -8e_3, \\
\sigma_q e_1 &= 3192e_1, & \sigma_q e_2 &= -8e_2, & \sigma_q e_3 &= 7e_3.
\end{aligned}$$

Since $(1+\sigma_p+\sigma_q)^2 = 3250(1+\sigma_p+\sigma_q)$, we have that $e_{\chi_1} = \frac{1}{3250}(1+\sigma_p+\sigma_q)$. Since $((399)1+49\sigma_p-\sigma_q)^2 = 750((399)1 + 49\sigma_p - \sigma_q)$, $e_{\chi_2} = \frac{1}{750}((399)1 + 49\sigma_p - \sigma_q)$. Finally, $((456)1 - 64\sigma_p + \sigma_q)^2 = 975((456)1 - 64\sigma_p + \sigma_q)$, we have $e_{\chi_3} = \frac{1}{975}((456)1 - 64\sigma_p + \sigma_q)$. Since $\mathbb{C}S$ is commutative, we know that each of the $n_\chi$'s has to be 1, so since $\frac{399}{750} = \frac{1729}{3250}$ and $\frac{456}{975} = \frac{1520}{3250}$, we can conclude that $m_{\chi_2} = 1729$ and $m_{\chi_3} = 1520$. Comparing with the other formula for the $e_\chi$'s allows us to complete the character table for $\mathbb{C}S$. For example, $(\frac{1729}{3250})(\frac{\chi_2(\sigma_p)}{n_p}) = \frac{49}{750}$ and $n_p = 57$ implies that $\chi_2(\sigma_p) = 7$, and we leave the others to the reader.

| | $1$ | $\sigma_p$ | $\sigma_q$ | $m_\chi$ |
|---|---|---|---|---|
| $\chi_1$ | 1 | 57 | 3192 | 1 |
| $\chi_2$ | 1 | 7 | $-8$ | 1729 |
| $\chi_3$ | 1 | $-8$ | 7 | 1520 |

It is no accident that the character values are closely related to the eigenvalues for association schemes. Indeed, if $\mathcal{X}$ is the representation affording the character $\chi$, then applying this to the equation $\widehat{\sigma_s} = \sum_s \hat{p}_{s,\psi} e_\psi$ gives $\mathcal{X}(\widehat{\sigma_s}) = \hat{p}_{s,\chi} I_{n_\chi}$, and taking traces we get $\chi(\widehat{\sigma_s}) = \hat{p}_{s,\chi} n_\chi$. When $\mathbb{C}S$ is commutative, every $n_\chi$ will be 1 and we can choose $\widehat{\sigma_s} = \sigma_s$ for each $s \in S$.

**Remark 3.2.** Though the parameters used in the above example may seem unnecessarily large for an introductory example, they become a bit more interesting when one takes in to account the fact that the existence of a 2-class association scheme of order 3250 with these structure constants would correspond to an extreme case of a strongly regular graph with valency $k$ and diameter $d$ for which the vertex/valency bound

$$n \leq 1 + k + k(\sum_{i=1}^{d-1} (k-1)^i)$$

holds with equality. Such graphs are called *Moore* graphs. If $k \geq 3$, then eigenvalue considerations imply that the diameter must be 2 and $k \in \{3, 7, 57\}$. The case $k = 3$ corresponds to the Petersen graph, and the case $k = 7$ to the Hoffman-Singleton graph. But a Moore graph with valency 57 *is not yet known to exist!* (All that is known about it is that if it does not exist, it corresponds to a nonschurian 2-class association scheme.) So what we have really computed is the character table of a table algebra which may or may not be realizable as an association scheme.

**Example 3.3.** Let $S$ be the scheme of order 12 with basic matrix

$$\sum_{i=0}^{d} i\sigma_i = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 & 4 & 5 & 5 & 6 & 6 & 7 & 7 \\ 1 & 0 & 3 & 2 & 4 & 4 & 5 & 5 & 6 & 6 & 7 & 7 \\ 2 & 3 & 0 & 1 & 6 & 6 & 7 & 7 & 4 & 4 & 5 & 5 \\ 3 & 2 & 1 & 0 & 6 & 6 & 7 & 7 & 4 & 4 & 5 & 5 \\ 4 & 4 & 7 & 7 & 0 & 1 & 6 & 6 & 5 & 5 & 2 & 3 \\ 4 & 4 & 7 & 7 & 1 & 0 & 6 & 6 & 5 & 5 & 3 & 2 \\ 5 & 5 & 6 & 6 & 7 & 7 & 0 & 1 & 2 & 3 & 4 & 4 \\ 5 & 5 & 6 & 6 & 7 & 7 & 1 & 0 & 3 & 2 & 4 & 4 \\ 7 & 7 & 4 & 4 & 5 & 5 & 2 & 3 & 0 & 1 & 6 & 6 \\ 7 & 7 & 4 & 4 & 5 & 5 & 3 & 2 & 1 & 0 & 6 & 6 \\ 6 & 6 & 5 & 5 & 2 & 3 & 4 & 4 & 7 & 7 & 0 & 1 \\ 6 & 6 & 5 & 5 & 3 & 2 & 4 & 4 & 7 & 7 & 1 & 0 \end{bmatrix}.$$

Since $\sigma_3\sigma_4 = \sigma_7$ and $\sigma_4\sigma_3 = \sigma_8$, $S$ is not commutative. The modified class sums are:

$$\begin{array}{rclrcl} \hat{\sigma}_0 & = & 8\sigma_0 + 4\sigma_1 & \hat{\sigma}_4 & = & \hat{\sigma}_2 + \hat{\sigma}_3 \\ \hat{\sigma}_1 & = & 4\sigma_0 + 8\sigma_1 & \hat{\sigma}_5 & = & \hat{\sigma}_2 + \hat{\sigma}_3 \\ \hat{\sigma}_2 & = & 4\sigma_2 + 2\sigma_4 + 2\sigma_5 & \hat{\sigma}_6 & = & 6\sigma_6 + 6\sigma_7 \\ \hat{\sigma}_3 & = & 4\sigma_3 + 2\sigma_4 + 2\sigma_5 & \hat{\sigma}_7 & = & \hat{\sigma}_6, \end{array}$$

so $B = \{\hat{\sigma}_0, \hat{\sigma}_1, \hat{\sigma}_2, \hat{\sigma}_3, \hat{\sigma}_6\}$ is a basis for $Z(\mathbb{C}S)$. The matrices for the elements of $B$ as left operators on $Z(\mathbb{C}S)$ in terms of the basis $B$ are:

$$\hat{\sigma}_0 = \begin{bmatrix} 8 & 4 & 0 & 0 & 0 \\ 4 & 8 & 0 & 0 & 0 \\ 0 & 0 & 8 & 4 & 0 \\ 0 & 0 & 4 & 8 & 0 \\ 0 & 0 & 0 & 0 & 12 \end{bmatrix} \qquad \hat{\sigma}_2 = \begin{bmatrix} 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 \\ 8 & 4 & 0 & 0 & 12 \\ 4 & 8 & 0 & 0 & 12 \\ 0 & 0 & 4 & 4 & 0 \end{bmatrix}$$

$$\hat{\sigma}_1 = \begin{bmatrix} 4 & 8 & 0 & 0 & 0 \\ 8 & 4 & 0 & 0 & 0 \\ 0 & 0 & 4 & 8 & 0 \\ 0 & 0 & 8 & 4 & 0 \\ 0 & 0 & 0 & 0 & 12 \end{bmatrix} \qquad \hat{\sigma}_3 = \begin{bmatrix} 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 \\ 4 & 8 & 0 & 0 & 12 \\ 8 & 4 & 0 & 0 & 12 \\ 0 & 0 & 4 & 4 & 0 \end{bmatrix}$$

$$\hat{\sigma}_6 = \begin{bmatrix} 0 & 0 & 0 & 0 & 12 \\ 0 & 0 & 0 & 0 & 12 \\ 0 & 0 & 12 & 12 & 0 \\ 0 & 0 & 12 & 12 & 0 \\ 12 & 12 & 0 & 0 & 12 \end{bmatrix}.$$

The five common eigenvectors to each of these matrices turn out to be: $e_1 = (1,1,3,3,2)^T$, $e_2 = (1,-1,1,-1,0)^T$, $e_3 = (1,1,0,0,-1)^T$, $e_4 = (1,-1,-1,1,0)^T$, and $e_5 = (1,1,-3,-3,2)^T$, and so the five primitive idempotents of $Z(\mathbb{C}S)$ are:

$$\begin{aligned} e_{\chi_1} &= \tfrac{1}{144}(\hat{\sigma}_0 + \hat{\sigma}_1 + 3\hat{\sigma}_2 + 3\hat{\sigma}_3 + 2\hat{\sigma}_6) \\ &= \tfrac{1}{12}(\sigma_0 + \sigma_1 + \sigma_2 + \sigma_3 + \sigma_4 + \sigma_5 + \sigma_6 + \sigma_7) \end{aligned}$$

$$\begin{aligned} e_{\chi_2} &= \tfrac{1}{16}(\hat{\sigma}_0 - \hat{\sigma}_1 + \hat{\sigma}_2 - \hat{\sigma}_3) \\ &= \tfrac{3}{12}(\sigma_0 - \sigma_1 + \sigma_2 - \sigma_3) \end{aligned}$$

$$\begin{aligned} e_{\chi_3} &= \tfrac{1}{36}(\hat{\sigma}_0 + \hat{\sigma}_1 - \hat{\sigma}_6) \\ &= \tfrac{2}{12}(2\sigma_0 + 2\sigma_1 - \sigma_6 - \sigma_7) \end{aligned}$$

$$\begin{aligned} e_{\chi_4} &= \tfrac{1}{16}(\hat{\sigma}_0 - \hat{\sigma}_1 - \hat{\sigma}_2 + \hat{\sigma}_3) \\ &= \tfrac{3}{12}(\sigma_0 - \sigma_1 - \sigma_2 + \sigma_3) \end{aligned}$$

$$\begin{aligned} e_{\chi_5} &= \tfrac{1}{144}(\hat{\sigma}_0 + \hat{\sigma}_1 - 3\hat{\sigma}_2 - 3\hat{\sigma}_3 + 2\hat{\sigma}_6) \\ &= \tfrac{1}{12}(\sigma_0 + \sigma_1 - \sigma_2 - \sigma_3 - \sigma_4 - \sigma_5 + \sigma_6 + \sigma_7). \end{aligned}$$

Therefore, the character table of $S$ is:

| | $\sigma_0$ | $\sigma_1$ | $\sigma_2$ | $\sigma_3$ | $\sigma_4$ | $\sigma_5$ | $\sigma_6$ | $\sigma_7$ | $m_\chi$ |
|---|---|---|---|---|---|---|---|---|---|
| $\chi_1$ | 1 | 1 | 1 | 1 | 2 | 2 | 2 | 2 | 1 |
| $\chi_5$ | 1 | 1 | $-1$ | $-1$ | $-2$ | $-2$ | 2 | 2 | 1 |
| $\chi_2$ | 1 | $-1$ | 1 | $-1$ | 0 | 0 | 0 | 0 | 3 |
| $\chi_4$ | 1 | $-1$ | $-1$ | 1 | 0 | 0 | 0 | 0 | 3 |
| $\chi_3$ | 2 | 2 | 0 | 0 | 0 | 0 | $-2$ | $-2$ | 2 |

What makes this an interesting example is that the number of character values is less than the number of collections of scheme relations that can take different character values. So the rule for groups that the number of character values equals the number of conjugacy classes does not hold for schemes. Another rule for groups that fails for schemes is that the irreducible character degrees of a scheme do not have to divide the order of the scheme. An example will be given in the exercises.

**Remark 3.4.** Much more is known about the eigenvalues and the dual eigenvalues in the case of commutative association schemes. If we let $\hat{P} = (\hat{p}_{b,\chi})_{b,\chi}$ and $\hat{Q} = (\hat{q}_{\chi,b})_{\chi,b}$, then

$$\begin{aligned} \sigma_b &= \sum_{\chi} \hat{p}_{b,\chi} e_\chi \\ &= \sum_{\chi} \hat{p}_{b,\chi} (\sum_{c} \hat{q}_{\chi,c} \widehat{\sigma_c}) \\ &= \sum_{c} (\sum_{\chi} \hat{p}_{b,\chi} \hat{q}_{\chi,c}) \widehat{\sigma_c}, \end{aligned}$$

so it follows that $\hat{P}\hat{Q} = I$.

If $S$ is a commutative scheme and we take $B = S$, then

$$e_\chi \circ \sigma_b = \hat{q}_{\chi,b} \sigma_b$$

for all $b \in B$, so the $\sigma_b$'s are eigenvectors common to the $e_\chi$'s under the pointwise product whose corresponding eigenvalues are the $\hat{q}_{\chi,b}$'s and whose entries are the $\hat{p}_{b,\chi}$'s. Hence the name "dual eigenvalues"!

(Dual eigenvalues of commutative schemes are usually defined to be the complex numbers $q_{\chi,s}$ for which $e_\chi = \frac{1}{n} \sum_{s \in S} q_{\chi,s} \sigma_s$. So our $\hat{q}_{\chi,s} = \frac{1}{n} q_{\chi,s}$.)

**Remark 3.5.** The central primitive idempotents of a CC will always be represented as positive semidefinite Hermitian matrices in their standard representation. To see this, first note that since $\mathbb{C}S$ is closed under the conjugate transpose $^*$, $A^*$ will lie in $Z(\mathbb{C}S)$ whenever $A \in Z(\mathbb{C}S)$. Therefore, there are constants $c_\psi \in \mathbb{C}$ such that $(e_\chi)^* = \sum_{\psi} c_\psi e_\psi$, and in particular $(e_\chi)^* e_\chi = c_\chi e_\chi$. Since $(e_\chi)^*$ must be one of the primitive idempotents of $Z(\mathbb{C}S)$, it suffices to show that $c_\chi \neq 0$. Taking the trace of both sides of the previous equation results in

$$c_\chi = tr(m_\chi^2 \sum_{s,t} \frac{\overline{\chi(\sigma_{s^*})}}{|s^*|} \frac{\chi(\sigma_{t^*})}{|t|} \sigma_{s^*} \sigma_t) = m_\chi^2 \sum_{s} \frac{|\chi(\sigma_{s^*})|^2}{|s|^2}.$$

The latter is greater than 0 because the value of an irreducible character of $S$ on the identity matrix will always be positive.

**Remark 3.6.** The dual eigenvalues and eigenvalues of a CC are intimately connected by means of the natural Hermitian form on $\mathbb{C}S$ that is given by

$$[A, B] = \frac{1}{n} tr(A\overline{B}) = \frac{1}{n} sum(A \circ \overline{B}^T),$$

for all $A, B \in \mathbb{C}S$, where $sum(A)$ is simply the sum of all of the entries of the $n \times n$ matrix $A$. This form has the properties:

(a) for all $A$, $B$, and $C \in \mathbb{C}S$, $[AC, B] = [A, BC^*]$ where $^*$ denotes the conjugate transpose, and

(b) for all $s, t \in S$, $[\sigma_s, \sigma_t] = \begin{cases} n_s, & \text{if } s = t, \\ 0, & \text{otherwise.} \end{cases}$

When $S$ is a commutative scheme, we have that for all $s \in S$ and $\chi \in Irr(S)$,

$$tr(\sigma_s e_\chi) = tr((\sum_\psi \hat{p}_{s,\psi} e_\psi) e_\chi) = \hat{p}_{s,\chi} tr(e_\chi) = \hat{p}_{s,\chi} m_\chi,$$

while at the same time

$$tr(\sigma_s e_\chi) = sum(\sigma_s \circ e_\chi^T) = sum(\sigma_s \circ (\sum_t \overline{\hat{q}_{\chi,t}} \sigma_t)) = \overline{\hat{q}_{\chi,s}} sum(\sigma_s) = \overline{\hat{q}_{\chi,s}} n_s n.$$

The latter can written in matrix form. Let $M$ and $\Lambda$ be the diagonal matricies $M = diag(m_\chi)_\chi$ and $\Lambda = diag(n_s)$. Then the latter identity can be written as $n\hat{Q}\Lambda = MP^*$. Combining this with the identity $\hat{P}\hat{Q} = I$ results in the *orthogonality relation*

$$\hat{P} M \hat{P}^* = n\Lambda.$$

**Remark 3.7.** The structure constants of a commutative association scheme are determined by its eigenvalues. Let $S$ be a commutative scheme, and let $r, s, t \in S$. Since $\sigma_r \circ (\sigma_s \sigma_t) = a_{str} \sigma_r$, we have that

$$tr(\sigma_r(\sigma_{t^*}\sigma_{s^*})) = sum(\sigma_r \circ (\sigma_{t^*}\sigma_{s^*})^T) = sum(\sigma_r \circ (\sigma_s \sigma_t)) = a_{str} n_r n,$$

and at the same time

$$tr(\sigma_r(\sigma_{t^*}\sigma_{s^*})) = tr(\sum_\chi \hat{p}_{r,\chi} \overline{\hat{p}_{t,\chi}\hat{p}_{s,\chi}} e_\chi) = \sum_\chi \hat{p}_{r,\chi} \overline{\hat{p}_{s,\chi}\hat{p}_{t,\chi}} m_\chi.$$

**Remark 3.8.** The most difficult part of the algorithm for computing character tables for CC's is the calculation of a common set of eigenvectors for the $\widehat{\sigma_b}$'s. The algorithms used by modern computer algebra systems are reasonably efficient as long as the character values of the CC lie in a cyclotomic extension of $\mathbb{Q}$. For schemes, however, it is unknown whether or not the character values will always lie in a cyclotomic extension of the rationals. For association schemes, this is a conjecture due to Norton that appears as Question (2), Section II.7 in the book *Algebraic Combinatorics I: Association Schemes* by Bannai and Ito.

**Remark 3.9.** If $R$ is an integral domain, the scheme ring $RS$ will be semisimple as long as char($R$) does not divide the order of $S$ nor any of the valencies $n_s$ for $s \in S$. To see this fact, let $\alpha \in J(RS)$, the Jacobson radical of $RS$, and write $\alpha = \sum_s \alpha_s \sigma_s$. Let $\{0\} = V_0 \subset V_1 \subset \cdots \subset V_m = RX$ be a composition series for the standard $RS$-module $RX$. Since $\alpha \in J(RS)$, we have $V_i \sigma_{s^*} \alpha \subseteq V_{i-1}$ for $i = 1, \ldots, m$, and so the matrix representing $\sigma_{s^*}\alpha$ in the standard representation will be similar to a strictly upper triangular matrix for every $s \in S$. Taking the traces of these matrices gives us the equations $\alpha_s n n_s = 0$, $s \in S$. So if the characteristic of $R$ does not divide the order of $S$, then the only elements $s \in S$ for which $\alpha_s$ can be nonzero are those for which the characteristic of $R$ divides $n_s$.

**EXERCISES.**

**Exercise 3.1.** Use Remark 3.4 and 3,6 to find a formula for the structure constants of a commutative association scheme in terms of the dual eigenvalues of the scheme.

**Exercise 3.2.** The *Krein parameters* of a commutative association scheme $S$ are the constants $\{\kappa_{\psi\phi\chi} : \psi, \phi, \chi \in Irr(S)\}$ defined by

$$e_\psi \circ e_\phi = \frac{1}{n} \sum_{\chi} \kappa_{\psi\phi\chi} e_\chi.$$

(As we will show later, these are nonnegative real numbers. However, they are not defined in general for non-commutative schemes, as $e_\chi \circ e_\psi$ need not lie in $Z(\mathbb{C}S)$.)

Find a formula for the Krein parameters of a commutative scheme $S$ in terms of the dual eigenvalues of $S$.

(Hint: Follow the dual procedure to that of Remark 3.7, starting with $(e_\psi \circ e_\phi)e_\chi$.)

**Exercise 3.3.** Find a formula for the Krein parameters of $S$ in terms of the eigenvalues of $S$. Use this to conclude that the Krein parameters are algebraic over $\mathbb{Q}$.

**Exercise 3.4.** Let $\chi, \psi \in Irr(S)$.

(a) Show that $e_\chi \otimes e_\psi$ is a positive semidefinite Hermitian matrix.

(b) Show that $e_\chi \circ e_\psi$ is a principal submatrix of the $n^2 \times n^2$ matrix $e_\chi \otimes e_\psi$.

(c) Show that the Krein parameters of a commutative association scheme are nonnegative real numbers.

**Exercise 3.5.** Let $S = \{\sigma_0 = I, \sigma_1, \dots, \sigma_4\}$ be the symmetric association scheme of order 28 whose multiplication is given by

$$\begin{array}{rclrcl}
\sigma_1^2 & = & 3\sigma_0 + \sigma_2, & \sigma_2\sigma_3 & = & 2\sigma_2 + 2\sigma_4 \\
\sigma_1\sigma_2 & = & 2\sigma_1 + \sigma_4, & \sigma_2\sigma_4 & = & 4\sigma_1 + 2\sigma_2 + 4\sigma_3 + 2\sigma + 4 \\
\sigma_1\sigma_3 & = & \sigma_3 + \sigma_4, & \sigma_3^2 & = & 6\sigma_0 + 2\sigma_1 + 2\sigma_3 + \sigma_4 \\
\sigma_1\sigma_4 & = & 2\sigma_2 + 2\sigma_3 + \sigma_4, & \sigma_3\sigma_4 & = & 4\sigma_1 + 4\sigma_2 + 2\sigma_3 + 2\sigma_4 \\
\sigma_2^2 & = & 6\sigma_0 + \sigma_2 + 4\sigma_3 + 2\sigma_4, & \sigma_4^2 & = & 12\sigma_0 + 4\sigma_1 + 4\sigma_2 + 4\sigma_3 + 6\sigma_4.
\end{array}$$

(a) Calculate the character table of $S$.

(b) Show that some of the Krein parameters of $S$ are irrational. (Some also have denominator 9, hence these are not algebraic integers over $\mathbb{Z}$.)

**Exercise 3.6.** Let $S$ be a commutative association scheme of order $n$ having $d$ classes. The *Frame number* of $S$ is

$$F_S = n^{d-1} \frac{\prod_{s \in S} n_s}{\prod_{\chi \in Irr(S)} m_\chi}.$$

(a) Show that if $\hat{P}$ is the matrix of eigenvalues of $S$, then $\frac{\det \hat{P}}{n}$ is an algebraic integer.

(Hint: Since every eigenvalue of $S$ is an algebraic integer, $\det \hat{P}$ is an algebraic integer. Since one of the $e_\chi$'s is $\frac{1}{n}J$, we have

$$e_\chi = \frac{1}{n}\sum_{s\in S}\sigma_s = \frac{1}{n}\sum_{\psi}(\sum_{s}\hat{p}_{s,\psi})e_\psi,$$

so $\sum_{s\in S}\hat{p}_{s,\psi} = n$ for $\psi = \chi$ and is 0 otherwise. Use this to perform a column operation on $\hat{P}$ and express $\frac{\det \hat{P}}{n}$ as a determinant of a submatrix of $P$.)

(b) Show that the Frame number of a commutative association scheme is a rational integer.

(Hint: Take the determinant of both sides of the orthogonality relation and apply the previous exercise.)

**Exercise 3.7.** Calculate the character table of the scheme of order 15 whose basic matrix is shown. What do you notice about the degrees of its irreducible characters?

$$\sum_{i=0}^{d} i\sigma_i = \begin{bmatrix} 0&1&2&3&3&3&3&4&4&4&4&5&5&5&5 \\ 2&0&1&4&4&4&4&5&5&5&5&3&3&3&3 \\ 1&2&0&5&5&5&5&3&3&3&3&4&4&4&4 \\ 3&4&5&0&3&4&5&1&3&4&5&2&3&4&5 \\ 3&4&5&3&0&5&4&4&5&1&3&5&4&3&2 \\ 3&4&5&4&5&0&3&5&4&3&1&3&2&5&4 \\ 3&4&5&5&4&3&0&3&1&5&4&4&5&2&3 \\ 4&5&3&2&4&5&3&0&4&5&3&1&4&5&3 \\ 4&5&3&3&5&4&2&4&0&3&5&5&3&1&4 \\ 4&5&3&4&2&3&5&5&3&0&4&3&5&4&1 \\ 4&5&3&5&3&2&4&3&5&4&0&4&1&3&5 \\ 5&3&4&1&5&3&4&2&5&3&4&0&5&3&4 \\ 5&3&4&3&4&1&5&4&3&5&2&5&0&4&3 \\ 5&3&4&4&3&5&1&5&2&4&3&3&4&0&5 \\ 5&3&4&5&1&4&3&3&4&2&5&4&3&5&0 \end{bmatrix}.$$

**Exercise 3.8.** In this exercise, we outline a proof for the fact that $Z(\mathbb{C}S)$ is equal to $span_{\mathbb{C}}(\{\widehat{\sigma_s} : s \in S\})$ for any scheme $S$.

(a) Let $M$ be a $\mathbb{C}S$-module. Let $\zeta_M : End_{\mathbb{C}}(M) \to End_{\mathbb{C}}(M)$ be the vector space homomorphism defined by

$$\zeta_M(\alpha) = \sum_{s\in S}\frac{1}{|s|}\sigma_{s^*}\alpha\sigma_s,$$

for all $\alpha \in End_{\mathbb{C}}(M)$. (Here the multiplication is composition and we think of $\sigma_s$ as a linear operator on $\mathbb{C}S$.) Show that for all $\alpha \in End_{\mathbb{C}}(M)$, $\zeta_M(\alpha) \in End_{\mathbb{C}S}(M)$; i.e. for all $\sigma_t \in S$, for all $m \in M$,

$$\zeta_M(\alpha)(\sigma_t m) = \sigma_t(\zeta_M(\alpha)(m)).$$

(b) Use the fact that $End_{\mathbb{C}S}(\mathbb{C}S) \cong Z(\mathbb{C}S)$ and the map $\zeta_{\mathbb{C}S}$ to show that $\widehat{\sigma_b}$ lies in $Z(\mathbb{C}S)$ for every $b \in S$.

(c) Let $S = \{s_0, s_1, \ldots, s_{d-1}\}$, and let $\phi \in End_{\mathbb{C}}(\mathbb{C}S)$ be given by $\phi(\sigma_{s_i}) = \delta_{0,i}\sigma_{s_0}$. Show that $\zeta_{\mathbb{C}S}(\phi) = \sigma_{s_0}$.

(d) Let $M_\chi$ be an irreducible $\mathbb{C}S$-module affording $\chi \in Irr(S)$. Viewing $M_\chi$ as a submodule of $\mathbb{C}S$, there is a module homomorphism $\mu : \mathbb{C}S \to M_\chi$ whose kernel is a complement to $M_\chi$ in $\mathbb{C}S$. Show that for all $m \in M_\chi$, $\zeta_{M_\chi}(\mu\phi\mu)(m) = m$.

(e) Identify $\mathbb{C}Se_\chi$ with $End_{\mathbb{C}}(M_\chi)$. By the previous part there is an $\alpha \in \mathbb{C}Se_\chi$ for which $\zeta_{M_\chi}(\alpha)(m) = m$, for all $m \in M$. Use this to conclude that $\zeta_{M_\chi}(\alpha) = e_\chi$.

(f) Show that $Z(\mathbb{C}S)$ is contained in the span of the $\widehat{\sigma_b}$'s.

# 4 Subschemes and Quotient Schemes

In this section we will start to lay the basis for a theory of schemes that parallels the theory of groups as closely as possible. We will begin develop a theory for schemes that includes analogs of subgroups, quotient groups, group homomorphisms, and important results concerning these concepts in group theory.

If $(X, S)$ is a CC, the *complex product* on $S$ is defined to be $st = \{u \in S : a_{stu} > 0\}$, for all $s, t \in S$. In other words, $st$ is the support of $\sigma_s \sigma_t$; i.e. those $u \in S$ for which $\sigma_u$ appears with nonzero coefficient in the product $\sigma_s \sigma_t = \sum_{u \in S} a_{stu} \sigma_u$. We can also think of the elements of $st$ as being those $u \in S$ containing points in the composition of the relations $s$ and $t$:

$$u \in st \iff \text{there exists an } (x, y) \in s \text{ and } (y, z) \in t \text{ such that } (x, z) \in u.$$

If $P$ and $Q$ are subsets of $S$, then their complex product is

$$PQ = \bigcup_{p \in P} \bigcup_{q \in Q} pq,$$

and

$$P^* = \{p^* : p \in P\}.$$

For any subset $R$ of $S$, we can define its *order* by summing the valency map on $R$; i.e. $n_R = \sum_{r \in R} n_r$. Note that $n_S = \sum_{s \in S} n_s = n$, the order of the CC.

**Definition 4.1.** *Let $(X, S)$ be an scheme. A nonempty subset $T$ of $S$ is a closed subset of $S$ if $T^* T \subseteq T$.*

Every scheme $(X, S)$ automatically contains the trivial closed subset $\{1_X\}$ and the closed subset $S$ itself. Several properties of closed subsets of schemes are immediate.

**Proposition 4.2.** *Let $T$ be a closed subset of an scheme $(X, S)$. Then*

*(a)* $1_X \in T$;

*(b)* $t \in T \implies t^* \in T$;

*(c)* $TT \subseteq T$; *and*

*(d) for all* $p, q \in T$, $\sigma_p \sigma_q = \sum_{t \in T} a_{pqt} \sigma_t$.

*In particular, $\mathbb{C}T$ is a subalgebra of $\mathbb{C}S$ under both ordinary and pointwise multiplication.*

*Proof.* (i) If $t \in T$, then $[\sigma_{t^*} \sigma_t, I] = n_t$ implies that $1_X \in T^* T \subseteq T$.

(ii) Since $1_X \in T$, $\{t^*\} = t^* 1_X \subseteq T^* T \subseteq T$.

(iii) It follows from (ii) that $T = T^*$, so $TT \subseteq T$ by definition.

(iv) We have $\sigma_p \sigma_q = \sum_{s \in S} a_{pqs} \sigma_s$ for all $p, q \in S$. If $T$ is a closed subset of $S$ and $p, q \in T$, then $pq \subseteq T$, so the only $s$'s in $S$ for which the $a_{pqs}$'s may be nonzero in this sum have to lie in $T$. $\square$

When $T$ is a closed subset of an scheme $S$, it follows from the above proposition that $\mathbb{C}T$ is a table subalgebra of $\mathbb{C}S$. The next proposition shows that this table algebra is realized as the adjacency algebra of an scheme, so closed subsets of schemes really are sub-schemes.

We will abuse the notation and say that a point $(x, y) \in X \times X$ belongs to a subset $R$ of $S$ if there is a relation $r \in R$ for which $(x, y) \in R$, and conveniently write this as $(x, y) \in R$.

**Proposition 4.3.** *Let $T$ be a closed subset of an scheme $(X, S)$. Fix an $x \in X$, and define*

$$Y := \{y \in X : (x, y) \in T\},$$

*and for each $s \in S$, set*

$$s_Y = s \cap Y \times Y.$$

*(Note that $s_Y$ will be empty unless $s \in T$.)*

*Let $T_Y = \{t_Y : t \in T\}$. Then $(Y, T_Y)$ is an scheme of order $n_T = \sum_{t \in T} n_t$.*

*Proof.* First, we claim that $T_Y$ is a partition of $Y \times Y$. Suppose $(y, z) \in Y \times Y$. Then we have that $(x, y) \in T$ and $(x, z) \in T$, so $(y, z) \in T^*T \subseteq T$. Therefore, there is a unique $t \in T$ for which $(y, z) \in t$, and thus a unique $t_Y \in T_Y$ such that $(y, z) \in T_Y$. Hence $T_Y$ is a partition of $Y \times Y$.

It is easy to see that $1_Y \in T_Y$, and that $(t_Y)^* = (t^*)_Y$, for all $t \in T$.

We have already shown above that the structure constants for $\mathbb{C}T$ are well-defined, and these will automatically be the intersection numbers $(Y, T_Y)$, so we have shown that this is an scheme.

The order of $(Y, T_Y)$ is $|Y| = |\{y \in X : (x, y) \in T\}| = \sum_{t \in T} n_t$. □

When $T$ is a closed subset of an scheme $S$, we can talk about the left cosets of $T$ in $S$ as being the complex products $sT := \{s\}T = \bigcup_{t \in T} st$, and, in a similar fashion, the right cosets $Ts$ and the double cosets $TsT$.

**Proposition 4.4.** *The collection of left cosets (right cosets, or double cosets, respectively) of $T$ in $S$ is a partition of $S$.*

*Proof.* Let $p, q \in S$ for which $p \in qT$. Then $pT \subseteq qT$, so it suffices to show that $q \in pT$. Since $p \in qT$, there exists a $t \in T$ for which $a_{qtp} \geq 1$. Therefore, $[\sigma_q \sigma_t, \sigma_p] = a_{qtp} n_p > 0$. But then $[\sigma_q, \sigma_p \sigma_{t^*}] > 0$, so we also have $a_{pt^*q} > 0$, and so $q \in pT$. This shows that the set of left cosets is a partition of $S$. Similarly, the right cosets will be a partition of $S$, and the double cosets will be unions of left cosets of $S$, and thus be a coarser partition than that of the left cosets. □

**Example 4.5.** The *thin radical* of an scheme $S$ is $\mathcal{O}_\vartheta(S) = \{s \in S : n_s = 1\}$. Since the thin radical consists of those elements of $S$ whose adjacency matrices are permutation matrices, and the product of two permutation matrices is another permutation matrix, it is automatic that the thin radical of $S$ is a closed subset of $S$. In fact, $\mathcal{O}_\vartheta(S)$ is a group under ordinary multiplication.

**Example 4.6.** If $P$ and $Q$ are closed subsets of an scheme $S$, then it is easy to see that $P \cap Q$ will be a closed subset of $S$. Other group-theoretic analogs such as center, centralizer, and normalizer can be defined, but these may not produce closed subsets.

**Example 4.7.** Let $T$ be a closed subset of A scheme $S$. The *strong normalizer* of $T$ is

$$K_S(T) = \{s \in S : s^*Ts \subseteq T\}.$$

The strong normalizer of a closed subset $T$ of A scheme $S$ of finite order is a closed subset of $S$, since $p, q \in K_S(T)$ implies that $(pq)^*Tpq = q^*p^*Tpq \subseteq q^*Tq \subseteq T$. If $K_S(T) = S$, then $T$ is said to be a *strongly normal* closed subset of $S$.

When $T$ is strongly normal in $S$, it follows that for all $s \in S$, $Ts = sT$ and $n_{sT} = n_T$. To see this, suppose that $(x, y) \in Ts$. Then there exists a $z \in X$ for which $(x, z) \in T$ and $(z, y) \in s$. Let $w \in X$ be such that $(x, w) \in s$. Then

$$(w, x) \in s^* \implies (w, z) \in s^*T \implies (w, y) \in s^*Ts \subseteq T,$$

and so $(x, y) \in sT$, and we can conclude that $Ts = sT$.

Furthermore, for any fixed $x \in X$ there are exactly $n_T$ choices $y_1, \ldots, y_{n_T} \in X$ for which $(x, y_i) \in T$. For the same $x$, there will be $n_s$ choices of $z_j \in X$ with $(x, z_j) \in s^*$. For each of these $z_j$'s, there will be $n_{Ts}$ choices of $w_{ji} \in X$ so that $(z_j, w_{ji}) \in Ts$. When $s \in K_S(T)$, each of the pairs $(x, w_{ji})$ must lie among the $(x, y_i)$'s because $T = s^*Ts$. Therefore, there are at most $n_T$ choices for the $w_{ji}$'s, so $n_{Ts} \leq n_T$. On the other hand, each $(v, x) \in s$ produces a $(v, y_i) \in sT = Ts$, and there are $n_T$ of these, so we must have $n_T \leq n_{Ts}$.

**Example 4.8.** When $S$ is A scheme and $\chi$ is a character of $\mathbb{C}S$, then the *kernel* of $\chi$ is

$$\ker\chi = \{s \in S : \chi(\sigma_s) = n_s n_\chi\}.$$

The kernel consists of the elements of $S$ whose adjacency matrices are mapped to $n_s$ times the identity matrix by an irreducible representation affording $\chi$. For instance, every element of $S$ lies in the kernel of the characer afforded by the valency representation. The kernel of a character is always a closed subset of $S$. To see this, let $\mathcal{X}$ be a representation affording the character $\chi$, and suppose $p, q \in \ker\chi$. Then

$$\chi(\sigma_p\sigma_q) = tr(\mathcal{X}(\sigma_p)\mathcal{X}(\sigma_q)) = tr(n_p n_q I) = n_p n_q n_\chi = \sum_{s\in S} a_{pqs} n_s n_\chi,$$

and

$$\chi(\sigma_p\sigma_q) = \chi(\sum_{s\in S} a_{pqs}\sigma_s) = \sum_{s\in S} a_{pqs}\chi(\sigma_s).$$

Since the $a_{pqs}$'s are all nonnegative, we are done if we can show that $\chi(\sigma_s) \leq n_s n_\chi$ for all $s \in S$, for then it will follow that $\chi(\sigma_s) = n_s n_\chi$ for every $s \in pq$, and so $\ker\chi$ will be a closed subset of $S$. Note that $\sigma_s$ is a nonnegative irreducible matrix, so by the Perron-Frobenius theorem it has a unique eigenvalue of largest modulus whose corresponding eigenvector is the unique one whose entries are all nonnegative. For $\sigma_s$, this eigenvalue is precisely $n_s$ and the eigenvector is the all 1's vector. Since $\chi(\sigma_s)$ is a sum of $n_\chi$ distinct eigenvalues of $\sigma_s$, we can conclude that $|\chi(\sigma_s)| \leq n_s n_\chi$, and that equality holds precisely when $\chi(\sigma_s) = n_s n_\chi$.

Unlike the kernel of the character of a finite group, the kernel of a character of A scheme is not always *normal*; i.e. the left cosets $s(\ker\chi)$ and right cosets $(\ker\chi)s$ need not agree for every $s \in S$.

When $T$ is a closed subset of A scheme $S$, then its adjacency algebra $\mathbb{C}T$ embeds as a semisimple subalgebra of the semisimple algebra $\mathbb{C}S$. In this situation, Frobenius reciprocity always applies, which says that if $V$ is a $\mathbb{C}S$-module and $W$ is a $\mathbb{C}T$-module, then

$$Hom_{\mathbb{C}S}(V, ind_{\mathbb{C}T}^{\mathbb{C}S}(W)) \cong Hom_{\mathbb{C}T}(res_{\mathbb{C}T}^{\mathbb{C}S}(V), W).$$

(See Curtis and Reiner, *Methods of Representation Theory, Volume I.*) In terms of characters, this says that if $\chi \in Irr(S)$ and $\psi \in Irr(T)$, then the multiplicity of $\chi$ in the induced character $\psi^S$ is the same as the multiplicity of $\psi$ in the restricted character $\chi_T$.

**Definition 4.9.** *Let $(X, S)$ be A scheme of finite order $n$, and let $T$ be a closed subset of $S$. Define*

$$\begin{aligned} xT &= \{y : (x,y) \in T\} \text{ for each } x \in X, \\ X/T &= \{xT : x \in X\}, \\ s^T &= \{(xT, yT) \in X/T \times X/T : (x,y) \in TsT\}, \text{ and} \\ S/\!\!/T &= \{s^T : s \in S\}. \end{aligned}$$

**Theorem 4.10.** *Let $(X, S)$ be A scheme of finite order $n$, and let $T$ be a closed subset of $S$. Then*

*(a) $\{xT : x \in X\}$ is a partition of $X$ for which each class in the partition has size $n_T$;*

*(b) $S/\!\!/T$ is a partition of $X/T \times X/T$; and*

*(c) $(X/T, S/\!\!/T)$ is A scheme of order $\frac{n}{n_T}$.*

*Proof.* (i) Suppose $x, y \in X$ for which $xT \cap yT \neq \emptyset$. If $z \in xT \cap yT$, then $(x,z), (y,z) \in T$, so $(x,y) \in TT^* = T$. Therefore, $y \in xT$, which implies that $yT \subseteq xT$. Similarly, $xT \subseteq yT$, so we can conclude that $xT = yT$. Therefore, $\{xT : x \in X\}$ is a partition of $X$. All of the sets in this partition have size $n_T$ since $n_T = |\{y \in X : (x,y) \in T\}| = |xT|$, for any fixed $x \in X$.

(ii) Let $p, q \in S$ for which $p^T \cap q^T \neq \emptyset$. If $(xT, yT) \in p^T \cap q^T$, then $(x,y) \in TpT \cap TqT$. Let $s$ be the unique relation in $S$ for which $(x,y) \in s$. Then $s \in TpT \cap TqT$. Since the double cosets of $T$ in $S$ are a partition of $S$, we conclude that $TpT = TqT$. But then $p^T = q^T$ since these sets are defined to contain exactly the same elements of $X/T \times X/T$, so the result follows.

(iii) We have that $S/\!\!/T$ is a partition of $X/T \times X/T$. $1_{X/T} = \{(xT, xT) : x \in X\} = (1_X)^T$, so $1_{X/T} \in S/\!\!/T$. If $s \in S$, then $(s^T)^* = \{(yT, xT) : (x,y) \in TsT\} = \{(yT, xT) : (y,x) \in T^*s^*T^* = T(s^*)T\} = (s^*)^T$, so $(s^T)^* \in S/\!\!/T$. We now need to show that the intersection numbers of $S/\!\!/T$ are well-defined. Let $p, q, r \in S$ and $(xT, zT) \in r^T$. Let $u \in TrT$ with $(x,z) \in u$. We have that

$$\begin{aligned} |\{yT \in X/T : (xT, yT) \in p^T, (yT, zT) \in q^T\}| &= \tfrac{1}{n_T}|\{y \in X : (x,y) \in TpT, (y,z) \in TqT\}| \\ &= \tfrac{1}{n_T} \sum_{s \in TpT} \sum_{t \in TqT} a_{stu}. \end{aligned}$$

This will be an integer since the first expression is an integer for fixed $u \in TrT$. If we replace $(x,z)$ by $(x', z')$ where $(x', x), (z, z') \in T$, then we will be counting the same collection of $y$'s in $X$ because $(x,y) \in TpT \iff (x', y) \in TpT$ and $(y,z) \in TqT \iff (y, z') \in TqT$. Therefore, this number does not depend on the initial choice of $(x,z) \in TrT$. Therefore, the intersection number $a_{p^T q^T r^T}$ is well-defined. This completes the proof that $(X/T, S/\!\!/T)$ is A scheme of order $\frac{n}{n_T}$. $\square$

**Remark 4.11.** Suppose $T$ is a strongly normal closed subset of A scheme $S$. Then every double coset $TsT$ for $s \in S$ will be equal to $sT$, and $n_{sT} = n_T$. From the proof that $S/\!\!/T$ is A scheme, we can see that the valency of the element $p^T$ is in general equal to

$$\begin{aligned} n_{p^T} &= \frac{1}{n_T} \sum_{s \in TpT} \sum_{t \in Tp^*T} a_{st1} \\ &= \frac{1}{n_T} \sum_{s \in TpT} n_s \\ &= \frac{n_{TpT}}{n_T}. \end{aligned}$$

Therefore, if $T$ is strongly normal in $S$, then every element of $S/\!\!/T$ has valency 1. So the quotient scheme of $S$ modulo a strongly normal closed subset will always be a thin scheme. The converse also holds, that is, if the quotient scheme $S/\!\!/T$ is thin, then $T$ is strongly normal in $S$.

**Remark 4.12.** If $T$ is a closed subset of a scheme $S$, then we will see in the exercises that $e_T = \frac{1}{n_T} \sum_{t \in T} \sigma_t$ is an idempotent of $\mathbb{C}S$. As in the construction of the double coset algebra, the algebra $e_T \mathbb{C}S e_T$ with unit element $e_T$ will be isomorphic to the adjacency algebra of the quotient scheme $\mathbb{C}[S/\!\!/T]$. It makes sense, therefore, to ask if the irreducible characters of $S/\!\!/T$ can be determined from the irreducible characters of $S$. The answer is given in Hanaki and Hirasaka's paper *Theory of Hecke algebras to association schemes*, *SUT Journal of Mathematics*, Vol. 38, No. 1, (2002), 61–66. Whenever $\chi$ is an irreducible character of $S$ for which $e = e_\chi e_T \neq 0$, $e\mathbb{C}Se$ will be a simple component of $e_T \mathbb{C}S e_T$, and all of the simple components of $e_T \mathbb{C}S e_T$ can be obtained in this way. So every irreducible character of $S/\!\!/T$ is obtained by restricting a unique $\chi \in \mathrm{Irr}(\mathbb{C}S)$ to $e_T \mathbb{C}S e_T$. Furthermore, if $\varphi \in \mathrm{Irr}(\mathbb{C}[S/\!\!/T])$ is the restriction of $\chi \in \mathrm{Irr}(\mathbb{C}S)$, then it turns out that $m_\varphi = m_\chi$. To see this, let $\Gamma_S$ be the standard character of $S$. Since $\Gamma_S(e_T) = \frac{1}{n_T}\Gamma_S(\sigma_{s_0}) = \frac{n_S}{n_T} = n_{S/\!\!/T}$, and $\Gamma_S(e_T \sigma_s e_T) = 0$ when $s \notin T$, the restriction of $\Gamma_S$ to $e_T \mathbb{C}S e_T$ agrees with the standard character of $S/\!\!/T$. Working out both sides of the equality

$$\Gamma_S(e_\chi e_T) = \Gamma_{S/\!\!/T}(e_\chi e_T)$$

gives us

$$m_\chi \chi(e_\chi e_T) = m_\varphi \varphi(e_\chi e_T),$$

so $m_\chi = m_\varphi$, as claimed.

**EXERCISES.**

**Exercise 4.1.** Let $T$ and $U$ be closed subsets of a scheme $S$.

(a) Show that the set of $T$-$U$-double cosets $\{TsU : s \in S\}$ is a partition of $S$.

(b) Show that if $U \subseteq K_S(T)$, then $TU$ is a closed subset of $S$.

**Exercise 4.2.** Show that the thin radical $\mathcal{O}_\vartheta(S)$ of a scheme $S$ is equal to the strong normalizer $K_S(\{1_X\})$ of the trivial closed subset.

**Exercise 4.3.** For any subset $R$ of a scheme $S$, define $e_R = \frac{1}{n_R} \sum_{r \in S} \sigma_r$.

(a) Show that $T$ is a closed subset of $S \iff e_T$ is an idempotent of $\mathbb{C}S$.

(b) Show that $T$ is a normal closed subset of $S \iff e_T$ is a central idempotent of $\mathbb{C}S$.

**Exercise 4.4.** Let $S$ be the scheme of order 12 from Chapter 3 whose character table is

| | $\sigma_0$ | $\sigma_1$ | $\sigma_2$ | $\sigma_3$ | $\sigma_4$ | $\sigma_5$ | $\sigma_6$ | $\sigma_7$ | $m_\chi$ |
|---|---|---|---|---|---|---|---|---|---|
| $\chi_1$ | 1 | 1 | 1 | 1 | 2 | 2 | 2 | 2 | 1 |
| $\chi_5$ | 1 | 1 | $-1$ | $-1$ | $-2$ | $-2$ | 2 | 2 | 1 |
| $\chi_2$ | 1 | $-1$ | 1 | $-1$ | 0 | 0 | 0 | 0 | 3 |
| $\chi_4$ | 1 | $-1$ | $-1$ | 1 | 0 | 0 | 0 | 0 | 3 |
| $\chi_3$ | 2 | 2 | 0 | 0 | 0 | 0 | $-2$ | $-2$ | 2 |

(a) Determine the kernels of all the irreducible characters of $S$.

(b) Determine whether or not each of the kernels is a normal closed subset of $S$. (You can find the basic matrix of $S$ in Chapter 3.)

(c) Are all of the normal closed subsets of $S$ strongly normal?

**Exercise 4.5.** Determine whether or not every closed subset of a commutative association scheme is normal. If so, are they also strongly normal?

**Exercise 4.6.** Show that if $T$ is a closed subset of a scheme $S$ for which $S/\!\!/T$ is thin, then $T$ is strongly normal in $S$.

**Exercise 4.7.** Let $S$ be a scheme of finite order.

(a) Show that the intersection of two strongly normal closed subsets of $S$ is strongly normal.

(b) Let $\mathcal{O}^\vartheta(S)$ be the intersection of all strongly normal closed subsets of $S$. Show that $\mathcal{O}^\vartheta(S)$ is a strongly normal closed subset of $S$. (This is known as the *thin residue* of $S$.)

**Exercise 4.8.** Suppose that $T$ is a strongly normal closed subset of a scheme $S$, and let $R$ be an integral domain.

(a) Show that

$$RS = \bigoplus_{s^T \in S/\!\!/T} R(TsT)$$

is an $S/\!\!/T$-graded $R$-algebra; i.e. $R(TpT)R(TqT) \subseteq R(TrT)$ whenever $p^T q^T = r^T$ in the group $S/\!\!/T$.

(b) Show that, if $s \notin T$, then it is not necessary for the $s^T$-component $R(TsT)$ in the grading to contain a unit of $RS$.

(Hint: It is possible for the adjacency matrices of every element in $TsT$ to have two identical rows.)

**Exercise 4.9.** Suppose $\varphi$ is an irreducible character of a Schurian scheme $S$, and let $G$ be the combinatorial automorphism group of $S$. Prove that the degree of $\varphi$ divides $|G|$.

# 5 Homomorphisms of schemes

**Definition 5.1.** *Let $(X, S)$ and $(\tilde{X}, \tilde{S})$ be schemes. An* ***scheme homomorphism*** *from $(X, S)$ to $(\tilde{X}, \tilde{S})$ is a pair $\phi = (\phi_X, \phi_S)$ of functions $\phi_X : X \to \tilde{X}$ and $\phi_S : S \to \tilde{S}$ satisfying*

*(a) $(x, y) \in s \implies (x\phi, y\phi) \in s\phi$, for all $s \in S$, and*

*(b) for all $w, z \in X$ and $s \in S$, $(w\phi, z\phi) \in s\phi \implies \exists (x, y) \in s$ such that $(x\phi, y\phi) = (w\phi, z\phi)$.*

**Proposition 5.2.** *Suppose $\phi : (X, S) \to (\tilde{X}, \tilde{S})$ is a scheme homomorphism. Then the following hold:*

*(a) $(pq)\phi \subseteq p\phi q\phi$, for all $p, q \in S$;*

*(b) $(s^*)\phi = (s\phi)^*$, for all $s \in S$;*

*(c) $(1_X)\phi = 1_{\tilde{X}}$;*

*(d) if $\tilde{T}$ is a closed subset of $\tilde{S}$, then $\tilde{T}\phi^{-1}$ is a closed subset of $S$.*

*(e) if $\phi_X$ is surjective, then $\phi_S$ is also surjective; and*

*(f) if $\phi_X$ is surjective, and $\tilde{\phi} : (\tilde{X}, \tilde{S}) \to (\tilde{\tilde{X}}, \tilde{\tilde{S}})$ is a scheme homomorphism of schemes, then the composition $\phi\tilde{\phi}$ is a scheme homomorphism.*

*Proof.* (i) If $\tilde{s} \in (pq)\phi$, then $\tilde{s} = s\phi$ for some $s \in pq$. If $(x, y) \in s$, then $(x\phi, y\phi) \in s\phi$, and there exists a $z \in X$ such that $(x, z) \in p$ and $(z, y) \in q$. Therefore, $(x\phi, z\phi) \in p\phi$ and $(z\phi, y\phi) \in q\phi$, so $(x\phi, y\phi) \in p\phi q\phi$ and thus $s\phi \in p\phi q\phi$.

(ii) If $(x, y) \in s$, then $(y\phi, x\phi) \in (s^*)\phi \cap (s\phi)^*$, so $(s^*)\phi = (s\phi)^*$.

(iii) If $x \in X$, then $(x\phi, x\phi) \in (1_X)\phi \cap 1_{\tilde{X}}$. Since $S$ is a scheme, this implies $(1_X)\phi = 1_{\tilde{X}}$.

(iv) If $p\phi, q\phi \in \tilde{T}$, then $(p\phi)(q\phi) \subseteq \tilde{T}$. By (i), it follows that $(pq)\phi \subseteq \tilde{T}$, as required.

(v) Suppose $\phi_X$ is surjective. Let $\tilde{s} \in \tilde{S}$, and suppose that $(\tilde{x}, \tilde{y}) \in \tilde{s}$. There exists an $(x, y) \in X \times X$ such that $(x\phi, y\phi) = (\tilde{x}, \tilde{y})$. If $s$ is the unique element of $S$ for which $(x, y) \in s$, then $(\tilde{x}, \tilde{y}) = (x\phi, y\phi) \in s\phi$, so we must have that $s\phi = \tilde{s}$.

(vi) The composition $(\phi\tilde{\phi})_X$ is well-defined since $\phi_X$ is surjective, and by (v), the composition $(\phi\tilde{\phi})_S$ will be well-defined because $\phi_S$ is surjective. For all $s \in S$, we have that $(x, y) \in s \implies (x\phi, y\phi) \in s\phi \implies (x\phi\tilde{\phi}, y\phi\tilde{\phi}) \in s\phi\tilde{\phi}$. If $(x\phi\tilde{\phi}, y\phi\tilde{\phi}) \in s\phi\tilde{\phi}$, then there exists a $(\tilde{x}, \tilde{y}) \in s\phi$ such that $(\tilde{x}\tilde{\phi}, \tilde{y}\tilde{\phi}) = (x\phi\tilde{\phi}, y\phi\tilde{\phi})$. Since $\phi$ is a homomorphism and $\phi_X$ is surjective, there exists a $(u, v) \in s$ such that $(u\phi, v\phi) = (\tilde{x}, \tilde{y})$. Therefore, there exists a $(u, v) \in s$ such that $(u\phi\tilde{\phi}, v\phi\tilde{\phi}) = (x\phi\tilde{\phi}, y\phi\tilde{\phi})$, which shows that $\phi\tilde{\phi}$ is a scheme homomorphism. $\square$

**Definition 5.3.** *Let $\phi : (X, S) \to (\tilde{X}, \tilde{S})$ be a scheme homomorphism. The kernel of $\phi$ is the closed subset $\phi^{-1}(1_{\tilde{X}})$ of $S$, which we will denote by* $\ker\phi$.

Let $T$ be a closed subset of a scheme $(X, S)$. Then the map $(X, S) \to (X/T, S/\!\!/T)$ given by $x \mapsto xT$ and $s \mapsto s^T$, for all $x \in X$ and $s \in S$ is a surjective scheme homomorphism with kernel equal to $T$. To see this, first note that

$$(x, y) \in s \implies (x, y) \in TsT \implies (xT, yT) \in s^T.$$

For the second property of scheme homomorphisms, we have that if $(xT, yT) \in s^T$, then $(x, y) \in TsT$, so there exists an $(a, b) \in s$ for which $(x, a) \in T$ and $(b, y) \in T$. Hence $xT = aT$ and $yT = bT$, so $(xT, yT) = (aT, bT)$ in $X/T$. Thus the map is a scheme homomorphism. It is clearly surjective. Finally, the kernel is $T$ because $s^T = 1_{X/T}$ implies that for all $(xT, Ty) \in s^T = 1_{X/T} = {1_X}^T$, we have $(x, y) \in T(1_X)T = T$, so $s \in T$. We will refer to this homomorphism as the *canonical homomorphism* associated with the closed subset $T$, and denote it by $\mod T$.

**Theorem 5.4.** *(First Isomorphism Theorem for schemes) Let $\phi : (X, S) \to (\tilde{X}, \tilde{S})$ be a scheme homomorphism with kernel $T$. Then*

*(a) for all $x, y \in X$, $x\phi = y\phi \iff xT = yT$;*

*(b) for all $p, q \in S$, $p^T = q^T \implies p\phi = q\phi$;*

*(c) $\phi$ is injective $\iff \ker \phi = \{1_X\}$; and*

*(d) the map $\bar{\phi} : (X/T, S/\!\!/T) \to (\tilde{X}, \tilde{S})$ given by $(xT)\bar{\phi} = x\phi$, $(s^T)\bar{\phi} = s\phi$, for all $x \in X$ and $s \in S$ is an injective scheme homomorphism.*

*Proof.* (i) Let $x, y \in X$ satisfying $xT = yT$. Then $xT = yT \implies (x, y) \in T \implies (x\phi, y\phi) \in T\phi = 1_{X'}$, so $x\phi = y\phi$.

If $x, y \in X$ for which $x\phi = y\phi$, then $(x\phi, y\phi) \in 1_{X'}$. So if $s$ is the element of $S$ for which $(x, y) \in s$, then we would have $s \in \ker \phi = T$, so $xT = yT$.

(ii) Let $p, q \in S$ for which $p^T = q^T$. Then for all $(x, y) \in p$, then $(xT, yT) \in p^T = q^T$, so $(x, y) \in TqT$. This implies that there exists $(a, b) \in q$ such that $(x, a), (b, y) \in T$. Therefore, $(x\phi, y\phi) \in p\phi$, $(x\phi, a\phi) \in T\phi = 1_{\tilde{X}}$, $(a\phi, b\phi) \in q\phi$, and $(b\phi, y\phi) \in T\phi = 1_{\tilde{X}}$. Thus $(x\phi, y\phi) = (a\phi, b\phi) = q\phi$ also, and we have that $p\phi = q\phi$.

(iii) If $\phi$ is injective, then $\phi_S$ is injective, so we have that for all $s \in S$, $s\phi = 1_{\tilde{X}} \implies s = 1_X$, so $\ker \phi = \{1_X\}$. Conversely, suppose $\ker \phi = \{1_X\}$. If $x\phi = y\phi$ and $s \in S$ with $(x, y) \in s$, then $(x\phi, y\phi) \in s\phi = 1_{\tilde{X}}$, so $s = 1_X$ and we must have $x = y$. Therefore, $\phi_X$ is injective. Now suppose $p, q \in S$ with $p\phi = q\phi$, and let $(x, y) \in p$. We have $(x\phi, y\phi) \in p\phi = q\phi$. There exists an $(a, b) \in q$ such that $(a\phi, b\phi) = (x\phi, y\phi)$. Since $\phi_X$ is injective, we must have $(a, b) = (x, y)$, so $p = q$ in $S$. Therefore, $\phi_S$ is also injective, and so we have that $\phi$ is injective.

(iv) Since $xT = yT \implies x\phi = y\phi$ and $p^T = q^T \implies p\phi = q\phi$ for all $x, y \in X$ and $p, q \in S$, we have that $\bar{\phi}$ is well-defined.

If $(xT, yT) \in s^T$, then $(x, y) \in TsT$. As before, there exists an $(a, b) \in s$ such that $(a\phi, b\phi) = (x\phi, y\phi)$. Therefore, $(xT\bar{\phi}, yT\bar{\phi}) = (x\phi, y\phi) \in s\phi = (s^T)\bar{\phi}$. Also, if $(xT\bar{\phi}, yT\bar{\phi}) \in (s^T)\bar{\phi}$, then $(x\phi, y\phi) \in s\phi$, so there exists an $(a, b) \in s$ for which $(a\phi, b\phi) = (x\phi, y\phi)$. Since $T = \ker \phi$, we have that $(aT, bT) = (xT, yT)$ and $(aT\bar{\phi}, bT\bar{\phi}) = (xT\bar{\phi}, yT\bar{\phi})$. Therefore, $\bar{\phi}$ is a scheme homomorphism.

If $s^T \in \ker \phi$, then $s\phi = 1_{X'}$, so $s \in \ker \phi = T$. If $(xT, yT) \in s^T$, then $(x, y) \in TsT = T$, so $xT = yT$, and thus $(xT, yT) \in 1_{X/T}$. Therefore, $\ker \bar{\phi} = \{1_{X/T}\}$, so $\bar{\phi}$ is injective. $\square$

**EXERCISES.**

**Exercise 5.1.** Prove the *Second Isomorphism Theorem for schemes:* If $T$ and $U$ are closed subsets of a scheme $S$ for which $U \subseteq K_S(T)$, then $TU$ is a closed subset of $S$ and

$$T/\!\!/(T \cap U) \cong (TU)/\!\!/U.$$

**Exercise 5.2.** Prove the *Third Isomorphism Theorem for schemes:* If $T$ and $U$ are closed subsets of $S$ for which $T \subseteq U \subseteq S$, then

$$S /\!\!/ U \cong (S /\!\!/ T) /\!\!/ (U /\!\!/ T).$$

**Exercise 5.3.** Give an example of a scheme homomorphism $\phi : S \to \tilde{S}$ and a closed subset $T$ of $S$ such that $\phi(T)$ is not a closed subset of $\tilde{S}$. (Hint: It is necessary that $\ker \phi$ be a non-normal subset of $S$.)

**Exercise 5.4.** Recall that the set of combinatorial automorphisms $\mathrm{Aut}(S)$ of a scheme $(X, S)$ was defined to be the set of permutations of $X$ for which $(x, y) \in s \implies (x\phi, y\phi) \in s$, for all $s \in S$. The set of all *scheme-automorphisms* of $(X, S)$ is the set of all bijective scheme homomorphisms $\phi : (X, S) \to (X, S)$ which preserve structure constants; i.e. $a_{p\phi, q\phi, s\phi} = a_{pqs}$, for all $p, q, s \in S$.

(a) Show that the set of all *scheme-automorphisms* of $(X, S)$ is a group under composition. (We will denote this by scheme-$\mathrm{Aut}(S)$.)

(b) Show that $\phi \mapsto \phi_S$ defines a group homomorphism from scheme-$\mathrm{Aut}(S)$ into $Sym(S)$ with kernel $\mathrm{Aut}(S)$.

(c) A *combinatorial isomorphism* of $S$ is a bijective scheme homomorphism with domain $S$ that is induced by letting a permutation $\tau \in Sym(X)$ act on $X \times X$ and setting $s\tau = \{(x\tau, y\tau) : (x, y) \in s\}$. Show that the scheme $(X, S\tau)$ will have structure constants satisfying $a_{p\tau, q\tau, s\tau} = a_{pqs}$, for all $p, q, s \in S$.

(d) Find an example of a combinatorial isomorphism of a scheme that is not an automorphism.

(e) Show that schemes that are combinatorially isomorphic will have the same character tables.

(f) Show that every combinatorial isomorphism of a thin scheme is a combinatorial automorphism (hence a group automorphism).

**Exercise 5.5.** Let $(X, S)$ be a Schurian association scheme. Let $G = Aut(S)$, $x \in X$, and let $H$ be the stabilizer of $x$ in $G$. Define a map $\phi : X \to G/H$ by $y\phi = gH \iff yg^{-1} = x$, for all $y \in X$.

Show that $\phi$ induces a combinatorial isomorphism from $(X, S) \to (G/H, G /\!\!/ H)$.

# 6 Categorical Considerations for schemes

The modern approach to algebraic objects is to consider them from the point of view of categories. The results of the previous two sections make it possible to do this for schemes. The result is a category which in many ways resembles the category of groups, and after a minor modification the category of schemes becomes a non-abelian exact category containing the category of groups as a subcategory.

Start by considering the category of schemes, whose objects are schemes and whose morphisms are scheme homomorphisms. For each pair of schemes $(X, S)$ and $(Y, T)$, there is a set (possibly empty) of scheme homomorphisms $\mathrm{Hom}((X, S), (Y, T))$. The composition of two scheme homomorphisms $\phi : (X, S) \to (Y, T)$ and $\psi : (Y, T) \to (Z, U)$ is defined whenever $\phi$ is surjective. (This composition law is similar to that of groups.)

One of the fundamental properties of the category of groups is that it has a *zero object*, which is a unique object 1 that is both an *initial object* and a *terminal object*. This means that for any object $A$ in the category, there are unique morphisms $1 \to A$ (initial object) and $A \to 1$ (terminal object). In the category of schemes, the trivial scheme $(1, 1) = (\{x\}, \{(x, x)\})$ is easily seen to be a terminal object, because the only possible scheme homomorphism from an arbitrary scheme $(X, S)$ into $(1, 1)$ sends every element of $X$ to $x$ and every element of $S$ to $\{(x, x)\}$. On the other hand, given a scheme $(X, S)$, for any fixed $x \in X$ we can define a scheme homomorphism $\phi_x : (\{x\}, \{(x, x\}) \to (X, S)$ by setting $x\phi_x = x \in X$ and $(x, x)\phi_x = 1_X$. Since $x \in X$ was arbitrary, this scheme homomorphism is not uniquely defined, so $(1, 1)$ is not an initial object. In other words, if we approach the category of schemes this way the category will not have a zero object.

However, the lack of a zero object can be alleviated after a minor modification. For every scheme $(X, S)$, instead of considering $X$ as simply being the set $\{x_1, \ldots, x_n\}$, we shall consider $X$ as the totally ordered set $(x_1, \ldots, x_n)$ with initial element $\star := x_1$, which we shall refer to as the *base point* of the scheme $(X, S)$. The *category of schemes* is then defined to be the category whose objects are the collection of finite schemes with base point and whose homomorphisms are the scheme homomorphisms $\phi : (X, \star, S) \to (Y, \star, T)$ for which $\phi(\star) = \star$. Since the composition of two such scheme homomorphisms will have this property, the composition law is basically the same as before. However, now the category will have a zero object, because for any scheme $(X, \star, S)$ with base point there will be a unique scheme homomorphism from $(1, \star, 1) \to (X, \star, S)$ that preserves base points.

The natural choice of base point for a thin scheme $(G, G)$ is the identity element of $G$. If $\varphi : G \to H$ is a homomorphism between finite groups, then the identity of $H$ is the image of the identity of $G$ under $\varphi$. Thus group homomorphisms can be regarded as homomorphisms in the category of schemes between thin schemes with base point. This means that the category of groups can be regarded as a subcategory of the category of schemes. Since every scheme homomorphism between two thin schemes is a group homomorphism, the category of groups is a *full* subcategory of the category of schemes.

Note that a combinatorial automorphism of a scheme $(X, S)$ can be regarded an isomorphism in the category of schemes between $(X, \star, S)$ and $(X, \star', S)$ by choosing the base point $\star'$ to be the image of $\star$.

The next fundamental property of the category of groups is the existence of a direct product.

**Definition 6.1.** *Let $(X, S)$ and $(Y, T)$ be two schemes. The direct product of $(X, S)$ and $(Y, T)$ is* *the scheme defined by*

$$(X,S) \times (Y,T) := (X \times Y, S \otimes T),$$

*where* $S \otimes T = \{s \otimes t : s \in S, t \in T\}$*, and we have that for each* $s \in S$ *and* $t \in T$*,*

$$((x_1,y_1),(x_2,y_2)) \in s \otimes t \iff (x_1,x_2) \in s \text{ and } (y_1,y_2) \in t.$$

The definition of $s \otimes t$ is motivated by the categorical product of the corresponding graphs, Indeed we have that the adjacency matrices $\sigma_{s\otimes t} = \sigma_s \otimes \sigma_t$ for all $s \in S$ and $t \in T$, and thus the adjacency algebras have the property (as with groups) that

$$\mathbb{C}[S \otimes T] \cong \mathbb{C}S \otimes_{\mathbb{C}} \mathbb{C}T.$$

If $(X, \star, S)$ and $(Y, \star, T)$ are schemes with base point, then the base point of the direct product $(X \times Y, \star, S \otimes T)$ should be chosen to be $(\star, \star)$.

We leave it to the reader to define the direct product and the direct sum of an infinite number of schemes, which can be done in a similar fashion to the way it is done with groups.

Once one has the idea of a direct product for finite schemes, one can define an *internal direct product*, and write a scheme $S$ as the direct product $T \otimes U$ of two of its closed subsets $T$ and $U$ if $S = TU$, $T \cap U = 1$, and every element $s$ of $S$ can be written uniquely as $s = tu = ut$ for $t \in T$ and $u \in U$. Under these conditions, it is routine to check that $S$ and $T \otimes U$ (and $U \otimes T$) are combinatorially isomorphic, so we can write $S = T \otimes U$.

Continuing on this line of ideas, we say that a finite scheme is *indecomposable* if it cannot be written as the direct product of two of its nontrivial closed subsets. In group theory, the *Krull-Schmidt theorem* asserts that every finite group can be written as the direct product of indecomposable finite groups in essentially only one way – the list of indecomposable direct summands of a finite group is uniquely determined up to group isomorphism. For commutative association schemes, this is a theorem of Ferguson and Turull. This seems to be still open for schemes in general, though it is routine to show that any scheme of finite order can be written as the direct product of indecomposable schemes. So the issue is to show that the list of indecomposable summands is unique up to scheme isomorphism.

Another equally important issue is the notion of a composition series for schemes. The most natural notion of a composition series for a scheme $(X,S)$ is that of a nested sequence of closed subsets

$$1 = T_0 \subsetneq T_1 \subsetneq T_2 \subsetneq \cdots \subsetneq T_k = S$$

with the property that each of the quotient schemes $T_i /\!\!/ T_{i-1}$ for $i \in \{1, \dots, k\}$ is a *primitive* scheme. A scheme is primitive if its only closed subsets are the trivial one and the scheme itself. It is routine to show that every scheme of finite order has at least one composition series. However, unlike the Jordan-Hölder theorem in finite group theory, the lists of composition factors occurring in two different composition series for a scheme $S$ do not have to agree up to scheme isomorphism. In fact, it follows from a result of Rao, Ray-Chaudhuri, and Singhi concerning commutative table algebras whose structure constants and Krein parameters are both nonnegative, which states that the composition factors of a commutative association scheme will agree up to algebraic isomorphism on their adjacency algebras. (This is Theorem II.9.11 in the book of Bannai and Ito.) But there are examples that show the composition factors do not have to agree up to combinatorial isomorphism.

One example is the scheme arising from the distance regular graph defined by the generalized 6-gon with 126 vertices.

An alternative notion of a composition series of schemes weakens the requirement on the factors to that of being *simple* schemes. In line with the notion of a simple group, $(X, S)$ is a *simple* scheme when 1 and $S$ are the only *normal* closed subsets of $S$. Zieschang has shown that the analog of the Jordan-Hölder theorem does hold in this setting: the list of simple composition factors of this type appearing in any composition series for a scheme of finite order is unique up to order and scheme isomorphism.

The most important property of the category of schemes has essentially already been established: *exactness*. We have seen in the previous section that for any injective scheme homomorphism $\phi : T \to S$, $T\phi$ will be a closed subset of $S$, and so there is a canonical surjective scheme homomorphism mod $T\phi : S \to S/\!\!/(T\phi)$. In the category of schemes, this means that any exact sequence of the form $1 \to T \to S$ can be canonically completed to a *short exact sequence*

$$1 \to T \xrightarrow{\phi} S \to S/\!\!/(T\phi) \to 1.$$

In addition, if we start with any surjective scheme homomorphism $\psi : S \to U$, then there is a canonical way to complete the exact sequence $S \to U \to 1$ to a short exact sequence

$$1 \to \ker \psi \to S \to U \to 1$$

in the category of schemes. These ideas are analogous to the notion of exactness of the category of abelian groups. (The expert reader will recall that usually when one speaks of an exact category, one starts with an *additive* category; i.e. one for which the set of homomorphims between objects has an abelian group structure, which does not hold for the category of schemes.) So we can conclude that the category of schemes is a non-additive exact category that contains the category of groups as a full subcategory.

**EXERCISES.**

**Exercise 6.1.** Suppose that a scheme $S = T \otimes U$ is the direct product of two closed subsets $T$ and $U$. Show that every $s \in S$ can be uniquely expressed as $s = tu$ for some $t \in T$ and $u \in U$, and that $tu = ut$.

**Exercise 6.2.** Suppose that we modify the category of commutative association schemes as we have done with the modified scheme-category. This gives us a category $\tilde{\mathcal{A}}$ whose objects are commutative association schemes with base point, and whose morphisms are given by the scheme homomorphisms defined on commutative association schemes that map base points to base points.

Show that $\tilde{\mathcal{A}}$ is a exact category. Is $\tilde{\mathcal{A}}$ an *additive* exact category?

**Exercise 6.3.** It is possible for two association schemes $(X, S)$ and $(X, S')$ to be algebraically isomorphic but not combinatorially isomorphic. A combinatorial isomorphism $\phi : (X, S) \to (X, S')$ arises from a permutation $\phi$ of $X$, which induces a bijection $(x, y) \mapsto (x\phi, y\phi)$ on $X \times X$, which then transforms the relations of $S$ into the relations of $S'$. Such a map will automatically preserve structure constants, i.e. $a_{pqs} = a_{p\phi q\phi s\phi}$ for all $p, q, s \in S$. An algebraic automorphism is simply a bijection from $S$ to $S'$ that preserves structure constants.

Show that the following pair of symmetric association schemes are algebraically, but not combinatorically, isomorphic.

$$\begin{bmatrix}
0 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 3 & 3 & 3 & 3 & 3 & 3 \\
1 & 0 & 1 & 1 & 2 & 2 & 3 & 3 & 3 & 3 & 2 & 2 & 2 & 2 & 3 & 3 \\
1 & 1 & 0 & 1 & 3 & 3 & 2 & 2 & 3 & 3 & 2 & 2 & 3 & 3 & 2 & 2 \\
1 & 1 & 1 & 0 & 3 & 3 & 3 & 3 & 2 & 2 & 3 & 3 & 2 & 2 & 2 & 2 \\
2 & 2 & 3 & 3 & 0 & 1 & 2 & 3 & 2 & 3 & 2 & 3 & 2 & 3 & 1 & 1 \\
2 & 2 & 3 & 3 & 1 & 0 & 3 & 2 & 3 & 2 & 3 & 2 & 3 & 2 & 1 & 1 \\
2 & 3 & 2 & 3 & 2 & 3 & 0 & 1 & 2 & 3 & 3 & 2 & 1 & 1 & 2 & 3 \\
2 & 3 & 2 & 3 & 3 & 2 & 1 & 0 & 3 & 2 & 2 & 3 & 1 & 1 & 3 & 2 \\
2 & 3 & 3 & 2 & 2 & 3 & 2 & 3 & 0 & 1 & 1 & 1 & 3 & 2 & 3 & 2 \\
2 & 3 & 3 & 2 & 3 & 2 & 3 & 2 & 1 & 0 & 1 & 1 & 2 & 3 & 2 & 3 \\
3 & 2 & 2 & 3 & 2 & 3 & 3 & 2 & 1 & 1 & 0 & 1 & 2 & 3 & 3 & 2 \\
3 & 2 & 2 & 3 & 3 & 2 & 2 & 3 & 1 & 1 & 1 & 0 & 3 & 2 & 2 & 3 \\
3 & 2 & 3 & 2 & 2 & 3 & 1 & 1 & 3 & 2 & 2 & 3 & 0 & 1 & 2 & 3 \\
3 & 2 & 3 & 2 & 3 & 2 & 1 & 1 & 2 & 3 & 3 & 2 & 1 & 0 & 3 & 2 \\
3 & 3 & 2 & 2 & 1 & 1 & 2 & 3 & 3 & 2 & 3 & 2 & 2 & 3 & 0 & 1 \\
3 & 3 & 2 & 2 & 1 & 1 & 3 & 2 & 2 & 3 & 2 & 3 & 3 & 2 & 1 & 0
\end{bmatrix}$$

and

$$\begin{bmatrix}
0 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 3 & 3 & 3 & 3 & 3 & 3 \\
1 & 0 & 1 & 1 & 2 & 2 & 3 & 3 & 3 & 3 & 2 & 2 & 2 & 2 & 3 & 3 \\
1 & 1 & 0 & 1 & 3 & 3 & 2 & 2 & 3 & 3 & 2 & 2 & 3 & 3 & 2 & 2 \\
1 & 1 & 1 & 0 & 3 & 3 & 3 & 3 & 2 & 2 & 3 & 3 & 2 & 2 & 2 & 2 \\
2 & 2 & 3 & 3 & 0 & 1 & 2 & 3 & 2 & 3 & 2 & 3 & 2 & 3 & 1 & 1 \\
2 & 2 & 3 & 3 & 1 & 0 & 3 & 2 & 3 & 2 & 3 & 2 & 3 & 2 & 1 & 1 \\
2 & 3 & 2 & 3 & 2 & 3 & 0 & 1 & 3 & 2 & 2 & 3 & 1 & 1 & 2 & 3 \\
2 & 3 & 2 & 3 & 3 & 2 & 1 & 0 & 2 & 3 & 3 & 2 & 1 & 1 & 3 & 2 \\
2 & 3 & 3 & 2 & 2 & 3 & 3 & 2 & 0 & 1 & 1 & 1 & 2 & 3 & 3 & 2 \\
2 & 3 & 3 & 2 & 3 & 2 & 2 & 3 & 1 & 0 & 1 & 1 & 3 & 2 & 2 & 3 \\
3 & 2 & 2 & 3 & 2 & 3 & 2 & 3 & 1 & 1 & 0 & 1 & 3 & 2 & 3 & 2 \\
3 & 2 & 2 & 3 & 3 & 2 & 3 & 2 & 1 & 1 & 1 & 0 & 2 & 3 & 2 & 3 \\
3 & 2 & 3 & 2 & 2 & 3 & 1 & 1 & 2 & 3 & 3 & 2 & 0 & 1 & 2 & 3 \\
3 & 2 & 3 & 2 & 3 & 2 & 1 & 1 & 3 & 2 & 2 & 3 & 1 & 0 & 3 & 2 \\
3 & 3 & 2 & 2 & 1 & 1 & 2 & 3 & 3 & 2 & 3 & 2 & 2 & 3 & 0 & 1 \\
3 & 3 & 2 & 2 & 1 & 1 & 3 & 2 & 2 & 3 & 2 & 3 & 3 & 2 & 1 & 0
\end{bmatrix}$$

**Exercise 6.4.** Show that schemes that are algebraically isomorphic have identical character tables (up to row and column permutation).

**Exercise 6.5:** Show that a finite scheme $(X, S)$ of order $n$ is primitive if and only if every non-identity relation $s \in S$ is the adjacency matrix of a connected graph with $n$ vertices.

**Exercise 6.6:** Suppose that $1 \to T \to S \to U \to 1$ is exact in the modified scheme category.

(a) Show that if $T$ and $S$ are Schurian, then $U$ is Schurian.

(b) Show that if $S$ and $U$ are Schurian, then $T$ is Schurian.

(c) Show that if $T$ and $U$ are Schurian, then $S$ need not be Schurian.

(Hint: There is a nonschurian scheme of order 16 with a nontrivial thin radical.)

**Exercise 6.7:** Show that any scheme homomorphism between thin schemes can be regarded as a group homomorphism between the respective groups.

# 7 The Category of Table Algebras

While the modified scheme-category provides a convenient perspective, it certainly does not provide a broad enough footing to work with all of the different aspects of schemes. For instance, both the valency map and embeddings of fusion subalgebras into adjacency algebras provide examples of natural algebra homomorphisms that do not arise from linear extension of scheme homomorphisms to the adjacency algebra $\mathbb{C}S$. In order to understand this phenomenon, the perspective of table algebras will be useful.

As with the development of association schemes, table algebras were first introduced only in the commutative case. The non-commutative table algebras we have introduced here were initially referred to as generalized real nonsingular table algebras by Arad, Fisman, and Muzychuk.

**Definition 7.1.** *A finite-dimensional unital $\mathbb{C}$-algebra with basis $B$ is a table algebra with table basis $B$, denoted $\mathbb{C}B$, if*

*(a)* $1 \in B$ *($1$ being the multiplicative identity in $\mathbb{C}B$);*

*(b) there is an involution $^*$ on $\mathbb{C}B$ for which $b^* \in B$, for all $b \in B$;*

*(c) all of the structure constants $\lambda_{bcd}$ $(b, c, d \in B)$ relative to the table basis $B$ are nonnegative real numbers;*

*(d) for all $b, c \in B$, $\lambda_{bc1} > 0 \iff c = b^*$;*

*(e) for all $b \in B$, $\lambda_{bb^*1} = \lambda_{b^*b1}$.*

If $x = \sum_{b \in B} x_b b \in \mathbb{C}B$ with each $x_b \in \mathbb{C}$, then we assume $x^* = \sum_{b \in B} \overline{x_b} b^*$, where $\overline{x_b}$ denotes complex conjugation, and we define the support of $x$ to be $supp(x) = \{b \in B : x_b \neq 0\}$.

It should be clear that the adjacency algebra of a scheme is an example of a table algebra. Indeed, many of the basic properties of table algebras are straightforward generalizations of the properties of the adjacency algebras of schemes that we have already established. We will record these here, leaving their proofs to the exercises.

**Proposition 7.2.** *Let $\mathbb{C}B$ be a table algebra with table basis $B$.*

*(a) For all $a, b, c \in B$, $\lambda_{abc} = \lambda_{b^*a^*c^*}$.*

*(b) For all $a, b, c, d \in B$, $\sum_{e \in B} \lambda_{abe}\lambda_{ecd} = \sum_{e \in B} \lambda_{aed}\lambda_{bce}$.*

*(c) For all $b, c \in B$, $\lambda_{bb^*c} = \lambda_{bb^*c^*}$*

*(d) The map $t : \mathbb{C}B \to \mathbb{C}$ given by $t(\sum_{b \in B} x_b b) = x_1$ is a trace on $\mathbb{C}B$; i.e. a $\mathbb{C}$-linear map satisfying $t(xy) = t(yx)$, for all $x, y \in \mathbb{C}B$.*

*(e) $\mathbb{C}B$ has a nondegenerate Hermitian form given by $[x, y] = t(xy^*)$, for all $x, y \in \mathbb{C}B$.*

*(f) For all $a, b, c \in B$, $[ab, c] = [a, cb^*]$ and $[a, bc] = [b^*a, c]$.*

(g) *For all* $a, b, c \in B$,

$$\lambda_{abc}\lambda_{cc^*1} = \lambda_{c^*ab^*}\lambda_{bb^*1} = \lambda_{cb^*a}\lambda_{aa^*1}.$$

(h) $\mathbb{C}B$ *is a semisimple algebra.*

Since $t$ is a *trace* on the semisimple algebra $\mathbb{C}B$ for which the corresponding Hermitian form $[x, y] = t(xy^*)$ is nondegenerate, the restriction of $t$ to any of the simple components in the Wedderburn decomposition $\mathbb{C}B = \bigoplus_{\chi\in\mathrm{Irr}(\mathbb{C}B)} \mathbb{C}Be_\chi$ has to be a nonzero scalar multiple of the usual trace on the full matrix ring $\mathbb{C}Be_\chi$. It follows that $t$ can be expressed as $\sum\limits_{\chi} z_\chi \chi$, for some nonzero scalars $z_\chi \in \mathbb{C}$. By an argument similar to the one we used for CC's, one can establish the following formula for the centrally primitive idempotents $e_\chi$ of $\mathbb{C}B$:

$$e_\chi = z_\chi \sum_{b\in B} \frac{\chi(b^*)}{\lambda_{bb^*1}} b, \text{ for all } \chi \in \mathrm{Irr}(\mathbb{C}B).$$

One important property of table algebras is a generalization of the valency map.

**Theorem 7.3.** *Let* $\mathbb{C}B$ *be a table algebra with table basis* $B$. *Then there exists a unique algebra homomorphism* $|\cdot| : \mathbb{C}B \to \mathbb{C}$ *for which* $|b| = |b^*| > 0$ *for all* $b \in B$.

*Proof.* Let $\hat{B} = \sum\limits_{b\in B} b$, and let $M_{\hat{B}}$ be the matrix representing $\hat{B}$ in the left regular representation of $\mathbb{C}B$. The entries of $M_{\hat{B}}$ are $(\sum\limits_{b\in B} \lambda_{bcd})_{d,c}$. We claim that all of these entries are positive. Indeed, for each pair $c, d \in B$, we have

$$[cd^*, cd^*] = [d^*d, c^*c] \geq \lambda_{dd^*1}\lambda_{cc^*1} > 0,$$

so there exists a $b \in B$ such that $\lambda_{cd^*b^*} \neq 0$. By the preceding proposition, it follows that $\lambda_{bcd} > 0$ for this $b$, so the claim follows.

By the Perron-Frobenius Theorem, $M_{\hat{B}}$ has a unique eigenvalue $\mu$ of largest modulus, and this $\mu$ is positive, has multiplicity 1, and its corresponding eigenvector $v = \sum\limits_{b\in B} v_b b$ has positive coordinates. Note that for each $c \in B$, we have

$$M_{\hat{B}}(vc) = \hat{B}(vc) = (\hat{B}v)c = \mu(vc),$$

so $vc$ is another eigenvector for $M_{\hat{B}}$ with eigenvalue $\mu$. Since $\mu$ has multiplicity 1, $vc$ has to be a scalar multiple of $v$ for each $c \in B$. Therefore, there exists a function $f : B \to \mathbb{C}$ such that $vc = f(c)v$. Since all of the coordinates of $v$ are positive real, we can show that for all $c \in B$,

$$f(c) = \frac{1}{v_d} \sum_{b\in B} v_b \lambda_{bcd}, \text{ for each } d \in B.$$

In particular, $f(c)$ is positive for all $c \in B$.

Being a projection onto a one-dimensional eigenspace, it is straightforward to show that $f$ extends to an algebra homomorphism from $\mathbb{C}B \to \mathbb{C}$, which we will also denote by $f$.

Since $f$ is a one-dimensional representation, $f$ is an irreducible representation of the semisimple algebra $\mathbb{C}B$. It follows then that $g(c) = f(c^*)$ is another irreduicble representation. If $f \neq g$, then

$f$ and $g$ are inequivalent irreducible representations of $\mathbb{C}B$. If $e_f$ and $e_g$ are the corresponding centrally primitive idempotents, then we will have

$$\begin{aligned} 0 = t(e_f e_g) &= z_f z_g \sum_{b,c \in B} \frac{f(b^*)g(c^*)}{\lambda_{bb^*1}\lambda_{cc^*1}} t(bc) \\ &= z_f z_g \sum_{b \in B} \frac{f(b^*)f(b)}{\lambda_{bb^*1}} \\ &= f(z_f z_g \sum_{b \in B} \frac{1}{\lambda_{bb^*1}} b^* b), \end{aligned}$$

which is a contradiction because $1 \in supp(b^*b)$ and $f(c) > 0$ for all $c \in B$. Therefore, we must have $f = g$, so we can conclude that $f(b) = f(b^*)$ for all $b \in B$. This concludes the proof of the theorem. $\square$

The algebra homomorphism described in the preceding theorem is called the *degree* map of the table algebra. If we rescale the table basis of $\mathbb{C}B$ by replacing every element of $B$ with $\hat{b} = \frac{|b|}{\lambda_{bb^*1}} b$, then we obtain a table algebra $\mathbb{C}\hat{B}$ with the property that $|\hat{b}| = \lambda_{\hat{b}\hat{b}^*1}$, for all $\hat{b} \in \hat{B}$. This operation is called *standardizing* the table algebra. The adjacency algebras of schemes provide examples of table algebras for which the basis of adjacency matrices is standardized.

For commutative table algebras $\mathbb{C}B$, one can define a Schur product abstractly by setting $b \circ c = \delta_{bc} b$ for all $b, c \in B$, and the two algebras $(\mathbb{C}B, \circ)$ and $(\mathbb{C}B, \cdot)$ are isomorphic commutative semisimple $\mathbb{C}$-algebras. As we did earlier, one can define the Krein parameters to be the structure constants for $(\mathbb{C}B, \circ)$ relative to its basis $E$ of centrally primitive idempotents under ordinary multiplication. If these Krein parameters are all nonnegative, then $(\mathbb{C}E, \circ)$ is a table algebra that is *dual* to $\mathbb{C}B$ (and, indeed, the dual of $\mathbb{C}E$ is isomorphic to $\mathbb{C}B$). However, it is an open problem to find necessary and sufficient conditions on the structure constants of $\mathbb{C}B$ that are equivalent to the Krein parameters being nonnegative.

Several categorical notions for schemes can be extended to the setting of table algebras, with similar properties. Here we will provide a list, leaving the properties of these notions to be established in the exercises.

Let $\mathbb{C}B$ be a table algebra with table basis $B$.

(a) A *sub-table algebra* is a table algebra $\mathbb{C}D$ that is a subalgebra of $\mathbb{C}B$ with the following properties:

   (a) for all $d \in D$, there are nonnegative real numbers $\mu_{db}$ such that $d = \sum_{b \in B} \mu_{db} b$;

   (b) the subsets $supp(d) := \{b \in B : \mu_{bd} > 0\}$ of $B$ as $d$ runs over $D$ are pairwise disjoint; and

   (c) $supp(1) = \{1\}$.
   (Note that sub-table algebras of group algebras include Schur subrings, and sub-table algebras of the adjacency algebras of schemes include fusion subrings.)

(b) If $b, c \in B$, then the support of $bc$ is $supp(bc) := \{d \in B : \lambda_{bcd} > 0\}$, and if $R, S \subseteq B$, then

$$RS = \bigcup_{r \in R} \bigcup_{s \in S} supp(rs).$$

If $R \subseteq B$, then we let $R^* = \{r^* : r \in R\}$, $R^+ = \sum_{r \in R} r$, and the *order* of $R$ is $o(R) = \sum_{r \in R} \frac{|r|^2}{\lambda_{rr^*1}}$.

(c) A *closed subset* of $B$ is a subset $C \subseteq B$ for which $C^*C \subseteq C$.

Closed subsets of table bases are analogous to closed subsets in schemes and subgroups of groups. Given a closed subset $C$ of $B$ and $b \in B$, the *left coset* $Cb$ is $\bigcup_{c \in C} supp(cb)$. Right cosets are defined similarly, and double cosets are defined to be the subsets

$$CbC = \bigcup_{c \in C} \bigcup_{d \in C} supp(cbd).$$

The set of left cosets (resp. right cosets, double cosets) of a closed subset of $B$ is a partition of $B$.

(d) Given any closed subset $C$ of a table basis $B$, the *quotient table algebra* $\mathbb{C}[B/\!\!/C]$ is the table algebra with table basis $B/\!\!/C = \{b/\!\!/C : b \in B\}$, where

$$b/\!\!/C := \frac{1}{o(C)}(CbC)^+.$$

If $C$ is a closed subset of a table basis $B$, then the structure constants of $\mathbb{C}[B/\!\!/C]$ are given in terms of the structure constants of $\mathbb{C}B$ by

$$\lambda_{a/\!\!/C, b/\!\!/C, d/\!\!/C} = \frac{1}{o(C)} \sum_{e \in CaC} \sum_{f \in CbC} \lambda_{efg},$$

for any $g \in CdC$.

One can check that the quotient table algebra of a group algebra $\mathbb{C}G$ relative to a subgroup $H$ is isomorphic to the ordinary Hecke algebra $\mathbb{C}[G/\!\!/H]$, and the quotient table algebra of the complex adjacency algebra of a scheme $S$ relative to a closed subset $T$ of $S$ will be isomorphic as algebras to $\mathbb{C}[S/\!\!/T]$.

(e) If $\mathbb{C}B$ and $\mathbb{C}D$ are table algebras, then $B \otimes D := \{b \otimes d : b \in B, d \in D\}$ is a table basis for a $\dim(B)\dim(D)$-dimensional table algebra $\mathbb{C}[B \otimes D] \cong \mathbb{C}B \otimes_{\mathbb{C}} \mathbb{C}D$ called the *tensor product* of the table algebras $\mathbb{C}B$ and $\mathbb{C}D$. (This is analogous to the direct product of groups and schemes.)

(f) If $\mathbb{C}B$ and $\mathbb{C}D$ are table algebras, we can form the *wreath product* $\mathbb{C}[B \wr D]$ with table basis $B \wr D$ given by

$$B \wr D = \{1 \otimes d : d \in D\} \bigcup \{b \otimes D^+ : b \in B \setminus \{1\}\}.$$

Note that the wreath product $\mathbb{C}[B \wr D]$ is defined to be a certain sub-table algebra of the tensor product $\mathbb{C}[B \otimes D]$ having dimension $\dim(D) + \dim(B) - 1$.

(g) An algebra homomorphism $\phi : \mathbb{C}B \to \mathbb{C}D$ between two table algebras with respective table bases $B$ and $D$ is a *table algebra homomorphism* if

(a) $\phi(1) = 1$;

(b) $\phi(b^*) = \phi(b)^*$,, for all $b \in B$;

(c) for all $b \in B$, there are nonnegative real numbers $\mu_{b,d}$ such that $\phi(b) = \sum\limits_{d \in D} \mu_{b,d} d$; and

(d) for all $b, c \in B$, $supp(\phi(b)) \cap supp(\phi(c)) \neq \emptyset \implies$ there exists $\rho_{b,c} > 0$ such that $\phi(b) = \rho_{b,c}\phi(c)$.

**Remark.** The definitions we have given for sub-table algebras and for table algebra homomorphisms are different than what has appeared in earlier literature. The motivation for the new definitions is to include both the degree map $\mathbb{C}B \to \mathbb{C}$ and the inclusion map $\mathbb{C}T \to \mathbb{C}S$ of a fusion of a scheme as examples of table algebra homomorphisms, and yet still have a definition which allowed a reasonable composition law for homomorphisms. In previous treatments, homomorphisms of table algebras have been restricted to those in which $supp(\phi(b))$ consists of a single element of $D$ in all cases, which is useful when dealing with problems concerning duality of table algebras. We will refer to this type of table algebra homomorphism as a *fission-free* table algebra homomorphism, and say that a table algebra homomorphism has *fission* if there is at least one table basis element for which the support of the image of this element has size larger than one.

It would be nice to know the conditions for a table algebra to arise from an association scheme. Of course, this is the case if there is an injective table algebra homomorphism $\phi : \mathbb{C}B \to M_n(\mathbb{C})$ for which $\phi(b)$ is a $(0,1)$-matrix for every $b \in B$ and and $\sum_b \phi(b) = J$.

**EXERCISES.**

**Exercise 7.1.** Let $\mathbb{C}B$ be a table algebra, and let $t$ be its trace. Let $t = \sum\limits_{\chi \in Irr(\mathbb{C}B)} z_\chi \chi$ be the expression of the trace $t$ as a linear combination of the irreducible characters of $\mathbb{C}B$, for some nonzero scalars $z_\chi \in \mathbb{C}$. Show that the formula for the unique centrally primitive idempotent $e_\chi$ for which $\chi(e_\chi) \neq 0$ is

$$e_\chi = z_\chi \sum_{b \in B} \frac{\chi(b^*)}{\lambda_{bb^*1}} b.$$

**Exercise 7.2.** Suppose that $\mathbb{C}D$ is a sub-table algebra of the table algebra $\mathbb{C}B$. Prove that the degree homomorphism of $\mathbb{C}D$ agrees with the restriction of the degree homomorphism of $\mathbb{C}B$ to $\mathbb{C}D$.

**Exercise 7.3.** Verify the following properties of closed subsets of a table basis $B$.

(a) If $C \subseteq B$ is a closed subset, then $1 \in C$.

(b) If $C \subseteq B$ is a closed subset, then $C^* = C$.

(c) If $C \subseteq B$ is a closed subset and $c, d \in C$, then $\lambda_{cdb} = 0$ whenever $b \in B \setminus C$.

(d) If $C \subseteq B$ is a closed subset, then $\mathbb{C}C$ is a sub-table algebra of $\mathbb{C}B$.

(e) If $C, D \subseteq B$ are closed subsets, then $C \cap D$ is a closed subset.

(f) If $C \subseteq B$ is a closed subset, then the set $\{bC : b \in B\}$ of left cosets of $C$ in $B$ is a partition of $B$.

(g) If $C \subseteq B$ is a closed subset, then the set $\{CbC : b \in B\}$ of double cosets of $C$ in $B$ is a partition of $B$.

**Exercise 7.4.** Let $C$ be a closed subset of $B$. If $\mu : B \to \mathbb{C}^{\times}$ and the table basis $B$ is re-scaled to $B' = \{\mu(b)b : b \in B\}$, show that the order of $C' = \{\mu(c)c : c \in C\}$ will be equal to the order of $C$.

**Exercise 7.5.** Suppose $\mathbb{C}B$ is a table algebra with standardized table basis $B$. Show that for any closed subset $C$ of $B$, $B/\!\!/C$ is a standardized table basis of $\mathbb{C}[B/\!\!/C]$, and $o(B/\!\!/C) = \frac{o(B)}{o(C)}$.

**Exercise 7.6.** Let $C$ be a closed subset of a table basis $B$. Show that the map $b \mapsto b/\!\!/C$ induces a fission-free table algebra homomorphism from $\mathbb{C}B$ onto $\mathbb{C}[B/\!\!/C]$.

**Exercise 7.7.** Since table algebra homomorphisms are algebra homomorphisms, the *kernel* of a table algebra homomorphism $\phi : \mathbb{C}B \to \mathbb{C}D$ is $\{\alpha \in \mathbb{C}B : \phi(\alpha) = 0\}$. A table algebra homomorphism will be injective when the dimension of the image is the equal to the dimension of the domain, which occurs precisely when the kernel is $\{0\}$.

(a) Find an example of a fission-free table algebra homomorphism that is not injective.

(b) Give an example of an injective table algebra homomorphism that has fission.

**Exercise 7.8.** Suppose that $C, D$ are closed subsets of a table basis $B$ that satisfy:

(a) for all $c \in C$ and $d \in D$, $cd = dc \in B$;

(b) $CD = B$; and

(c) $C \cap D = \{1\}$.

Prove that $\mathbb{C}B \cong \mathbb{C}[C \otimes D]$ as table algebras.

**Exercise 7.9.** Show that the composition of table algebra homomorphisms is a table algebra homomorphism.

**Exercise 7.10.** Let $B$ be a standardized table basis. The set of *linear* elements of $B$ is $L(B) = \{b \in B : bb^* = \lambda_{bb^*1}1\}$.

(a) Show that $|b| \geq 1$, and equality holds if and only if $b$ is linear.

(b) Show that $L(B)$ is a closed subset of $B$. Is $L(B)$ a group?

**Exercise 7.11.** Let $\mathbb{C}B$ be a table algebra with table basis $B$. We say that two subsets $S, T \subseteq B$ are *conjugate* in $B$ if there exists a $b \in B$ such that $bSb^* \subseteq T$ and $b^*Tb \subseteq S$.

a) Show that if $S, T \subseteq B$ are conjugate in $B$ and $b \in B$ for which $bSb^* \subseteq T$ and $b^*Tb \subseteq S$, then $bSb^* = T$ and $b^*Tb = S$.

b) Show that conjugacy is an equivalence relation on the family of subsets of $B$.

**Exercise 7.12.** Let $\mathbb{C}B$ be a table algebra with table basis $B$. If $S \subseteq B$, then the *normalizer* of $S$ is $N_B(S) = \{b \in B : bSb^* \subseteq S\}$. Show that if $S$ is a subset of $B$, then $N_B(S)$ is a closed subset of $B$, and $bSb^* = S$ for all $b \in N_B(S)$.

**Exercise 7.13.** Show that the modified category of schemes is a *full* subcategory of the restricted category of table algebras whose morphisms consist only of fission-free homomorphisms. Show that the category of schemes whose morphisms are table algebra homorphisms between schemes is a full subcategory of the category of table algebras.

# 8 Sylow Theory for Table Algebras

In this section we present an analog of Sylow's theorem for certain types of table algebras that includes Sylow's theorem for finite groups, following the exposition of Blau and Zieschang that is based on properties of standardized table algebras due to Arad, Fisman, and Muzychuk.

**Proposition 8.1.** *Let $\mathbb{C}B$ be a table algebra with standardized table basis $B$. Let $b \in B$ and let $C$ and $D$ closed subsets of $B$. Then the following hold:*

*(a) For all $d \in B$, $\sum\limits_{c \in B} \lambda_{bcd} = |b|$;*

*(b) $bC^+ = |b|C^+ \iff bC \subseteq C$;*

*(c) $bC^+ = |b|(bC)^+ \iff supp(b^*b) \subseteq C$;*

*(d) $bC^+ = \beta(bC)^+$ for some positive $\beta \leq |b|$;*

*(e) $C^+bD^+ = \mu(CbD)^+$, for some positive integer $\mu$; and*

*(f) $o(CbD) = o(D) \iff b^*Cb \subseteq D$.*

*Proof.* (i): Since $B$ is a standard table basis, we have

$$\sum_{c \in B} \lambda_{bcd}|d| = \sum_{c \in B} \lambda_{d^*bc^*}|c^*| = |d^*b| = |d||b|,$$

so (i) follows by cancelling $|d|$.

(ii): $bC^+ = |b|C^+ \implies supp(bc) \subseteq C$ for every $c \in C \implies bC \subseteq C$.

On the other hand, suppose $bC \subseteq C$. Then $bC^+ = \sum\limits_{c \in C}(\sum\limits_{d \in C} \lambda_{bcd}d) = \sum\limits_{d \in C} \mu_d d$, where $0 < \mu_d = \sum\limits_{c \in C} \lambda_{bcd} \leq |b|$ by part (i). Since $|bC^+| = |b||C^+| = |b| \sum\limits_{d \in C} |d|$, we can conclude that $\mu_d = |b|$ for every $d \in C$, and (ii) follows.

(iii): From the proof of (ii), we see that $bC^+ = |b|(bC)^+ \iff \sum\limits_{c \in C} \lambda_{bcd} = |b|$ for every $d \in bC$. By (i), this is happens if and only iff $\lambda_{bed} = 0$ for all $e \in B \setminus C$, for all $d \in bC$. But $\lambda_{bed} = 0 \iff \lambda_{b^*de} = 0$, so this is equivalent to $b^*d \subseteq C$ for all $d \in bC$. Finally, $b^*d \subseteq C$, for all $d \in bC$ is equivalent to $b^*bC \subseteq C$, which is equivalent to $supp(b^*b) \subseteq C$, so (iii) holds.

(iv): Write $bC = \{b = b_1, b_2, ..., b_m\}$. Since $b_iC = bC$ for all $b_i \in bC$, we have that $b_iC^+ = \sum\limits_{j=1}^{m} \mu_{ij}b_j$ with $\mu_{ij} > 0$, for all $i, j \in \{1, \dots, m\}$. If we set $M = (\mu_{ij})_{ij}$, then the fact that $(b_iC^+)C^+ = |C^+|b_iC^+$ implies that $M^2 = |C^+|M$. Since every entry of $M$ is positive, this implies that each column of $M$ is an eigenvector for the Perron-Frobenius eigenvalue of $M$. Since this eigenvalue has multiplicity 1, this implies that $M$ has rank 1. Also, $(bC)^+C^+ = |C^+|(bC)^+$ by (ii), so $|C^+| \sum\limits_j b_j = \sum\limits_i b_iC^+ = \sum\limits_i \sum\limits_j \mu_{ij}b_j = \sum\limits_j (\sum\limits_i \mu_{ij})b_j$, which implies that every column of $M$ has the same column sum $|C^+|$. Since $M$ has rank 1, every column of $M$ has to be the same vector. Therefore, all of the entries $\mu_{1j}$ are equal to the same positive number $\beta$, and we can conclude that $bC^+ = \sum\limits_j \mu_{1j}b_j =$

$\beta(bC)^+$. Since it is easy to see that $\beta = \sum_{c\in C} \lambda_{bcd}$ for all $d \in bC$, it follows from (i) that $\beta \leq |b|$, which proves (iv).

(v): $C^+bD^+ = \sum_{d\in CbD} \mu_d d$ for some $\mu_d > 0$, so it suffices to show that all of these $\mu_d$ are equal to one another. By (iv), we have $C^+bC^+ = \beta C^+(bD)^+ = \beta \sum_{d\in bD} \alpha_d (Cd)^+$. Therefore, $\mu_e = \mu_f$ whenever $e, f$ lie in the same right coset of $C$. Similarly, we can show that $\mu_e$ will be equal to $\mu_f$ whenever $e, f$ lie in the same left coset of $D$. If $e, f \in CbD$ are arbitrary, then there exists a $c \in C$ and $d \in D$ such that $f \in supp(ced)$. But then there exists a $g \in B$ such that $g \in supp(ce)$ and $f \in supp(gd)$, so we have $\mu_e = \mu_g = \mu_f$, for all $e, f \in CbC$, as required.

(vi): We have that $o(Cb)o(D) = |(Cb)^+||D^+| = |\sum_{e\in CbD} \sum_{a\in Cb} (\sum_{d\in D} \lambda_{ade})e| \leq |(Cb)^+||(CbD)^+|$. Therefore, $o(D) \leq o(CbD)$, and equality holds if and only if $\lambda_{abe} = 0$ for all $a \in Cb$, $b \in B \setminus D$, and $e \in CbD$. Since $\lambda_{abe} = 0 \iff \lambda_{e^*ab^*} = 0$, this is equivalent to $(CbD)^*(Cb) \subseteq D$, which is equivalent to $b^*Cb \subseteq D$, as required. $\square$

It goes without saying that the analogous properties for right cosets can be established in a similar fashion, as it can be shown that $o(CbD) = o(C) \iff bDb^* \subseteq C$. The analog of Sylow's theorem we seek applies to table algebras that are $p$-fractional and $p$-valenced for some prime $p$. We now define these notions.

**Definition 8.2.** *Let $\mathbb{C}B$ be a table algebra with standardized table basis $B$. Let $p$ be a prime. We say that $\mathbb{C}B$ is $p$-fractional if for all $b, c, d \in B$, there exists nonnegative integers $n, m$ such that $\lambda_{bcd} = \frac{n}{p^m}$, and $\mathbb{C}B$ is $p$-valenced if for all $b \in B$, there exists a nonnegative integer $\ell$ such that $|b| = \lambda_{bb^*1} = p^\ell$.*

Note that if $G$ is a finite group, then the group algebra $\mathbb{C}G$ is a $p$-fractional $p$-valenced table algebra for any prime $p$.

**Definition 8.3.** *Let $\mathbb{C}B$ be a table algebra with standardized table basis $B$. A closed subset $C$ of $B$ is a closed $p$-subset of $B$ if $\mathbb{C}C$ is $p$-valenced and $o(C)$ is a power of $p$. A closed $p$ subset $P$ of $B$ is a Sylow $p$-subset of $B$ if $P$ is a closed $p$-subset of $B$ for which $o(B/\!\!/P)$ is not divisible by $p$. The set of Sylow $p$-subsets of $B$ is denoted by $\mathrm{Syl}_p(B)$.*

**Lemma 8.4.** *Let $p$ be a prime, and suppose $\mathbb{C}B$ is a $p$-fractional $p$-valenced table algebra with standardized table basis $B$. Suppose $C$ and $D$ are closed $p$-subsets of $B$.*

*(a) For all $b \in B$, $o(bC)$, $o(Cb)$, $o(CbD)$ are powers of $p$ that are divisible by $o(C)$.*

*(b) $\mathbb{C}[B/\!\!/C]$ is a $p$-fractional $p$-valenced table algebra with standardized table basis $B/\!\!/C$.*

*(c) If $p$ divides $o(C)$, then $p$ divides the order of the group $L(C)$ consisting of linear elements of $C$.*

*Proof.* (i): Let $b \in B$. It suffices to show that $o(CbD)$ is a power of $p$ that is divisible by $o(C)$. Since $\mathbb{C}B$ is $p$-fractional, we have by the previous lemma that there exists a positive $\mu \in \mathbb{Z}[\frac{1}{p}]$ such that $C^+bD^+ = \mu(CbD)^+$. If $\mu = \frac{m}{p^k}$ for some positive integer $m$ coprime to $p$ and nonnegative integer $k$, then we have that $|C^+||b||D^+| = \frac{m}{p^k}|(CbD)^+|$. Since $\mathbb{C}B$ is $p$-valenced, the left hand side of this equation is a power of $p$. Therefore, $m$ has to be 1 and $|(CbD)^+| = o(CbD)$ is a power of $p$ that is divisible by $|C^+| = o(C)$.

(ii): In a previoius exercise we saw that $\mathbb{C}[B/\!\!/C]$ is a table algebra with standardized table basis $B/\!\!/C$. The degree of each $b/\!\!/C \in B/\!\!/C$ is $|b/\!\!/C| = \frac{o(CbC)}{o(C)}$, which is a power of $p$ by (i). Therefore, $\mathbb{C}[B/\!\!/C]$ is $p$-valenced. Since $o(C)$ is a power of $p$ and $\mathbb{C}B$ is $p$-fractional, it follows from the formula for structure constants of $\mathbb{C}B$ that each of these is an element of $\mathbb{Z}[\frac{1}{p}]$.

(iii): For $x \in C$, we have that $x \in L(C) \iff |x| = 1$, so $o(L(C))$ is the order of the group $L(C)$. Since $\mathbb{C}B$ is $p$-valenced, $|y|$ will be a positive power of $p$ for every $y \in C \setminus L(C)$. Therefore, $o(C)$ and $o(C \setminus L(C)) = o(C) - o(L(C))$ are both divisible by $p$, so $o(L(C))$ is also divisible by $p$. □

**Theorem 8.5.** ***(Sylow's theorem for Table Algebras)*** *Let $p$ be a prime, and suppose $\mathbb{C}B$ is a $p$-fractional $p$-valenced table algebra with standardized table basis $B$. Then the following hold.*

*(a)* $\mathrm{Syl}_p(B) \neq \emptyset$.

*(b) If $P$ is a closed $p$-subset of $B$ for which $p$ divides $o(B/\!\!/P)$, then there exists a closed $p$-subset $P'$ such that $P \subseteq P' \subseteq N_B(P)$ and $o(P') = po(P)$.*

*(c) Any two Sylow $p$-subsets of $B$ are conjugate in $B$.*

*(d)* $|\mathrm{Syl}_p(B)| \equiv 1 \mod p$.

*Proof.* (i): If $p$ does not divide $o(B)$, then the closed subset $\{1\}$ of $B$ is a Sylow $p$-subset of $B$ by definition. If $p$ divides $o(B)$, then by the previous lemma, $o(L(B))$ is divible by $p$. Therefore, $B$ has closed $p$-subsets, and so since $B$ is finite it must have maximal closed $p$-subsets. The fact that every maximal closed $p$-subset is a Sylow $p$-subset in this case will be a consequence of (ii).

(ii): Since $p$ divides $o(B/\!\!/P)$, the order of the group $L(B/\!\!/P)$ must be divisible by $p$. By Cauchy's theorem, $L(B/\!\!/P)$ contains a subgroup $H$ of order $p$. Since $H$ is a closed subset of $B/\!\!/P$, there exists a closed subset $P'$ of $B$ such that $P \subseteq P'$ and $P'/\!\!/P = H$. Since $P'/\!\!/P$ consists of linear elements of $B/\!\!/P$, we have that $P' \subseteq N_B(P)$. Furthermore, $p = o(P'/\!\!/P) = o(P')/o(P)$, proving (ii).

(iii): Suppose $P, Q \in \mathrm{Syl}_p(B)$. For each $P$-$Q$-double coset $PbQ$ for $b \in B$, we have that $o(PbQ)$ is a power of $p$ that is divisible by $o(P)$. Since the set of $P$-$Q$-double cosets is a partition of $B$, there is at least one $P$-$Q$-double coset $Pb_0Q$ for which $o(P) = o(Pb_0Q) = o(Q)$. But we know that these equalities can hold if and only if $b_0Qb_0^* \subseteq P$ and $b_0^*Pb_0 \subseteq Q$. Therefore, $P$ and $Q$ are conjugate in $B$.

(iv): Let $P$ be a Sylow $p$-subset of $B$. If $P = \{1\}$, then it is the unique Sylow $p$-subset of $B$, so we are done. If $P \neq \{1\}$, then $o(L(P))$ is divisible by $p$, and hence $L(P)$ contains a subgroup $H$ of order $p$. $H$ acts by conjugation on the set of all Sylow $p$-subsets of $B$. Since the orbits of this action have sizes either 1 or $p$, it suffices to show that the number of Sylow $p$-subsets that are fixed under conjugation by $H$ is congruent to 1 modulo $p$.

If $Q \in \mathrm{Syl}_p(B)$ is fixed under conjugation by $H$, then $H \subseteq N_B(Q)$, and thus $HQ$ is a closed subset of $B$. If $H \not\subseteq Q$, then $o(HQ/\!\!/Q) = o(H) = p$, which is a contradiction because $B$ does not contain a closed subset of order $po(Q)$. So we must have that $H \subseteq Q$ whenever $Q$ is a Sylow $p$-subset of $B$ that is fixed under conjugation by $H$.

This implies that there is a bijection between the set of Sylow $p$-subsets of $B$ that are fixed under conjugation by $H$ with the set of Sylow $p$-subsets of $B/\!\!/H$. By induction on $o(B)$, this number is congruent to 1 modulo $p$, so the result follows. $\square$

**EXERCISES.**

**Exercise 8.1.** Let $p$ be a prime, and suppose $\mathbb{C}B$ is a $p$-fractional $p$-valenced table algebra with standardized table basis $B$. Let $C$ be a fixed Sylow $p$-subset of $B$. Show that the number of Sylow $p$-subsets of $B$ is $|\{b \in B : \mathrm{supp}(b^*b) \subseteq C\}|$.

**Exercise 8.2.** Let $p$ be a prime, and suppose $\mathbb{C}B$ is a $p$-fractional $p$-valenced table algebra with standardized table basis $B$. Show that $B$ has a unique Sylow $p$-subset.

**Exercise 8.3.** Suppose $(X,S)$ is a $p$-valenced scheme. Let $P$ be a Sylow $p$-subset of $S$, and let $K = K_S(P)$ be the strong normalizer of $P$ in $S$; i.e. $s^*Ps = P$ for all $s \in K$.

a) Show that, for all $s \in S$, $s^*s \subseteq K \implies s^*s \subseteq P$.
   Hint: $K/\!\!/P = O_\vartheta(S/\!\!/P)$.

b) Show that the number of Sylow $p$-subsets of $S$ is bounded above by $\frac{n_{S/\!\!/K}}{n_{O_\vartheta(S/\!\!/K)}}$.

**Exercise 8.4.** Determine all of the Sylow 2-subsets of the scheme of order 12 with basic matrix

$$\sum_{i=0}^{d} i\sigma_i = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 & 4 & 5 & 5 & 6 & 6 & 7 & 7 \\ 1 & 0 & 3 & 2 & 4 & 4 & 5 & 5 & 6 & 6 & 7 & 7 \\ 2 & 3 & 0 & 1 & 6 & 6 & 7 & 7 & 4 & 4 & 5 & 5 \\ 3 & 2 & 1 & 0 & 6 & 6 & 7 & 7 & 4 & 4 & 5 & 5 \\ 4 & 4 & 7 & 7 & 0 & 1 & 6 & 6 & 5 & 5 & 2 & 3 \\ 4 & 4 & 7 & 7 & 1 & 0 & 6 & 6 & 5 & 5 & 3 & 2 \\ 5 & 5 & 6 & 6 & 7 & 7 & 0 & 1 & 2 & 3 & 4 & 4 \\ 5 & 5 & 6 & 6 & 7 & 7 & 1 & 0 & 3 & 2 & 4 & 4 \\ 7 & 7 & 4 & 4 & 5 & 5 & 2 & 3 & 0 & 1 & 6 & 6 \\ 7 & 7 & 4 & 4 & 5 & 5 & 3 & 2 & 1 & 0 & 6 & 6 \\ 6 & 6 & 5 & 5 & 2 & 3 & 4 & 4 & 7 & 7 & 0 & 1 \\ 6 & 6 & 5 & 5 & 3 & 2 & 4 & 4 & 7 & 7 & 1 & 0 \end{bmatrix}.$$

**Exercise 8.4.** Determine all of the Sylow 3-subsets of the scheme of order 18 with basic matrix

$$\sum_{i=0}^{d} i\sigma_i = \begin{bmatrix}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 6 & 6 & 7 & 7 & 7 & 8 & 8 & 8 & 9 & 9 & 9 \\
1 & 0 & 4 & 5 & 2 & 3 & 8 & 8 & 8 & 9 & 9 & 9 & 6 & 6 & 6 & 7 & 7 & 7 \\
2 & 5 & 0 & 4 & 3 & 1 & 8 & 8 & 8 & 9 & 9 & 9 & 6 & 6 & 6 & 7 & 7 & 7 \\
3 & 4 & 5 & 0 & 1 & 2 & 8 & 8 & 8 & 9 & 9 & 9 & 6 & 6 & 6 & 7 & 7 & 7 \\
5 & 2 & 3 & 1 & 0 & 4 & 6 & 6 & 6 & 7 & 7 & 7 & 8 & 8 & 8 & 9 & 9 & 9 \\
4 & 3 & 1 & 2 & 5 & 0 & 6 & 6 & 6 & 7 & 7 & 7 & 8 & 8 & 8 & 9 & 9 & 9 \\
6 & 9 & 9 & 9 & 6 & 6 & 0 & 4 & 5 & 8 & 8 & 8 & 7 & 7 & 7 & 1 & 2 & 3 \\
6 & 9 & 9 & 9 & 6 & 6 & 5 & 0 & 4 & 8 & 8 & 8 & 7 & 7 & 7 & 2 & 3 & 1 \\
6 & 9 & 9 & 9 & 6 & 6 & 4 & 5 & 0 & 8 & 8 & 8 & 7 & 7 & 7 & 3 & 1 & 2 \\
7 & 8 & 8 & 8 & 7 & 7 & 9 & 9 & 9 & 0 & 4 & 5 & 1 & 2 & 3 & 6 & 6 & 6 \\
7 & 8 & 8 & 8 & 7 & 7 & 9 & 9 & 9 & 5 & 0 & 4 & 2 & 3 & 1 & 6 & 6 & 6 \\
7 & 8 & 8 & 8 & 7 & 7 & 9 & 9 & 9 & 4 & 5 & 0 & 3 & 1 & 2 & 6 & 6 & 6 \\
9 & 6 & 6 & 6 & 9 & 9 & 7 & 7 & 7 & 1 & 2 & 3 & 0 & 4 & 5 & 8 & 8 & 8 \\
9 & 6 & 6 & 6 & 9 & 9 & 7 & 7 & 7 & 2 & 3 & 1 & 5 & 0 & 4 & 8 & 8 & 8 \\
9 & 6 & 6 & 6 & 9 & 9 & 7 & 7 & 7 & 3 & 1 & 2 & 4 & 5 & 0 & 8 & 8 & 8 \\
8 & 7 & 7 & 7 & 8 & 8 & 1 & 2 & 3 & 6 & 6 & 6 & 9 & 9 & 9 & 0 & 4 & 5 \\
8 & 7 & 7 & 7 & 8 & 8 & 2 & 3 & 1 & 6 & 6 & 6 & 9 & 9 & 9 & 5 & 0 & 4 \\
8 & 7 & 7 & 7 & 8 & 8 & 3 & 1 & 2 & 6 & 6 & 6 & 9 & 9 & 9 & 4 & 5 & 0
\end{bmatrix}.$$