Algebras of Linear Transformations: Corrections

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Introduction

These corrections are the result of feedback I received from Tim Bouma, Allan Donsig, Jonathon Dorfman, Alan Hopenwasser, Michael Muskulus, and Lewis Robinson since the book was published. I remain very appreciative of their remarks. I also am indebted to Tanya Gerasymova and Nadya Zharko for their assistance in preparing these "fix ups."

Typographical and other errors are dealt with first, followed by a simplified proof of Proposition 3.14 (page 88). Replacements for Proposition 3.23 and Corollary 3.24 (pages 99–100) and Theorem 4.23 (pages 141–142) are attached at the end of this document.

Typographical and Minor Mathematical Corrections

p. 15 In Line 7 in proof of 1.15, "By the First Isomorphism Theorem..." should be "By the Rank Nullity theorem...".

p. 43 Line 7 should be

$$\langle S\rangle = \{\sum_{i=1}^{m_1} s_i + \sum_{j=1}^{m_2} a_j s_j + \sum_{k=1}^{m_3} b_k s_k + \sum_{l=1}^{m_4} e_l s_l f_l \,:\, m_1, m_2, m_3, m_4 \in \mathbb{Z}_0^+, \\ s_i, s_j, s_k, s_l \in S,\ a_j, b_k, e_l, f_l \in \mathfrak{A}\}.$$

Line 10 should be

$$\langle a\rangle = \{\sum_{j=1}^{m} bac \,:\, m \in \mathbb{Z}^+, b, c \in \mathfrak{A}\}$$

p. 44 Line -6 should be $[x] = \{y \in \mathfrak{A} \,:\, y - x \in \mathfrak{J}\}$.

p. 46 In Line 6 "leads to a product that is nonassociative" should be "leads to a product that is associative".

p. 48 Line -3 should be $\frac{5}{6} + \frac{1}{3}\omega + \omega^2 + \frac{3}{2}\omega^3$,
In Line -13 the counter in the second sum on the left side of the formula should be h, not g.

p. 59 The displayed equation in line 8 should be

$$f_i(x) \;=\; \prod_{j \neq i} (\zeta_i - \zeta_j)^{-1}(x - \zeta_j)\,.$$

p. 68 Line 2 should be "It is readily seen that S is a real linear transformation",
Line 7 should be "S is a real linear isomorphism",
Line 8 should be "2-dimensional real vector space", and in
Line 10 "$\dim_{\mathbb{C}} \mathfrak{D}^- = 1$" should be removed.

p. 82 In Proposition 3.8(1), the equation should be $ETE = TE$.

p. 83 The displayed equation in line 8 should be

$$(1-E)T(1-E) = T - TE - ET + ETE = T - TE = T(1-E)\,.$$

p. 78 In Line -10, $\Psi(w) = 1$ should be $\Psi(u) = 1$.

p. 87 In Line 4 $\langle \xi_1 + \eta_1, \xi_1 \rangle$ should be $\langle \xi_1 + \eta_1, \xi_2 \rangle$.

p. 89 In Line 5 in proof of 3.16, $\vartheta_i \neq \mathcal{S}$ should be $\vartheta_i \notin \mathcal{S}$.

p. 90 In Line -4 the modifier "such that $\vartheta(v_0) \neq 0$" can be omitted.

p. 102 The displayed equation in line 11 should be

$$xy = x(bz) = (xb)z = (\pi(x)b)\,z = 0z = 0\,.$$

p. 105 In Lines 14-15 the "α_j" should be "$\overline{\alpha_j}$",
In Line 16 "$f(g(T))$" should be "$\widetilde{f}(g(T))$",
In Line -6 "$\operatorname{ran} P_j$" should be "$\operatorname{Ran} Q_j$".

p. 106 In Line 1 "$l \neq i$" should be "$j \neq i$".

p. 107 In Line 3 "and let g be it any function" should be "and let g be any function".

p. 109 In Line 3, the word *orthonormal* ought to be *eigenvalues*.

p. 124 In Line -2 and -1, Rad A should be Rad $\mathfrak{A}$.

p. 126 In Line 1, $J(\mathfrak{A})^k b \in \Omega$ should be $J(\mathfrak{A})^k b \subset \Omega$.

p. 129 In the proof of 4.14, the latter does [...] that $\mathfrak{J} = \mathfrak{M}$ should be the former does [...] that $\mathfrak{J} = \mathfrak{A}$.

p. 132 Line 10f should be:

$$ca = (ba)^{2^n-1}(ba) = (ba)^{2^n},$$

and consider $(ba)^{2^{n+1}}$, which is a nonzero element of $\mathcal{J}_n^2$ [...]

p. 137 In the second line of the proof of Lemma 4.20, the element r should be of the form $r = \sum_{j=1}^{k} s_j e t_j$, for some $s_j, t_j \in R$.

p. 143 In Line -1 remove "exist".

p. 144 Line 8 should be: $\cong \mathbb{Q}[x]/\langle x-1\rangle \times \mathbb{Q}[x]/\langle x+1\rangle$.

p. 145 Line -8 should be: (ii) $\mathbb{Q}C_2 \cong Q[x]/\langle x-1\rangle \times \mathbb{Q}[x]/\langle x+1\rangle$.

p. 145 Line -2 insert "the" before "case".

p. 148 Eliminate problem 12b.

p. 153 Line 12 should be:

$$= \left\langle \xi, \sum_{z\in G} \left(\sum_{g\in G} \bar{\alpha}_{gz^{-1}} \eta_g \right) z \right\rangle = \left\langle \xi, \sum_{g\in G} \left(\sum_{z\in G} \bar{\alpha}_{(gz^{-1})^{-1}} \eta_z \right) g \right\rangle$$

p. 156 In line 12, replace $\psi_1, \ldots, \psi_q$ with $\phi_1, \ldots, \phi_q$.

p. 157 In line 4, U should be V.

p. 159 In line 4, $UR_{ij}^{(\ell)}U^*$ should be $U_\ell R_{ij}^{(\ell)} U_\ell^*$.

p. 161 Line -2 in the proof of 5.7: Remove "Then $a^*a \in \mathrm{Rad}\,\mathfrak{A}$,".

p. 164 Line 11 should be:

$$p^* = z^{-1}ee^* = ez^{-1}e^* = ee^*z^{-1} = p,$$

p. 169 Line 4 remove second "that".

p. 170 Equation below (5.15.1) should be:

$$h = \varphi^{-1}\left(\sum_{i=1}^{k} \mu_i P_i\right) = \sum_{i=1}^{k} \mu_i p_i$$

p. 184 Remove second "algebras".

p. 188 The first sentence in Problem 5 is mysteriously absent. It should be: Consider the real algebra $\mathbb{C}$ under the involution $z^* = z$, for all $z \in \mathbb{C}$.

p. 194 In Proposition 6.10, X should be $\mathfrak{X}$.

p. 214 Line 11 should be: All that remains to prove is that [...]

p. 216 In Line 8, $S_k = \sum_i \sum_j \zeta_{ij}^{(k)} E_{ij} \otimes a_k$ should be: $S_k = \sum_i \sum_j \zeta_{ij}^{(k)} E_{ij}$.

p. 216 Line -13 should be: 6.19, Theorem 4.16 and [...]

p. 217 In lines 8 and 9, θ_1 should replace each occurence of θ and ϱ.

p. 222 In Line 5, (Exercise 17) should be (Exercise 15).

p. 224 In line 15, $\mathfrak{H}_1 \times \ \times \mathfrak{H}_2$ should be $\mathfrak{H}_1 \times \mathfrak{H}_2$.

p. 224 In line 22, $\mathfrak{H}_1 \otimes \ \ldots \otimes \mathfrak{H}_2$ should be $\mathfrak{H}_1 \otimes \mathfrak{H}_2$.

p. 225 Line 18ff should be:
Hence,

$$\begin{aligned} \langle \xi, \eta \rangle &= \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{l=1}^{m} \sum_{k=1}^{n} \alpha_{ij} \overline{\beta_{lk}} \langle e_i \otimes e_j, e_l \otimes e_k \rangle \\ &= \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{l=1}^{m} \sum_{k=1}^{n} \alpha_{ij} \overline{\beta_{lk}} \langle e_i, e_l \rangle \langle e_j, e_k \rangle \end{aligned}$$

The simplified proof of Proposition 3.14

Though the original proof is correct, the forward direction has a simpler proof by contradiction.
Let $L \in \operatorname{Lat}\mathfrak{A}$ be nontrivial and choose vectors $w \in V \setminus L$ and $0 \neq u \in L$. Because $\mathfrak{A}$ is a transitive algebra, there exists $A \in \mathfrak{A}$ such that $Au = w$. As L is A-invariant, this implies that $w \in L$, which is a contradiction.

Additional Comment

The remaining pages herein concern a more significant slip. My original claim in Proposition 3.23 was false. What was required in Proposition 3.23 was a minimal left ideal, not simply a single matrix column. Thus, Proposition 3.23 and Corollary 3.24 required rewriting, as did Theorem 4.23. These revisions are on the pages that follow.

In other words, the invariant subspace lattice of $\varrho(\mathfrak{A})$ is precisely the set of left ideals of $\mathfrak{A}$.

The concept of left ideal is, therefore, naturally related that of an invariant subspace, and a few of these connections are explored here.

Suppose that $\mathfrak{G}$ and $\mathfrak{N}$ are left ideals of an algebra $\mathfrak{A}$.

1. The left ideal $\mathfrak{N}$ is called a **maximal left ideal** if
   (i) $\mathfrak{N} \neq \mathfrak{A}$, and
   (ii) for every left ideal $\mathfrak{W}$ such that $\mathfrak{W} \supseteq \mathfrak{N}$, either $\mathfrak{W} = \mathfrak{N}$ or $\mathfrak{W} = \mathfrak{A}$.
2. The left ideal $\mathfrak{G}$ is called a **minimal left ideal** if
   (i) $\mathfrak{G} \neq \{0\}$, and
   (ii) for any left ideal $\mathfrak{K}$ of $\mathfrak{A}$ with $\mathfrak{K} \subseteq \mathfrak{G}$, either $\mathfrak{K} = \{0\}$ or $\mathfrak{K} = \mathfrak{G}$.

In this book, the minimal left ideals that we are most interested in are those in matrix algebras over division algebras. The following example shows how some of these ideals can be constructed.

3.23 Example. (A minimal left ideal of $M_n(\mathfrak{D})$.) Assume that $\mathfrak{D}$ is a division algebra over a field $\mathbb{F}$. Fix l and consider the set

$$\mathfrak{G}_l = \left\{ \begin{pmatrix} 0 & \dots & 0 & d_{1l} & 0 & \dots & 0 \\ 0 & \dots & 0 & d_{2l} & 0 & \dots & 0 \\ \vdots & & & \vdots & & & \vdots \\ 0 & \dots & 0 & d_{nl} & 0 & \dots & 0 \end{pmatrix} : d_{il} \in \mathfrak{D} \text{ for all } i \right\}.$$

Straightforward matrix multiplication reveals that $AX \in \mathfrak{G}_l$ for every $A \in M_n(\mathfrak{D})$ and every $X \in \mathfrak{G}_l$; thus, $\mathfrak{G}_l$ is a left ideal. To show that $\mathfrak{G}_l$ is minimal, first adopt the notation that was introduced in the proof of Proposition 2.32: $E_{ij} \otimes d_{ij}$ shall denote the matrix in $M_n(\mathfrak{D})$ whose (i,j)-entry is $d_{ij} \in \mathfrak{D}$ and whose other entries are zero. Thus, $\mathfrak{G}_l = \{\sum_{i=1}^n E_{il} \otimes d_{il} : d_{il} \in \mathfrak{D}, 1 \leq i \leq n\}$. Fix l and assume that $\mathfrak{W}$ is a nonzero left ideal such that $\mathfrak{W} \subseteq \mathfrak{G}_l$. We aim to prove that $\mathfrak{W} = \mathfrak{G}_l$.

Choose a nonzero $x \in \mathfrak{W}$. Then $x = \sum_{m=1}^n E_{il} \otimes s_{ml}$ and there is at least one i such that $s_{il} \neq 0$. Consider $y = E_{ii} \otimes s_{il}^{-1} \in M_n(\mathfrak{D})$. Because $\mathfrak{W}$ is a left ideal, we have $E_{il} \otimes 1 = yx \in \mathfrak{W}$. Likewise, for any k, $E_{kl} \otimes 1 = z(E_{il} \otimes 1) \in \mathfrak{W}$, where $z = E_{ik} \otimes 1$. Now let $d_{1l}, \dots, d_{nl} \in \mathfrak{D}$ be arbitrary. Then

$$\sum_{i=1}^n E_{il} \otimes d_{il} = \left(\sum_{i=1}^n E_{ii} \otimes d_{il}\right) \left(\sum_{i=1}^n E_{il} \otimes 1\right) \in \mathfrak{W}.$$

Hence, $\mathfrak{W} \supseteq \mathfrak{G}_l$. $\diamondsuit$

The left ideals $\mathfrak{G}_l$ in Example 3.23 are not the only minimal left ideals of $M_n(\mathfrak{D})$. For example,

$$\left\{ \begin{pmatrix} d_1 & d_1 & 0 \\ d_2 & d_2 & 0 \\ d_3 & d_3 & 0 \end{pmatrix} : d_1, d_2, d_3 \in \mathfrak{D} \right\}$$

is also a minimal left ideal of $M_3(\mathfrak{D})$. The link between this example and $\mathfrak{G}_1$ is that both are sets of the form $\{ae : a \in M_n(\mathfrak{D})\}$, where $e \in M_n(\mathfrak{D})$ is a fixed nonzero idempotent. This observation leads to a quite general characterisation of the minimal left ideals of $\mathfrak{L}(V)$.

3.24 Proposition. *If $\mathfrak{W} \subseteq \mathfrak{L}(V)$ and V is a nonzero finite-dimensional vector space, then $\mathfrak{W}$ is a minimal left ideal of $\mathfrak{L}(V)$ if and only if $\mathfrak{W} = \{ae : a \in \mathfrak{L}(V)\}$ for some rank-1 idempotent $e \in \mathfrak{L}(V)$.*

Proof. Assume that $e \in \mathfrak{L}(V)$ is an idempotent with one-dimensional range. Then there is a basis $\{v_1, \ldots, v_n\}$ of V such that $\operatorname{ran} e = \operatorname{Span}\{v_1\}$ and $\ker e = \operatorname{Span}\{v_2, \ldots, v_n\}$. The matrix representation of e with respect to the basis $\{v_1, \ldots, v_n\}$ is simply the matrix unit E_{11} in $M_n(\mathbb{F})$. Thus, $\{ae : a \in \mathfrak{L}(V)\}$ is represented as $\{AE_{11} : A \in M_n(\mathbb{F})\}$—in other words, as $\mathfrak{G}_1$. By Example 3.23, the left ideal $\mathfrak{G}_1$ is minimal; hence, $\{ae : a \in \mathfrak{L}(V)\}$ is a minimal left ideal of $\mathfrak{L}(V)$.

Conversely, assume that $\mathfrak{W}$ is a minimal left ideal of $\mathfrak{L}(V)$. Let $I \subseteq \mathfrak{L}(V)$ be the subspace of all finite sums of all linear transformations of the form wa, where $w \in \mathfrak{W}$ and $a \in \mathfrak{L}(V)$. It is plain to see that I is a nonzero ideal of $\mathfrak{L}(V)$; because $\mathfrak{L}(V)$ is simple, we conclude that $I = \mathfrak{L}(V)$. Therefore,

$$\begin{aligned} \mathfrak{L}(V) = I &= \left\{ \sum_{i=1}^{m} \tilde{w}_i y_i \,:\, m \in \mathbb{Z}^+, \tilde{w}_i \in \mathfrak{W}, y_i \in \mathfrak{L}(V) \right\} \\ &= \left\{ \sum_{i=1}^{m} \sum_{j=1}^{n} \tilde{w}_i w_j a_j \,:\, m, n \in \mathbb{Z}^+, \tilde{w}_i, w_j \in \mathfrak{W}, a_j \in \mathfrak{L}(V) \right\}. \end{aligned}$$

Hence, there is at least one pair $x, y \in \mathfrak{W}$ for which $xy \neq 0$.

With such a pair, consider now the left ideal $\mathfrak{L}_y = \{wy : w \in \mathfrak{W}\}$ of $\mathfrak{L}(V)$. The previous paragraph shows that $\mathfrak{L}_y$ is nonzero; thus, by the minimality of $\mathfrak{W}$, $\mathfrak{L}_y = \mathfrak{W}$. Consequently, there is a nonzero $e \in \mathfrak{W}$ with $ey = y$. From $e^2 y = e(ey) = ey = y$ follows $(e^2 - e)y = 0$. Set $\mathfrak{W}_0 = \{z \in \mathfrak{W} : zy = 0\}$; so $e^2 - e \in \mathfrak{W}_0$ and $\mathfrak{W}_0$ is a left ideal of $\mathfrak{L}(V)$ contained in $\mathfrak{W}$. However, $e \in \mathfrak{W} \backslash \mathfrak{W}_0$ implies that $\mathfrak{W}_0 = \{0\}$. Therefore, $e^2 = e$ and the left ideal $\{ae : a \in \mathfrak{L}(V)\}$ of $\mathfrak{L}(V)$ is nonzero and contained in $\mathfrak{W}$. Hence, $\mathfrak{W} = \{ae : a \in \mathfrak{L}(V)\}$. The proof that e has rank 1 is left as an exercise. $\square$

Thus, $M_n(\mathfrak{D})$ has exactly n distinct nonzero minimal left ideals. The dimension of each such left ideal is simply $n(\dim \mathfrak{D})$, because any one column has n positions and each entry in this column lies in a space of dimension equal to the dimension of $\mathfrak{D}$.

Now, what is a possible choice for the idempotent e whose role is so important in the proof of Theorem 4.21? Fix a minimal nonzero left ideal $\mathfrak{G}_l$. Basic matrix multiplication reveals that the matrix unit E_{ll} serves well for e. For any $A \in M_n(\mathfrak{D})$, $E_{ll}AE_{ll}$ is the matrix with the (l,l)-element of A in position (l,l) and zeros elsewhere. So clearly $E_{ll}(M_n(\mathfrak{D}))E_{ll} \cong \mathfrak{D}$, which is precisely to be expected from the proof of Theorem 4.21. Moreover, multiplication of any matrix A on the right by E_{ll} simply leaves column l of A fixed and sends the other columns to zero, hence, $\mathfrak{G}_l = M_n(\mathfrak{D})\, E_{ll}$, as predicted by the proof of the Theorem 4.21.

Further analysis of simple algebras must, evidently, involve a study of division algebras. The fascinating theory of division algebras is a highly nontrivial subject in itself, and we do not attempt to touch upon it in greater detail in this book.

4.4 Isomorphism Classes of Semisimple Algebras

The results concerning the structure of simple and semisimple algebras lead to a classification of such algebras up to isomorphism.

4.23 Isomorphism of Simple Algebras. *If $\mathfrak{D}_1$ and $\mathfrak{D}_2$ are division algebras over $\mathbb{F}$, and if $n_1, n_2 \in \mathbb{Z}^+$, then $M_{n_1}(\mathfrak{D}_1) \cong M_{n_2}(\mathfrak{D}_2)$ if and only if $n_1 = n_2$ and $\mathfrak{D}_1 \cong \mathfrak{D}_2$.*

Proof. Suppose that $\varphi : M_{n_1}(\mathfrak{D}_1) \to M_{n_2}(\mathfrak{D}_2)$ is an isomorphism. Let $\{e_{ij} : 1 \le i,j \le n_1\}$ and $\{f_{st} : 1 \le s,t \le n_2\}$ be the standard matrix units of $M_{n_1}(\mathbb{F})$ and $M_{n_2}(\mathbb{F})$, respectively. Using the notation of Proposition 2.32 and Example 3.23, every element $x \in M_{n_1}(\mathfrak{D}_1)$ has the form $x = \sum_{i,j} \delta_{ij} \otimes e_{ij}$, for some $\{\delta_{ij}\}_{i,j=1}^{n_1} \subset \mathfrak{D}_1$. Let $E_{ij} = 1 \otimes e_{ij} \in M_{n_1}(\mathfrak{D}_1)$. The matrices E_{ij} satisfy the usual relations for matrix units and, hence, so do the matrices $F_{ij} = \varphi(E_{ij}) \in M_{n_2}(\mathfrak{D}_2)$.

Because $\varphi(1) = 1$, we have that

$$1 \;=\; \sum_{i=1}^{n_1} F_{ii} \;=\; \sum_{t=1}^{n_2} 1 \otimes f_{tt}\,.$$

Further, there are $d_{st} \in \mathfrak{D}_2$ such that

$$1 \otimes f_{11} \;=\; \sum_{i,j=1}^{n_1} d_{st} \otimes F_{ij}\,.$$

Fix a pair (p,q) for which $d_{pq} \neq 0$. Consider $y = F_{1p}(d_{pq}^{-1} \otimes f_{11})$ and $z = (1 \otimes f_{11})F_{q1}$ in $M_{n_2}(\mathfrak{D}_2)$. Because $f_{11}^2 = f_{11}$, we have, on the one hand,

$$yz \;=\; F_{1p}(d_{pq}^{-1} \otimes f_{11})F_{q1} \;=\; F_{1p}\left(\sum_{i,j=1}^{n_1} d_{pq}^{-1} d_{st} \otimes F_{ij}\right) F_{q1} \;=\; \varphi(E_{11})\,.$$

On the other hand,

$$zy \;=\; (d_{pq}^{-1} \otimes f_{11})F_{11}(1 \otimes f_{11}) \;\in\; \{d \otimes f_{11} \,:\, d \in \mathfrak{D}_2\} \cong \mathfrak{D}_2\,,$$

where the identity of $\mathfrak{D}_2$ is identified with $1 \otimes f_{11}$. As $F_{11} = y(zy)z$, the element $zy \in \mathfrak{D}_2$ is nonzero. Moreover, $(zy)^2 = zy$; hence, via cancellation in $\mathfrak{D}_2$, $zy = 1 \otimes f_{11}$.

Next, let

$$a \;=\; \sum_{i=1}^{n_1} F_{ii} y (1 \otimes f_{1i})\,, \qquad b \;=\; \sum_{j=1}^{n_1} (1 \otimes f_{j1}) z F_{1t}\,.$$

Thus, $ab = 1$ and so

$$1 \;=\; ba \;=\; \sum_{j=1}^{n_1} (1 \otimes f_{jj}) \;=\; \sum_{t=1}^{n_2} (1 \otimes f_{tt})\,.$$

Hence, $n_1 = n_2$.

Finally,

$$\begin{aligned}
\mathfrak{D}_1 \cong E_{11}(M_{n_1}(\mathfrak{D}_1))E_{11} &\cong \varphi\left(E_{11}(M_{n_1}(\mathfrak{D}_1))E_{11}\right)\\
&= F_{11}(M_{n_2}(D_2))F_{11}\\
&= (1 \otimes f_{11})M_{n_2}(D_2)(1 \otimes f_{11})\\
&\cong \mathfrak{D}_2\,.
\end{aligned}$$

Conversely, the proof that $M_n(\mathfrak{D}_1) \cong M_n(\mathfrak{D}_2)$, if $\mathfrak{D}_1$ and $\mathfrak{D}_2$ are isomorphic division algebras, is easy and so is omitted. $\square$

4.24 Isomorphism of Semisimple Algebras. *Suppose that $\mathfrak{A}$ and $\mathfrak{B}$ are the semisimple algebras*

$$\begin{aligned}
\mathfrak{A} &\cong (M_{k_1}(\mathfrak{D}_1)) \times \cdots \times (M_{k_n}(\mathfrak{D}_n))\;,\\
\mathfrak{B} &\cong (M_{l_1}(\mathfrak{D}'_1)) \times \cdots \times (M_{l_m}(\mathfrak{D}'_m))\;.
\end{aligned}$$

Then $\mathfrak{A}$ and $\mathfrak{B}$ are isomorphic if and only if $m = n$ and there is a permutation $\tau \in S_n$ such that $l_i = k_{\tau(i)}$ and $\mathfrak{D}'_i \cong \mathfrak{D}_{\tau(i)}$ for all $i = 1, \ldots, n$.